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The Bounded Approximation Property for the Weighted (LB)-Spaces of Holomorphic Mappings on Banach Spaces

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Abstract. In this paper, we consider the bounded approximation property for the weighted (LB)-spaces $VH(E)$ and $VH_0(E)$ of entire functions defined corresponding to a decreasing family V of weights on a Banach space E. For a suitably restricted family V of weights, the bounded approximation property for a Banach space E has been characterized in terms of the bounded approximation property for the space $VG(E)$, the predual of the space $VH(E)$.

Keywords: (LB)-spaces, weighted spaces of holomorphic functions, bounded approximation property, Hörmander algebras.

MSC 2000 classification: 46B28, 46E50, 46G20

Introduction

The present paper deals with the study of the bounded approximation property(BAP) for weighted (LB)-spaces of holomorphic mappings on a Banach space. The study of the approximation properties for spaces of holomorphic functions on infinite dimensional spaces was initiated by R. Aron and M. Schottenloher in [1] and later continued in [11], [12], [13], [14], [15], [20], [21], [28]. However, by using the properties of Cesáro means operators, Bierstedt [7] studied the BAP for the projective and inductive limits of weighted spaces defined on a balanced open subset of a finite dimensional space given by a countable family V of weights. In case of infinite dimension, D. Carando, D. García, M. Maestre, P. Rueda and P. Sevilla-Peris in [16], [22], [23] carried out investigations on structural properties, spectrum, composition operators defined on the projective limit of weighted spaces of holomorphic mappings corresponding to an increasing sequence of weights. On the other hand, M.J. Beltrán [4, 5, 6] studied the inductive limits of such spaces defined by a decreasing family of weights which are referred as weighted (LB)-spaces. In our earlier work, we initiated the study of the approximation properties in [24, 25] for such spaces and continue the same in this paper. After having given preliminaries in Section 2, we prove in Section 3 that a separable Banach space E has the BAP if and only if $\mathcal{G}_w(E)$ has the BAP, where w is a weight defined with the help of strictly positive continuous function η on $[0,\infty)$, as $w(x) = \eta(\|x\|)$, $x \in E$. Finally, in Section 4, an $\mathcal{S}$-absolute decomposition is obtained for the space $VG(E)$, and the BAP for the spaces $VH(E)$ and $VG(E)$ is investigated.

1 Preliminaries

In this section, we recall some definitions and results from the theory of infinite dimensional holomorphy and approximation properties for which the references are [3], [17], [18], [27], [29], [34]. The symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{C}$ respectively denote the set of natural numbers, $\mathbb{N} \cup \{0\}$ and the complex plane. The letters E and F stand for complex Banach spaces and the symbols E' and E^* denote respectively the algebraic dual and topological dual of E. The symbols X and Y are used for locally convex spaces and $X_b^* = (X^*, \beta(X^*, X))$ denotes the strong dual of X, where $\beta(X^*, X)$ is the topology of uniform convergence on all bounded subsets of X.

For each $m \in \mathbb{N}$, $\mathcal{L}(^mE; F)$ denotes the Banach space of all continuous m-linear mappings from E to F endowed with its natural sup norm. For $m = 1$, $\mathcal{L}(^1E; F) = \mathcal{L}(E; F)$ is the class of all continuous linear operators from E to F. The topology of uniform convergence on compact subsets of E is denoted by τ_c. A mapping $P : E \to F$ is said to be a *continuous m-homogeneous polynomial* if there exists a continuous m-linear map $A \in \mathcal{L}(^mE; F)$ such that $P(x) = A(x, \dots, x)$, $x \in E$. The space of all m-homogeneous continuous polynomials from E to F is denoted by $\mathcal{P}(^mE; F)$ which is a Banach space endowed with the norm

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\|.$$

A polynomial $P \in \mathcal{P}(^mE, F)$ is said to be of *finite type* if it is of the form

$$P(x) = \sum_{j=1}^{k} \phi_j^m(x) y_j, \ x \in E$$

where $\phi_j \in E^*$ and $y_j \in F$, $1 \leq j \leq k$. We denote by $\mathcal{P}_f(^mE, F)$, the space of all finite type m-homogeneous polynomials from E into F. For $m = 1$, $\mathcal{P}_f(^mE, F)$

is the space of all finite rank operators from E to F, denoted by $\mathcal{F}(E,F)$. $\mathcal{P}_w({}^mE,F)$ denotes the space of all m-homogeneous polynomials which are *weakly uniformly continuous on bounded subsets* of E (a polynomial P is weakly uniformly continuous on bounded subsets if for any bounded subset B of E and $\epsilon > 0$, there exist $\phi_i \in E^*$, $i = 1, 2, \dots, k$ and $\delta > 0$ such that $\|P(x) - P(y)\| < \epsilon$ whenever $|\phi_i(x) - \phi_i(y)| < \delta$, $i = 1, 2, \dots, k$ and $x, y \in B$).

A mapping $f : U \to F$ is said to be *holomorphic*, if for each $\xi \in U$, there exists a ball $B(\xi, r)$ with center at ξ and radius $r > 0$, contained in U and a sequence $\{P_m\}_{m=1}^{\infty}$ of polynomials with $P_m \in \mathcal{P}({}^mE;F)$, $m \in \mathbb{N}_0$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - \xi) \tag{1}$$

where the series converges uniformly for each $x \in B(\xi, r)$. The space of all holomorphic mappings from U to F is denoted by $\mathcal{H}(U;F)$ and it is usually endowed with τ_0, the topology of uniform convergence on compact subsets of U. In case $U = E$, the class $\mathcal{H}(E,F)$ is the space of entire mappings from E into F. For $F = \mathbb{C}$, we write $\mathcal{H}(U)$ for $\mathcal{H}(U;\mathbb{C})$. A subset A of U is *U-bounded* if it is a bounded set and has a positive distance from the boundary of U. A holomorphic mapping f is of *bounded type* if it maps U-bounded sets to bounded sets. The space of holomorphic mappings of bounded type is denoted by $\mathcal{H}_b(U;F)$.

