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Homogeneous pregeodesics and the orbits neighbouring a lightlike one

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Abstract. The spacelike and timelike orbits of isometric actions on Lorentz manifolds have invariant normal bundles, a fact which permits to study the neighbourhoods of such orbits. As the lightlike orbits do not admit an invariant normal bundle in general, other means are required. It is shown that the concept of homogeneous pregeodesics is helpful in this respect.

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Introduction

In studying isometric actions on Riemannian manifolds the fact is fundamental that any orbit has such an invariant neighbourhood where the orbits are closely related to the given one. The construction of such invariant neighbourhoods is based on the existence of invariant normal bundles of the orbits. In case of isometric actions on Lorentz manifolds the spacelike and the timelike orbits have invariant normal bundles, while the lightlike ones, in general, do not admit invariant normal bundles. Yet, as it is shown below, in some important cases of lightlike orbits it is possible to describe the neighbouring orbits. The simplest one of these cases is the the canonical action of the Lorentz group on the 4-dimensional Minkowski space where the lightlike orbits do not have invariant normal bundles, yet their neighbouring orbits are easy to describe. The objective of this paper is to mimic the above situation in as general settings as possible.

The results presented below are based on the concept of homogeneous pregeodesic. In fact, the lightlike tangent vectors to a lightlike orbit generate a 1-dimensional distribution on this orbit and lightlike geodesics yield the integral curves of this distribution, provided that the orbit has codimension 1. On the other hand, the integral curves of the above distribution are also obtainable as trajectories of some infinitesimal generators of the isometric action. If

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the parametrization of such a trajectory does not coincide with any admissible affine parametrization of the above geodesic, then this trajectory is called a genuine homogeneous pregeodesic. Those lightlike orbits of codimension 1 are considered below where all the above trajectories are genuine homogeneous pregeodesics and results are presented concerning the orbits neighbouring such lightlike ones.

1 Homogeneous pregeodesics

1 Definition. Let $(M, <,>)$ be a Lorentz manifold and $X : M \to TM$ its Killing field. If $X(z) \neq 0_z$ and

$$\nabla_{X(z)}X = 0_z$$

is valid for a $z \in M$, then the integral curve of X with initial point z is obviously a geodesic, and accordingly it is called a *homogeneous geodesic*.

Several results concerning homogeneous geodesics of Lorentz manifolds have been obtained during the past years. Recently lightlike homogeneous geodesics were studied by O. Kowalski, his coworkers and others [1], [2], [3], [5]. It seems that a related more general concept, that of homogeneous pregeodesic is also useful in case of lightlike geodesics which will be motivated and introduced subsequently.

2 Definition. Let $(M, <,>)$ be a Lorentz manifold and $X : M \to TM$ a Killing field, then the smooth function

$$\kappa : M \ni z \mapsto \frac{1}{2} < X(z), X(z) > \in \mathbb{R}$$

is called the *energy* of X. A simple calculation shows that

$$\mathrm{grad}\kappa(z) = -\nabla_{X(z)}X = A_{X(z)}X, \ z \in M$$

holds with the skew-symmetric tensor field $A_X = \mathcal{L}_X - \nabla_X$ (for the proof in the analogous Riemannian case see e. g. [4], vol. I, pp. 249-250). Consider also the *zero set*

$$\mathcal{Z}(X) = \{ \ z \in M \mid X(z) = 0_z \ \}$$

of X and the open set $M' = M - \mathcal{Z}(X)$. It will be assumed in what follows that $\mathrm{grad}\kappa$ is nowhere zero on M', therefore the connected components of the level sets of the restricted energy function $\widehat{\kappa} = \kappa\lceil M'$ are smooth hypersurfaces.

3 Lemma. *Let X be a Killing field of a Lorentz manifold, then the hypersurfaces given by the connected components of the zero level set $\widehat{\kappa}^{-1}(0)$ of the restricted energy function of X are lightlike if and only if*

$$\mathrm{grad}\widehat{\kappa}(z) = \lambda(z)X(z),\ z \in \widehat{\kappa}^{-1}(0)$$

holds with a smooth function $\lambda : \widehat{\kappa}^{-1}(0) \to \mathbb{R}$ which is nowhere 0.

Proof. Consider a connected component C of $\widehat{\kappa}^{-1}(0)$ and assume that $\mathrm{grad}\widehat{\kappa} = \lambda X$ holds with a smooth function λ which is nowhere 0 on C. Then $\mathrm{grad}\widehat{\kappa}$ is lightlike on C since X is lightlike there. But then

$$< \mathrm{grad}\widehat{\kappa}(z), T_zC >= 0,\ z \in C$$

implies that $T_zC,\ z \in C$ is lightlike, and thus the hypersurface C is lightlike.