A weight v (a continuous and strictly positive function) defined on an open balanced subset U of E is said to be *radial* if $v(tx) = v(x)$ for all $x \in U$ and $t \in \mathbb{C}$ with $|t| = 1$; and on E it is said to be *rapidly decreasing* if $\sup_{x \in E} v(x)\|x\|^m < \infty$ for each $m \in \mathbb{N}_0$. Let us recall from [36], weighted spaces of entire functions defined as

$$\mathcal{H}_v(U;F) = \{f \in \mathcal{H}(U;F) : \|f\|_v = \sup_{x \in U} v(x)\|f(x)\| < \infty\}$$

and

$$\mathcal{H}_v^0(U;F) = \{f \in \mathcal{H}(U;F) : \text{for given } \epsilon > 0 \text{ there exists a U-bounded set } A \text{ such that } \sup_{x \in U \setminus A} v(x)\|f(x)\| < \epsilon\}.$$

The space $(\mathcal{H}_v(U;F), \|\cdot\|_v)$ is a Banach space and $\mathcal{H}_v^0(U;F)$ is a closed subspace of $\mathcal{H}_v(U;F)$. For $F = \mathbb{C}$, we write $\mathcal{H}_v(U) = \mathcal{H}_v(U;\mathbb{C})$ and $\mathcal{H}_v^0(U) = \mathcal{H}_v^0(U;\mathbb{C})$. The symbol B_v denotes the closed unit ball of $\mathcal{H}_v(U)$. It has been proved in [24], p.131 that the class $\mathcal{P}(E)$ of polynomials is a subspace of $H_v(E)$ if and only if the weight v is rapidly decreasing.

Following Beltran [4], let $V = \{v_n\}$ denote a countable decreasing family ($v_{n+1} \le v_n$ for each n) of weights on E. Corresponding to V, let us consider the weighted (LB)-spaces, *i.e.*, the inductive limit of weighted spaces $\mathcal{H}_{v_n}(E)$, $n \in \mathbb{N}$ defined as

$$VH(E) = \bigcup_{n\geq 1} \mathcal{H}_{v_n}(E)$$

and

$$VH_0(E) = \bigcup_{n\geq 1} \mathcal{H}^0_{v_n}(E)$$

and endowed with the locally convex inductive topologies $\tau_{\mathcal{I}}$ and $\tau'_{\mathcal{I}}$ respectively. Since the closed unit ball B_{v_n} of each $\mathcal{H}_{v_n}(E)$ is τ_0-compact, it follows by Mujica's completeness Theorem, cf. [32], $VH(E)$ is complete and its predual

$$VG(E) = \{\phi \in VH(E)' : \phi|B_{v_n} \text{ is } \tau_0 - \text{continuous for each } n \in \mathbb{N}\}$$

is endowed with the topology of uniform convergence on the sets B_{v_n}. In case V consists of a single weight v, $VG(E)$ is written as $\mathcal{G}_v(E)$.

Let Δ be a mapping from E to $VG(E)$ defined as $\Delta(x) = \delta_x$ where $\delta_x(f) = f(x)$, $f \in VH(E)$ and $x \in E$. Then $\Delta \in \mathcal{H}(E; VG(E))$. Let us recall the following linearization theorem from [4]

Theorem 1.1. (Linearization Theorem) Let E be a Banach space and V be a decreasing sequence of radial rapidly decreasing weights on E. For every Banach space F and every $f \in VH(E,F)$, there exists a unique continuous operator $T_f \in \mathcal{L}(VG(E),F)$ such that $T_f \circ \Delta = f$. Moreover the mapping $\psi : VH(E,F) \to \mathcal{L}_i(VG(E),F)$, $\psi(f) = T_f$ is a topological isomorphism.

In case V is a singleton set $\{v\}$, we write Δ_v for Δ. The mapping ψ becomes an isometric isomorphism between $\mathcal{H}_v(E,F)$ and $\mathcal{L}(G_v(E),F)$ with $\Delta_v \in \mathcal{H}_v(E; VG(E))$ and $\|\Delta_v\|_v \le 1$. Note that the above linearization theorem also holds for a weight v defined on an open subset U of E. As a consequence of Theorem 1.1, we have

Proposition 1.2. [25] Let v be a radial rapidly decreasing weight on a Banach space E. Then E is topologically isomorphic to a 1-complemented subspace of $\mathcal{G}_v(E)$.

Concerning the representation of members of $\mathcal{G}_v(U)$, we have

Proposition 1.3. [25] Let v be a weight on an open subset U of a Banach space E. Then each u in $\mathcal{G}_v(U)$ has a representation of the form

$$u = \sum_{n\geq 1} \alpha_n v(x_n) \Delta_v(x_n)$$

for $(\alpha_n) \in l_1$ and $(x_n) \subset U$.

A locally convex space X is said to have the *approximation property* if for any continuous semi-norm p on X, for a compact set K of X and $\epsilon > 0$, there exists a finite rank operator $T = T_{\epsilon,K}$ such that $\sup_{x\in K} p(T(x) - x) < \epsilon$. If $\{T_{\epsilon,K} : \epsilon > 0 \textit{ and } K$ varies over compact subsets of $X\}$ is an equicontinuous subset of $\mathcal{F}(X, X)$, X is said to have the BAP. When X is normed and there exists a $\lambda \geq 1$ such that $\|T_{\epsilon,K}\| \leq \lambda$ for each $\epsilon > 0$ and each compact set K, X is said to have the λ-BAP. In case $\lambda = 1$, we say X has the metric approximation property(MAP).

The following characterization of the BAP for a separable Banach space due to Pelczyński [35], is quoted from [33].

Theorem 1.4. For a separable Banach space E, the following assertions are equivalent:

(a) E has the bounded approximation property.