Assume now conversely that the hypersurface C is lightlike. Then by the above mentioned orthogonality

$$\mathrm{grad}\widehat{\kappa}(z) \in T_zC$$

holds, and $\mathrm{grad}\widehat{\kappa}(z) \neq 0_z$ by assumption, therefore this vector is lightlike for $z \in C$. On the other hand $X(z)$ is lightlike, and since

$$0 = X < X, X >= 2 < \nabla_X X, X >= 2 < \mathrm{grad}\widehat{\kappa}, X >$$

is valid, $X(z) \in T_zC$ holds too. Consequently,

$$\mathrm{grad}\widehat{\kappa} = \lambda(z)X(z),\ z \in C$$

holds where $\lambda(z) \neq 0$ for $z \in C$, and since both X and $\mathrm{grad}\widehat{\kappa}$ are smooth, λ is smooth as well. *QED*

4 Definition. If $\gamma : I \to M$ is a geodesic of a Lorentz manifold $(M, <, >)$ and $\mu : J \to I$ is a smooth bijection where $J \subset \mathbb{R}$ is an interval, then

$$\gamma \circ \mu : J \to M$$

is called a *pregeodesic* of the Lorentz manifold.

Let X be a Killing field of a Lorentz manifold $(M, <, >)$, if $X(z) \neq 0_z$ and

$$\nabla_{X(z)} X = \nu X(z)$$

hold with a $\nu \in \mathbb{R}$ for a $z \in M$, then as an obvious simple argument shows, the integral curve of X with initial point z is a pregeodesic (see e. g. [6], pp. 69, 95-96) and it is said to be a *homogeneous pregeodesic*. If in particular $\nu \neq 0$ holds, then the integral curve is called a *genuine homogeneous pregeodesic*.

5 Remark. A genuine homogeneous pregeodesic can be only lightlike.

In fact, since $\nu \neq 0$, the following equality

$$0 = X(z) < X, X >= 2 < \nabla_{X(z)} X, X(z) >= 2 < \nu X(z), X(z) >$$

implies that $< X(z), X(z) >= 0$ is valid. But since $X(z) \neq 0_z$ is assumed, $X(z)$ has to be lightlike.

6 Corollary. *Let $(M, <,>)$ be a Lorentz manifold and $X : M \to TM$ a Killing field. Then the connected components of the zero level set $\widehat{\kappa}^{-1}(0)$ of the restricted energy function $\widehat{\kappa}$ of X are lightlike hypersurfaces if and only if those integral curves of X which have their starting points in these components are genuine homogeneous pregeodesics.*

7 Lemma. *Let $(M, <,>)$ be a Lorentz manifold, $\Phi : G \times M \to M$ isometric action of a connected Lie group, $\gamma : I \to M$ a maximal lightlike geodesic and $X \in \mathfrak{g}$ an infinitesimal generator such that*

$$\Phi(\mathrm{Exp}(\tau X), z) = \gamma \circ \mu(\tau), \ \tau \in \mathbb{R}$$

holds with a smooth bijection $\mu : \mathbb{R} \to I$ for some $z \in \gamma(I)$. Then one of the following two cases is valid:

$$\mu(\tau) = \alpha\tau + \beta, \ \tau \in \mathbb{R}, \text{with some } \alpha \in \mathbb{R} - \{0\}, \ \beta \in \mathbb{R};$$

$$\mu(\tau) = \frac{\varepsilon}{\lambda} e^{\lambda\tau} + \alpha\tau + \beta, \ \tau \in \mathbb{R}, \text{with some } \varepsilon, \ \lambda \in \mathbb{R} - \{0\}, \text{and } \alpha, \ \beta \in \mathbb{R}.$$

In the first case $\gamma \circ \mu$ is a homogeneous geodesic and in the second one a genuine homogeneous pregeodesic under the action Φ.

Proof. Consider the smooth curve $\psi = \gamma \circ \mu : \mathbb{R} \to M$, then for its tangent field the following holds

$$\psi' = \mu' \cdot (\dot{\gamma} \circ \mu),$$

$$\nabla_{\psi'} \psi' = \nabla_{\psi'} \big(\mu' \cdot (\dot{\gamma} \circ \mu)\big) = \mu'' \cdot (\dot{\gamma} \circ \mu) + (\mu')^2 \cdot \big((\nabla_{\dot{\gamma}} \dot{\gamma}) \circ \mu\big) =$$

$$= \frac{\mu''}{\mu'} \cdot \psi'.$$

Moreover, since the action Φ is isometric, it is affine with respect to the Levi-Cività covariant derivation of the Lorentz manifold. Therefore the following are also valid

$$\Phi_{\mathrm{Exp}(\vartheta X)} \psi(\tau) = \psi(\tau + \vartheta), \ \tau, \vartheta \in \mathbb{R},$$