(b) There is a sequence $\{T_n\} \subset \mathcal{F}(E, E)$ such that $T_n(x) \to x$ for every $x \in E$.

(c) E is topologically isomorphic to a complemented subspace of a Banach space with a monotone Schauder basis.

Improving the above result, Mujica and Vieira [33] proved

Theorem 1.5. Let E be a separable Banach space with the λ-bounded approximation property. Then for each $\epsilon > 0$, there is a Banach space F with a Schauder basis such that E is isometrically isomorphic to a 1-complemented subspace of the Banach space F and, the sequence $\{T_n\}$ of canonical projections in F has the properties $\sup_n \|T_n\| \leq \lambda + \epsilon$ and $\limsup_{n\to\infty} \|T_n\| \leq \lambda$.

We also make use of the following result from [27], which can be easily verified.

Proposition 1.6. Let E be a Banach space with the λ-bounded approximation property. Then each complemented subspace of E with the projection map P has the $\lambda\|P\|$-bounded approximation property.

The following result proved for the MAP for any Banach space E in [25], is useful for the proof of the main result of this paper.

Theorem 1.7. Let v be a radial rapidly decreasing weight on a Banach space E satisfying $v(x) \leq v(y)$ whenever $\|x\| \geq \|y\|$, $x, y \in E$. Then the following assertions are equivalent:

(a) E has the MAP.

(b) $\overline{\{P \in \mathcal{P}_f(E, F) : \|P\|_v \leq 1\}}^{\tau_c} = B_{\mathcal{H}_v(E,F)}$ for any Banach space F.

(c) $\overline{B_{\mathcal{H}_v(E)\bigotimes F}}^{\tau_c} = B_{\mathcal{H}_v(E,F)}$ for any Banach space F.

(d) $\Delta_v \in \overline{B_{\mathcal{H}_v(E)\bigotimes\mathcal{G}_v(E)}}^{\tau_c}$.
(e) $\mathcal{G}_v(E)$ has the MAP.

A sequence of subspaces $\{X_n\}_{n=1}^{\infty}$ of a locally convex space X is called a *Schauder decomposition* of X if for each $x \in X$, there exists a unique sequence $\{x_n\}$ of vectors $x_n \in X_n$ for all n, such that

$$x = \sum_{n=1}^{\infty} x_n = \lim_{m\to\infty} u_m(x)$$

where the projection maps $\{u_m\}_{m=1}^{\infty}$ defined by $u_m(x) = \sum_{j=1}^{m} x_j$, $m \geq 1$ are continuous.

Corresponding to a sequence space $\mathcal{S} = \{(\alpha_n)_{n=1}^{\infty} : \alpha_n \in \mathbb{C}$ and $\limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \leq 1\}$, a Schauder decomposition $\{X_n\}_n$ is said to be $\mathcal{S}$-absolute if for each $\beta = (\beta_j) \in \mathcal{S}$

(i) $\beta \cdot x = \sum_{j=1}^{\infty} \beta_j x_j \in X$ for each $x = \sum_{j=1}^{\infty} x_j \in E$, and

(ii) for a continuous semi-norm p on X, $p_\beta(x) = \sum_{j=1}^{\infty} |\beta_j| p(x_j)$ defines a continuous semi-norm on X.

If each X_n is one dimensional and $X_n = \overline{span}\{x_n\}$, then $\{x_n\}_{n\geq 1}$ is a Schauder basis of X. A Schauder basis $\{x_n\}_{n\geq 1}$ of a Banach space X is called *monotone* if $\|u_m(x)\| \leq \|u_{m+1}(x)\|$ for each $m \in \mathbb{N}$ and each $x \in X$.

Proposition 1.8. [18] If a sequence $\{X_n\}_{n=0}^{\infty}$ of subspaces of a locally convex space X is an $\mathcal{S}$-absolute decomposition for X, then $\{(X_n)_b{}^*\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for X_b^*.

We quote from [11],[14], the following

Proposition 1.9. If $\{X_n\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition of a locally convex space X, then X has the AP(BAP) if and only if each X_n has the AP(BAP).

Theorem 1.10. [11] Let E be a Banach space. Then E^* has the BAP if and only if $\mathcal{P}_w(^nE)$ has the BAP for each $n \in \mathbb{N}$.

2 The Bounded Approximation Property for the space $\mathcal{G}_\eta(E)$

Throughout this section $\eta : [0, \infty) \to (0, \infty)$ denotes a continuous decreasing function satisfying $\eta(r)r^k \to 0$ for each $k \in \mathbb{N}_0$, i.e. η is rapidly decreasing.

For such a function η, consider a weight w on a Banach space E as $w(x) = \eta(\|x\|)$, $x \in E$. Clearly w is a radial rapidly decreasing weight on E satisfying $w(x) \le w(y)$ whenever $\|x\| \ge \|y\|$, $x, y \in E$. For a weight given by η, we write $\mathcal{H}_\eta$ for $\mathcal{H}_w$ and $\mathcal{G}_\eta$ for $\mathcal{G}_w$. In this section, we characterize the bounded approximation property for the space E in terms of the bounded approximation property for the space $\mathcal{G}_\eta(E)$.

Theorem 2.1. Let $(E, \|\cdot\|)$ be a Banach space with a monotone Schauder basis $\{x_n : n \in \mathbb{N}\}$ and w be a radial rapidly decreasing weight on E satisfying $w(x) \le w(y)$ whenever $\|x\| \ge \|y\|$, $x, y \in E$. Then, for each Banach space F and $f \in \mathcal{H}_w(E, F)$, there exists a sequence $\{f_n\} \subset \mathcal{H}_w(E) \bigotimes F$ such that $f_n \xrightarrow{\tau_c} f$ and $\|f_n\|_w \le \|f\|_w$, and a fortiori, $B_{\mathcal{H}_w(E) \bigotimes F}$ is τ_c-sequentially dense in $B_{\mathcal{H}_w(E,F)}$.

Proof. Let $f \in B_{\mathcal{H}_w(E,F)}$. For each $n \in \mathbb{N}$, write $B_n = E_n \cap nB_E$, where $E_n = \overline{span}\{x_1, x_2, \ldots, x_n\}$. Clearly each B_n is a compact subset of E. Thus, by implication $(a) \Rightarrow (c)$ of Theorem 1.7, there exists a sequence $(f_n) \subset B_{\mathcal{H}_w(E) \bigotimes F}$ with $\|f_n\|_w \le \|f\|_w$ for each $n \in \mathbb{N}$ such that

$$\sup_{x \in B_n} \|f_n(y) - f(y)\| < \frac{1}{n} \tag{2}$$

Since $\{f_n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{H}_w(E, F)$, for $\epsilon > 0$, there exists a $\delta > 0$ such that for each $n \in \mathbb{N}$ and $\|y - x\| < \delta$.

$$\|f_n(y) - f_n(x)\| < \epsilon. \tag{3}$$

As the basis $\{x_n\}$ is monotone, $E = \overline{\bigcup B_n}$. Fix $y_0 \in E$. Then for the above $\delta > 0$, choose $n_0 \in \mathbb{N}$ and $x \in B_{n_0}$ such that $\|y_0 - x\| < \delta$. By (2), $\{f_n(x)\}$ is a Cauchy sequence, and so $\{f_n(y_0)\}$ is Cauchy by (3). Thus $f_n(y) \to f(y)$, for each $y \in E$. As the sequence $\{f_n\}$ is equicontinuous, it converges in the τ_c topology. QED

A particular case of the above theorem for the weight given by η, is stated in

Theorem 2.2. Let E be Banach space with a monotone Schauder basis. Then $B_{\mathcal{H}_\eta(E) \bigotimes F}$ is τ_c-sequential dense in $B_{\mathcal{H}_\eta(E,F)}$ for each Banach space F.

The next proposition is a particular case of [33, Therorem 1.2]. Its proof makes use of the following lemma from [33].

Lemma 2.3. Let E be a finite dimensional Banach space of dimension n. Then for each $\epsilon > 0$, there exist $m \in \mathbb{N}$ and operators $U_1, \ldots, U_{mn} \in \mathcal{L}(E; E)$ of rank one such that

$$\sum_{j=1}^{mn} U_j(x) = x \quad \text{for each } x \in E$$

and

$$\|\sum_{j=1}^{k} U_j\| + \|\sum_{j=k+1}^{mn} U_j\| \leq 1+\epsilon \quad \text{for each} \quad 1 \leq k \leq mn.$$

Proposition 2.4. Let E be a separable Banach space with the bounded approximation property. Then there is a Banach space F having Schauder basis such that E is topologically isomorphic to a 1-complemented subspace of F.

Proof. To facilitate the reading, we outline its proof from [33], cf. the proof of Theorem 1.2. Assume that E has the λ-BAP for some $\lambda > 0$ and $\{y_n\}_{n=1}^{\infty}$ is a dense subset of E. Then for each $n \in \mathbb{N}$, there exist $T_n \in \mathcal{F}(E;E)$ such that $\|T_n\| \leq \lambda$ and $\|T_n y_i - y_i\| \leq \frac{1}{n}$ for $1 \leq i \leq n$. Let S_n be the sequence of operators defined as $S_1 = T_1$ and $S_n = T_n - T_{n-1}$, $n \geq 2$. Then $\sum_{n=1}^{\infty} S_n(x) = x$ for each $x \in E$.

Choose $\epsilon > 0$ arbitrarily and a strictly decreasing sequence $\{\epsilon_n\}$ of positive numbers tending to zero with $\epsilon_1 = \frac{\epsilon}{\lambda}$. Then for each $n \in \mathbb{N}$, there are operators $U_{n,1}, U_{n,2}, \ldots, U_{n,m_n}$ on $S_n(E)$ of rank one such that $\sum_{j=1}^{m_n} U_{n,j}x = x$ for each $x \in S_n(E)$ and $\|\sum_{j=1}^{k} U_{n,j}\| + \|\sum_{j=k+1}^{m_n} U_{n,j}\| \leq (1+\epsilon_n)$ for each $1 \leq k \leq m_n$ by Lemma 2.3.

For each $k \in \mathbb{N}$, we define the operators A_k and numbers δ_k as follows:

$$A_k = U_{n,i} \circ S_n \text{ and } \delta_k = \epsilon_n$$

where i, n are chosen such that k has the unique representation in the form $k = m_1 + \cdots + m_{n-1} + i$ with $1 \leq i \leq m_n$. Then

$$\|\sum_{j=1}^{k} A_j(x) - x\| \leq \|T_{n-1}x - x\| + (1+\epsilon_n)\|(T_n - T_{n-1})(x)\| \to 0 \quad \text{as } n \to \infty.$$

Since each A_j is of rank one, there exist $\phi_j \in E^*$ and $a_j \in E$ such that $A_j(x) = \phi_j(x)a_j$. Thus $\sum_{j=1}^{\infty} A_j(x) = \sum_{j=1}^{\infty} \phi_j(x)a_j = x$ and $\|\sum_{j=1}^{\infty} A_j\| \leq \lambda(1+\delta_k)$.

Now consider the sequence space $F = \{(\alpha_j) : \sum_{j=1}^{\infty} \alpha_j a_j$ converges *in* $E\}$. It is a Banach space endowed with the norm $|||(\alpha_j)||| = \sup_k \| \sum_{j=1}^{k} \alpha_j a_j\|$ for which unit vectors form a monotone Schauder basis. The mappings $A : E \to F$ and $B : F \to E$ defined as $A(x) = (\phi_j(x))$, $x \in E$ and $B((\alpha_j)) = \sum_{j=1}^{\infty} \alpha_j a_j$ satisfy

(i) $B \circ A = I_E$, the identity map on E, (ii) $\|B\| \leq 1$; and

(iii)$A \circ B : F \to F$ is a projection map.