$$T_{\psi(\tau)} \Phi_{\mathrm{Exp}(\vartheta X)} \nabla_{\psi'(\tau)} \psi' = \nabla_{\psi'(\tau+\vartheta)} \psi' = \frac{\mu''(\tau + \vartheta)}{\mu'(\tau + \vartheta)} \psi'(\tau + \vartheta),$$

$$T_{\psi(\tau)}\Phi_{\mathrm{Exp}(\vartheta X)}\nabla_{\psi'(\tau)}\psi' = T_{\psi(\tau)}\Phi_{\mathrm{Exp}(\vartheta X)}\big(\frac{\mu''(\tau)}{\mu'(\tau)}\cdot\psi'(\tau)\big)$$
$$= \frac{\mu''(\tau)}{\mu'(\tau)}T_{\psi(\tau)}\Phi_{\mathrm{Exp}(\vartheta X)}\psi'(\tau) = \frac{\mu''(\tau)}{\mu'(\tau)}\cdot\psi'(\tau+\vartheta).$$

Now the above equalities imply that the function

$$\frac{\mu''}{\mu'} : \mathbb{R}\to\mathbb{R}$$

is constant. Let $\lambda\in\mathbb{R}$ be the constant value of the above function. Then considering the function μ the following two cases are possible:

1^{st} case: If $\lambda=0$, then $\mu(\tau)=\alpha\tau+\beta,\ \tau\in\mathbb{R}$ with some $\alpha\in\mathbb{R}-\{0\},\ \beta\in\mathbb{R}$.

2^{nd} case: If $\lambda\neq 0$, then $\mu(\tau)=\frac{\varrho}{\lambda}e^{\lambda\tau}+\delta,\ \tau\in\mathbb{R}$ with some $\varrho,\ \lambda\in\mathbb{R}-\{0\}$ and $\delta\in\mathbb{R}$. QED

8 Corollary. *In the above specified first case the lightlike geodesic γ has to be complete, and in the second one it has to include an open half of a complete lightlike geodesic.*

9 Definition. Let $(M,<,>)$ be a lightlike geodesically complete Lorentz manifold, $\Phi : G\times M\to M$ isometric action of a connected Lie group and $X\in\mathfrak{g}-\{0\}$ an infinitesimal generator such that a lightlike geodesic $\gamma:\mathbb{R}\to M$ exists with

$$\Phi(\mathrm{Exp}(\tau X),z)=\gamma(\frac{\varrho}{\lambda}e^{\lambda\tau}+\delta),\ \tau\in\mathbb{R},$$

where $\varrho,\ \lambda\in\mathbb{R}-\{0\}$ and $\delta\in\mathbb{R}$. Put now $\varepsilon=\mathrm{sign}(\lambda)$, then the point

$$s(\gamma)=\gamma(\delta)=\lim_{\varepsilon\tau\to-\infty}\gamma(\frac{\varrho}{\lambda}e^{\lambda\tau}+\delta)$$

exists and it is called the *stationary point* of the genuine homogeneous pregeodesic.

10 Lemma. *Let $(M,<,>)$ be a lightlike geodesically complete Lorentz manifold, $\Phi : G\times M\to M$ isometric action of a connected Lie group, $X\in\mathfrak{g}-\{0\}$ an infinitesimal generator and $\gamma:\mathbb{R}\to M$ a lightlike geodesic such that*

$$\Phi(\mathrm{Exp}(\tau X),z)=\gamma(\frac{\varrho}{\lambda}e^{\lambda\tau}+\delta),\ \tau\in\mathbb{R}$$

holds with $\varrho,\ \lambda\in\mathbb{R}-\{0\},\ \delta\in\mathbb{R}$. Then the stationary point $s(\gamma)$ of the above genuine homogeneous pregeodesic is invariant under each

$$\Phi_{\mathrm{Exp}(\sigma X)} : M\to M,\ \sigma\in\mathbb{R}.$$

Proof. In fact, since the action Φ is continuous, the following holds

$$\Phi(\mathrm{Exp}(\sigma X), s(\gamma)) = \Phi(\mathrm{Exp}(\sigma X), \lim_{\varepsilon\tau\to-\infty} \Phi(\mathrm{Exp}(\tau X), z)) =$$

$$= \lim_{\varepsilon\tau\to-\infty} \Phi(\mathrm{Exp}((\sigma+\tau)X, z) = s(\gamma).$$

QED

2 The orbits neighbouring a lightlike one

In case of the canonical action of the Lorentz group $O(1;3)$ on the 4-dimensional Minkowski space $\mathbb{R}^4_1$ a lightcone is the orbit of the identity component of $O(1;3)$ and its generators are genuine homogeneous pregeodesics. The orbits neighbouring the light cone are the origin, which is included in the closure of the lightcone, furthermore spacelike and timelike orbits of codimension 1. It will be shown subsequently that the above phenomenon is quite general. Namely, in case of a lightlike orbit of codimension 1 with genuine homogeneous pregeodesics there is an orbit of lower dimension included in the closure of the given lightlike one, moreover spacelike and timelike ones also of codimension 1 neighbouring the given one.