Define $W = \{(\alpha_j) \in F : \| \sum_{j=1}^{\infty} \alpha_j a_j\| \leq 1, \| \sum_{j=1}^{k} \alpha_j a_j\| \leq \lambda(1 + \delta_k)$ for each k$\}$. Then

$$\{(\alpha_j) \in F : \|B((\alpha_j))\| \leq 1 \text{ and } \|y\| \leq \lambda\} \subset W \subset \{(\alpha_j) \in F : \|(\alpha_j)\| \leq (1 + \epsilon_1)\}.$$

Thus W is an absolutely convex, absorbing and bounded set and so its Minkowski functional $\| \cdot \|_W$ defines a norm on F. We write $F_W = (F, \| \cdot \|_W)$. Moreover, $|||\cdot||| \leq \|\cdot\|_W$ yields the identity $I : (F, \|\cdot\|_W) \to (F, |||\cdot|||)$ is a homeomorphism and so F_W is a Banach space having unit vectors as Schauder basis. It is shown in [33] that $A : E \to F_W$ is an isometric embedding and $A \circ B : F_W \to F_W$ is a projection map with $\|A \circ B\|_W \leq 1$. Thus $A(E)$ is 1-complemented in F_W. QED

Theorem 2.5. Let E be a separable Banach space with the BAP. Then $B_{\mathcal{H}_\eta(E) \bigotimes G}$ is τ_c-sequentially dense in $B_{\mathcal{H}_\eta(E,G)}$ for each Banach space G.

Proof. Consider $f \in B_{\mathcal{H}_\eta(A(E),G)}$. Recalling the notations from the proof of the preceeding proposition, we have $f \circ A \circ B \circ I \in B_{\mathcal{H}_\eta(F_W,G)}$; indeed,

$$\sup_{y \in F} \eta(\|y\|_W)\|f \circ A \circ B \circ I(y)\|_G \leq \sup_{y \in F} \eta(\|A \circ B \circ I(y)\|_W)\|f(A \circ B \circ I(y))\|_G \leq 1$$

as $\|A \circ B \circ I(y)\|_W \leq \|y\|_W$ for each $y \in F$ and η is decreasing. Thus, in view of Theorem 2.2, there exists a sequence $(f_n) \subset B_{\mathcal{H}_\eta(F_W) \otimes G}$ such that $f_n \xrightarrow{\tau_c} f \circ A \circ B \circ I$.

Write f'_n for the restriction of f_n on $A(E)$, *.i.e.* $f'_n = f_n|A(E)$.

In order to show the convergence of f'_n to f with respect to τ_c, consider a compact subset K of $A(E)$. Then $K = A(K')$ for some compact set K' of E. Consequently,

$$\begin{aligned}\sup_{y \in K} \|(f'_n - f)(y)\|_G &= \sup_{x \in K'} \|f'_n(A(x)) - f(A(x))\|_G \\ &= \sup_{x \in K'} \|f'_n(A(x)) - f \circ A \circ B \circ I \circ A(x)\|_G \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

QED

Next, we prove

Proposition 2.6. For a given weight w on an open subset U of a Banach space E, if the closed unit ball $B_{\mathcal{H}_w(U)\bigotimes F}$ is τ_c-sequentially dense in $B_{\mathcal{H}_w(U,F)}$ for some Banach space F, then $\mathcal{F}(\mathcal{G}_w(U),F)$ is sequentially dense in $(\mathcal{L}(\mathcal{G}_w(U),F),\tau_p)$ where τ_p is the the topology of pointwise convergence.

Proof. Choose $T\in\mathcal{L}(\mathcal{G}_w(U),F)$ and $\epsilon>0$ arbitrarily. Then by Theorem 1.1, there exist $f\in\mathcal{H}_w(U,F)$ such that $T=f\circ\Delta_w$. By the hypothesis, there is a sequence $(f_n)\subset H_w(U)\bigotimes F$ with $\|f_n\|_w\le\|f\|_w=\|T\|$ such that $f_n\xrightarrow{\tau_c} f$. Again, applying Theorem 1.1, there exist $T_n\in\mathcal{L}(\mathcal{G}_w(U),F)$ for each $n\in\mathbb{N}$ such that $T_n=f_n\circ\Delta_w$.

Now, fix $u\in\mathcal{G}_w(U)$. Then by Proposition 1.3,

$$u=\sum_{i\ge1}\alpha_i w(x_i)\delta_{x_i}$$

for $(\alpha_i)\in l_1$ and $(x_i)\subset U$. Choose $i_0\in\mathbb{N}$ such that $\sum\limits_{i>i_0}|\alpha_i|<\epsilon$.

$$\|(T_n-T)(u)\|\le\sum_{i=1}^{i_0}|\alpha_i|w(x_i)\|(f_n-f)(x_i)\|+2\epsilon\|T\|$$

The set $K=\{x_1,x_2,\dots,x_{i_0}\}$ is a compact subset of U. Hence, for $\epsilon'=\frac{\epsilon}{\sum\limits_{i\le i_0}|\alpha_i|w(x_i)}$, there exist $n_0\in\mathbb{N}$ such that $\sup\limits_{x\in K}\|f_n(x)-f(x)\|<\epsilon'$ for all $n\ge n_0$. Thus

$$\|(T_n-T)(u)\|<\epsilon(2\|T\|+1)$$

for all $n\ge n_0$. This completes the proof. QED

The above results lead to

Theorem 2.7. A separable Banach space E has the BAP if and only if $\mathcal{G}_\eta(E)$ has the BAP

Proof. Using Theorem 2.5 and Proposition 2.6 for $F=\mathcal{G}_\eta(E)$, the identity on $\mathcal{G}_\eta(E)$ can be approximated pointwise by finite rank operators. Hence by Theorem 1.4, $\mathcal{G}_\eta(E)$ has the BAP. The other implication follows immediately by Theorem 1.2 and 1.6. QED

3 The Approximation Properties for the space $VH(E)$ and its predual

In [37], R. Ryan has constructed the predual $\mathcal{Q}(^mE)$ of $\mathcal{P}(^mE), m \in \mathbb{N}$ which is defined as

$$\mathcal{Q}(^mE) = \{\phi \in \mathcal{P}(^mE)' : \phi|B_m \text{ is } \tau_0 - \text{continuous}\}$$

where B_m is the unit ball of $\mathcal{P}(^mE)$. The space $\mathcal{Q}(^mE)$ is endowed with the topology of uniform convergence on B_m. Analogous to condition (II) considered by García, Maestre and Rueda [22] for an increasing family of weights, Beltran [4] introduced a condition for a decreasing family V; indeed, a decreasing family V of weights satisfies condition (A) if for each $m \in \mathbb{N}$, there exist $D > 0$, $R > 1$ and $n \in \mathbb{N}$, $n \geq m$ such that

$$\|P^j f(0)\|_{v_n} \leq \frac{D}{R^j}\|f\|_{v_m}$$

for each $j \in \mathbb{N}$ and $f \in \mathcal{H}_{v_m}(E)$.

In this section, we show that the sequence of spaces $\{\mathcal{Q}(^mE)\}_{m=1}^{\infty}$ form an $\mathcal{S}$-absolute decomposition for $VG(E)$ when the family V satisfies condition (A) and consequently derive that a Banach space E has the BAP if and only if $VG(E)$ has the BAP.

Let us begin with

Lemma 3.1. Let V be a decreasing family of weights satisfying condition (A). Then for $\beta \in \mathcal{S}$ and $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ and $C > 0$ such that

$$\{\beta \cdot f = \sum_{j=0}^{\infty} \beta_j P^j f(0) : f \in B_{v_m}\} \subset CB_{v_n}.$$

Also,

$$\|\sum_{j=0}^{\infty} \beta_j P^j f(0)\|_n \leq \sum_{j=0}^{\infty} |\beta_j| \|P^j f(0)\|_n \leq C \ for\ each\ f \in B_{v_m} \tag{4}$$

Proof. Fix $\beta \in \mathcal{S}$ and $m \in \mathbb{N}$. Then, by condition (A), there exist $D > 0$, $R > 1$ and $n \in \mathbb{N}$, $n \geq m$ such that

$$\|P^j f(0)\|_{v_n} \leq \frac{D}{R^j}\|f\|_{v_m}$$

for each $j \in \mathbb{N}$ and $f \in \mathcal{H}_{v_m}(U)$. Now, for $R > 1$, we can choose $M > 0$ such that $|\beta_j| \le M(\frac{1+R}{2})^j$ for each $j \in \mathbb{N}$. Then, for each $f \in B_{v_m}$

$$\|\beta \cdot f\|_n \le \sum_{j=0}^{\infty} |\beta_j| \|P^j f(0)\|_{v_n} \le C\|f\|_{v_m} \tag{5}$$

where $C = \sum_{j=0}^{\infty} M(\frac{1+R}{2R})^j D$. Thus $\{\beta \cdot f : f \in B_{v_m}\} \subset CB_{v_n}$. Clearly (4) follows from (5). QED

Theorem 3.2. If $V = \{v_n\}$ is a decreasing family of radial rapidly decreasing weights satisfying condition (A), then the sequence of spaces $\{\mathcal{Q}(^m E)\}_{m=1}^{\infty}$ forms an $\mathcal{S}$-absolute decomposition for $VG(E)$ endowed with the topology of uniform convergence on B_{v_m} for each $m \in \mathbb{N}$, where B_{v_m} denotes the unit ball of $\mathcal{H}_{v_m}(E)$ for each m.

Proof. For $\phi \in VG(E)$ and $m \in \mathbb{N}$, define $\phi_m : VH(E) \to \mathbb{C}$ as

$$\phi_m(f) = \phi(P^m f(0)), \quad f \in VH(E)$$

Since $B_m \subset B_{v_m}$, $\phi_m|B_m$ is τ_0-continuous. Thus $\phi_m \in \mathcal{Q}(^m E)$. In order to show that $\phi = \sum_{j=0}^{\infty} \phi_j$ in $VG(E)$, fix $m \in \mathbb{N}$ and consider

$$\|\phi - \sum_{j=0}^{k-1} \phi_j\|_{B_{v_m}} = \sup_{f \in B_{v_m}} |\phi(\sum_{j=k}^{\infty} P^j f(0))| = \sup_{f \in B_{v_m}} \frac{1}{k^2} |\phi(k^2 \sum_{j=k}^{\infty} P^j f(0))|$$

Now, taking $\beta = (j^2)$ in (4), we have

$$\|k^2 \sum_{j=k}^{\infty} P^j f(0)\|_{v_n} \le \sum_{j=k}^{\infty} j^2 \|P^j f(0)\|_{v_n} \le C$$

for each $f \in B_{v_m}$. Thus, $\{k^2 \sum_{j=k}^{\infty} P^j f(0) : f \in B_{v_m}, k \in \mathbb{N}\} \subset CB_{v_n}$ and so

$$\|\phi - \sum_{j=0}^{k-1} \phi_j\|_{B_{v_m}} \le \frac{1}{k^2} \|\phi\|_{CB_{v_n}} \to 0$$

as $k \to \infty$.

For the continuity of the projection map $R_j : VG(E) \to \mathcal{Q}(^jE)$, $j \in \mathbb{N}$, consider a net (ϕ^η) in $VG(E)$, converging to 0. Then

$$\begin{aligned}\|R_j(\phi^\eta)\|_{B_{v_m}\bigcap \mathcal{P}(^jE)} &= \sup_{B_{v_m}\bigcap \mathcal{P}(^jE)} |\phi_j^\eta(P)| = \sup_{B_{v_m}\bigcap \mathcal{P}(^jE)} |\phi^\eta(P)| \\ &\leq \|\phi^\eta(P)\|_{B_{v_m}} \to 0.\end{aligned}$$

Finally, we show that this decomposition for $VG(E)$ is $\mathcal{S}$-absolute. Let $\beta = (\beta_j) \in \mathcal{S}$ and $\phi = (\phi_j) \in VG(E)$ and $l, k \in \mathbb{N}$ with $l \geq k$. Since $(j^2\beta_j) \in \mathcal{S}$, for $m \in \mathbb{N}$, by Lemma 3.1, there exist $C > 0$ and $n \in \mathbb{N}$ such that $\{\sum_{j=k}^{l} j^2\beta_j P^j f(0) : f \in B_{v_m}, k,\ l \in \mathbb{N}\} \subset CB_{v_n}$. Thus

$$\begin{aligned}\|\sum_{j=k}^{l}\beta_j\phi_j\|_{B_{v_m}} &= \sup_{f\in B_{v_m}} |\sum_{j=k}^{l}\beta_j\phi_j(f)| \leq \sum_{j=k}^{l}\frac{1}{j^2}\|\phi(j^2\beta_j P^j f(0))\|_{B_{v_m}} \\ &\leq \sum_{j=k}^{l}\frac{1}{j^2}\|\phi\|_{CB_{v_n}} \to 0 \quad \text{as } k,\ l \to \infty.\end{aligned}$$

Thus $\sum_{j=0}^{\infty}\beta_j\phi_j$ converges in $VG(E)$. Similar arguments would yield that the decomposition is absolute. $\boxed{QED}$

Making use of Proposition 1.9 and Theorem 3.2, we get

Theorem 3.3. Let V be a decreasing family of radial rapidly decreasing weights satisfying condition (A). Then the following are equivalent:

(a). E has the BAP(AP).

(b). $\mathcal{Q}(^mE)$ has the BAP(AP) for each $m \in \mathbb{N}$.

(c). $VG(E)$ has the BAP(AP).

Proof. $(a) \Leftrightarrow (b)$ This result is proved in [28] for the AP and in [13] for the BAP.

$(b) \Leftrightarrow (c)$ is a consequence of Proposition 1.9 and Theorem 3.2. $\boxed{QED}$

Remark 3.4. We would like to point out that Theorem 2.7 is not a particular case of Theorem 3.3 as the family V consisting of a single weight v never satisfies condition (A); indeed, if there exist some $D > 0$ and $R > 1$ such that for each $j \in \mathbb{N}$ and $f \in \mathcal{H}_v(U)$

$$\|P^j f(0)\|_v \leq \frac{D}{R^j}\|P^j f(0)\|_v$$

would yield $R^j \leq D$ for each j, which is not possible.

The following result of Beltran [4] is an immediate consequence of Theorem 3.2 and Proposition 1.8.

Proposition 3.5. If V is a decreasing family of radial rapidly decreasing weights satisfying condition (A), then $\{\mathcal{P}(^mE)\}_{m=0}^{\infty}$ is an $\mathcal{S}-$absolute decomposition for $VH(E)$.

Next, we prove

Theorem 3.6. Let E be a Banach space with $\mathcal{P}(^mE) = \mathcal{P}_w(^mE)$, for each $m \in \mathbb{N}$. Then, for a decreasing family V of radial rapidly decreasing weights on E satisfying condition (A), the following are equivalent:

(a). E^* has the BAP.

(b). $\mathcal{P}(^mE)$ has the BAP for each $m \in \mathbb{N}$.

(c). $VH(E)$ has the BAP.

Proof. $(a) \Leftrightarrow (b)$ follows from Theorem 1.10; and $(b) \Leftrightarrow (c)$ is a consequence of Propositions 1.9 and 3.5. QED

Since $\mathcal{P}(^m\mathbb{C}^N) = \mathcal{P}_w(^m\mathbb{C}^N)$ for each $m \in \mathbb{N}$ and for given $N \in \mathbb{N}$, we have

Corollary 3.7. The spaces $VH(\mathbb{C}^N)$ have the BAP for any family V of radial rapidly decreasing weights satisfying (A) and $N \in \mathbb{N}$.

Remark 3.8. In case the family V of radial rapidly decreasing weights satisfies condition (A), it has been proved in [5], $VH(E) = VH_0(E)$ for any Banach space E. The above corollary, thus, yields the result of Bierstedt, namely $(VH_0(\mathbb{C}^N), \tau_V)$ has the BAP for the restricted family V satisfying condition (A).

The Hörmander algebra $A_p(E)$ of entire functions defined on a Banach space has been studied by Beltran [5] who considered the sequence V of weights $v_n(x) = e^{-np(\|x\|)}$, $x \in E$, defined with the help of a growth condition p, *i.e.* an increasing continuous function $p : [0, \infty) \to [0, \infty)$ with the following properties:

(a) $\phi : r \to p(e^r)$ is convex.

(b) $log(1 + r^2) = o(p(r))$ as $r \to \infty$.

(c) there exists $\lambda \geq 0$ such that $p(2r) \leq \lambda(p(r) + 1)$ for each $r \geq 0$.

Then $V = \{v_n\}$ is a family of rapidly decreasing weights satisfying condition (A). This observation leads to

Example 3.9. Under the hypothesis of Theorem 3.6 for E, *i.e.*, $\mathcal{P}(^mE) = \mathcal{P}_w(^mE)$, for each $m \in \mathbb{N}$, $A_p(E)$ has the BAP whenever E^* has the BAP. In particular, $A_p(\mathbb{C}^N)$ has the BAP for each growth condition p and $N \in \mathbb{N}$.

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References

[1] R. Aron, M. Schottenloher: *Compact holomorphic mappings on Banach spaces and the approximation property*, J. Funct. Anal., 21 (1976), 7-30.

[2] R. Aron, J. B. Prolla: *Polynomial approximation of differentiable functions on Banach spaces*, J. Reine Angew. Math., 313 (1980), 195-216.

[3] J. A. Barroso: *Introduction to Holomorphy*, North-Holland Math. Studies, 106, North-Holland, Amsterdam, 1985.

[4] M. J. Beltrán: *Linearization of weighted (LB)-spaces of entire functions on Banach spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., RACSAM 106, 1 (2012), 275-286.

[5] M. J. Beltrán: *Spectra of weighted (LB)-algebras of entire functions on Banach spaces*, J. Math Anal. Appl., 387, no. 2 (2012), 604-617.

[6] M. J. Beltrán: Operators on weighted spaces of holomorphic functions, PhD Thesis, Universitat Politècnica de València, València, 2014.

[7] K. D. Bierstedt, J. Bonet, A. Galbis: *Weighted spaces of holomorphic functions on balanced domains*, Michigan Math. J., 40(1993), 271-297.

[8] K. D. Bierstedt, J. Bonet: *Biduality in Fréchet and (LB)-spaces*, in: Progress in Functional Anlysis, K. D. Bierstedt et. al.(eds.), North Holland Math. Stud. 170, North Holland, Amsterdam, 1992, 113-133.

[9] K. D. Bierstedt, R. G. Miese, W. H. Summers: *A projective description of weighted inductive limits*, Trans. Amer. Math. Soc., 272 (1982), 107-160.

[10] K. D. Bierstedt: *Gewichtete Raume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt II*, J. Reine Angew. Math., 260 (1973), 133-146.

[11] C. Boyd, S. Dineen, P. Rueda: *Weakly uniformly continuous holomorphic functions and the approximation property*, Indag. Math.(N.S.), 12 (2001), 147-156.

[12] E. Çalişkan: *Approximation of holomorphic mappings on infinite dimensional spaces*, Rev. Mat. Complut., 17 (2004), 411-434.

[13] E. Çalişkan: *The bounded approximation property for the predual of the space of bounded holomorphic mappings*, Studia Math., 177 (2006), 225-233.

[14] E. Çalişkan : *The bounded approximation property for spaces of holomorphic mappings on infinite dimensional spaces*, Math. Nachr., 279 (2006), 705-715.

[15] E. Çalişkan: *The bounded approximation property for weakly uniformly continuous type holomorphic mappings*, Extracta Math., 22(2007), 157-177.

[16] D. Carando, D. Garcia, M. Maestre, P. Sevilla-Peris: *On the spectra of algebras of analytic functions*, Contemp. Math., 561 (2012), 165-198.

[17] P. G. Casazza: Approximation properties, in: W.B. Johnson, J. Lindenstrauss (Eds.), Handbook of the geometry of Banach spaces 1 (2001): 271-316.

[18] S. Dineen: Complex Analysis in Locally Convex Spaces, North-Holland Math. Studies, vol. 57, North-Holland, Amsterdam, 1981.

[19] S. Dineen: Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag, London, 1999.

[20] S. Dineen, J. Mujica: *The approximation property for spaces of holomorphic functions on infinite dimensional spaces I*, J. App. Theory, 126 (2004), 141-156.

[21] S. Dineen, J. Mujica: *The approximation property for spaces of holomorphic functions on infinite dimensional spaces II*, J. Funct. Anal., 259 (2010), 545-560.

[22] D. García, M. Maestre, P. Rueda: *Weighted spaces of holomorphic functions on Banach spaces*, Studia Math., 138(1) (2000),1-24.

[23] D. García, M. Maestre, P. Sevilla- peris: *Weakly compact composition operators between weighted spaces*, Note Mat., 25(2005/06), 205-220.

[24] M. Gupta, D. Baweja: *Weighted Spaces of holomorphic functions on Banach Spaces and the approximation property*, Extracta Math., 31(2)(2016), 123-144.

[25] M. Gupta, D. Baweja: *The bounded approximation property for the weighted spaces of holomorphic mappings on Banach spaces*, Glasgow Math. J. (to appear).

[26] E. Jorda: *Weighted vector-valued holomorphic functions on Banach spaces*, Abstract and Applied Analysis, Vol. 2013. Hindawi Publishing Corporation, (2013).

[27] J. Lindenstrauss, L. Tzafriri: Classical Banach spaces I, Springer, Berlin 1977.

[28] J. Mujica: *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc., 324, 2 (1991), 867-887.

[29] J. Mujica: Complex Analysis in Banach Spaces, North-Holland Math. Studies, vol. 120, North-Holland, Amsterdam, 1986.

[30] J. Mujica: *Linearization of holomorphic mappings of bounded type*, In: Progress in Functional Analysis, K.D. Biersted et al.(eds.), North Holland Math. Stud. 170, North-Holland, Amsterdam (1992), 149-162.

[31] J. Mujica, L. Nachbin: *Linearization of holomorphic mappings on locally convex spaces*, J. Math. Pures. Appl., 71 (1992), 543-560.

[32] J. Mujica: *A completeness criteria for inductive limits of Banach spaces*. In Functional Analysis: Holomorphy and Approximation Theory II, North Holland Math. Studies, vol. 86 North-Holland, Amsterdam, 1984, 319-329.

[33] J. Mujica, D. M. Vieira: *Schauder basis and the bounded approximation property in Separable Banach spaces*, Studia Math., 196(1)(2010), 1-12.

[34] L. Nachbin: Topology on Spaces of Holomorphic Mappings, Springer-Verlag, Berlin 1969.

[35] A. Pelczyński: *Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis*, Studia Math., 40(1971), 239-242.

[36] P. Rueda: *On the Banach Dieudonne theorem for spaces of holomorphic functions*, Quaest. Math., 19(1996), 341-352.

[37] R. Ryan: Applications of topological tensor products to infinite dimensional holomorphy, PhD Thesis, Trinity College, Dublin 1980.