11 Definition. Let $(M, <,>)$ be a Lorentz manifold, $L \subset M$ an immersed lightlike submanifold, and $E_z \subset T_zL$ the unique 1-dimensional lightlike subspace for $z \in L$. Then a smooth 1-dimensional distribution $\mathcal{E}$ is defined by

$$\mathcal{E} : L \ni z \mapsto E_z \in T_zL.$$

A maximal integral manifold of $\mathcal{E}$ is called a *generator* of L. The submanifold L is obviously obtainable as disjoint union of its generators. If $A \subset L$ is a generator of L, then a smooth bijection

$$\xi : I \to A$$

of the interval $I \subset \mathbb{R}$ is called a *parametric representation* of A.

12 Lemma. *Let $L \subset M$ be a lightlike hypersurface of a Lorentz manifold and $A \subset L$ its generator. Then any parametric representation*

$$\xi : I \to A$$

is a pregeodesic of the Lorentz manifold.

Proof. Fix $z \in L$ and let X be a smooth vector field defined on a neighbourhood W of z in M such that its values in $W \cap L$ span the distribution $\mathcal{E}\lceil(W \cap L)$. Let U be a smooth vector field on W such that its values on $W \cap L$

are in TL. Then the values of $[U,X]$ on $W \cap L$ are also in TL. Thus the following holds on $W \cap L$

$$0 = U < X, X >= 2 < \nabla_U X, X >= 2 < \nabla_X U + [U,X], X >= 2 < \nabla_X U, X >=$$

$$= 2(X < U, X > - < U, \nabla_X X >) = -2 < U, \nabla_X X > .$$

The above calculation yields now that on $W \cap L$ the following holds

$$\nabla_X X = \phi X$$

where $\phi : W \cap L \to \mathbb{R}$ is a smooth function. But then an integral curve $\xi : I \to L$ of X is a pregeodesic by a basic result (see e. g. [6], p. 69). QED

13 Corollary. *Let $G(z) \subset M$ be a lightlike orbit of codimension 1 of an isometric action in a Lorentz manifold and $X \in \mathfrak{g}$ an infinitesimal generator such that $X(z) \in T_z G(z)$ is lightlike. Then the integral curve of X with initial point z is a homogeneous pregeodesic.*

14 Lemma. *Let $(M, <,>)$ be a Lorentz manifold, $\Phi : G \times M \to M$ isometric action of a connected Lie group and $G(z) \subset M$ its lightlike orbit of codimension 1. The action Φ is transitive on the set of generators of $G(z)$.*

Proof. Since a generator of $G(z)$ is such a lightlike geodesic which is a unique one in each of its points, the action Φ maps a generator of $G(z)$ to a generator of $G(z)$. As the action Φ is transitive on $G(z)$ it is transitive on the set of its generators as well. QED

15 Theorem. *Let $(M, <,>)$ be a lightlike geodesically complete Lorentz manifold, $\Phi : G \times M \to M$ isometric action of a connected Lie group and $G(z)$ a lightlike orbit of codimension 1 such that each generator of $G(z)$ is a genuine homogeneous pregeodesics by the action of a 1-parameter subgroup of G. Then the following hold:*

(1) *The set $S(G(z)) = \{ \; s(A) \; | A \subset G(z) \; \}$ of stationary points of generators A of $G(z)$ is an orbit of Φ, and the map*

$$\varpi : G(z) \supset A \ni z' \mapsto s(A) \in S(G(z)),$$

which maps a point of $G(z)$ to the stationary point of the generator containing this point, is equivariant.

(2) *The map ϖ induces a smooth equivariant map*

$$\omega : G/G_z \to G/G_x$$

of the above coset manifolds associated with the orbits $G(z)$, $G(x)$, where $x = \varpi(z)$.

(3) *The inclusion* $G(x) \subset \overline{G(z)}$ *holds and the inequality*

$$\dim G(x) < \dim G(z)$$

is valid.

PROOF. (1) Since the generators of $G(z)$ are genuine homogeneous pregeodesics, each generator A of $G(z)$ has a stationary point $s(A)$. In order to show that the map $\varpi : G(z) \to S(G(z))$ is equivariant, consider a point $z' \in G(z)$ and the generator A containing z' which is a genuine homogeneous pregeodesic by a 1-parameter subgroup $\mathrm{Exp}(\tau X),\ \tau \in \mathbb{R}$ of G. Then for $g \in G$ the following holds

$$\Phi(g, \varpi(z')) = \Phi(g, \lim_{\varepsilon\tau \to -\infty} \Phi(\mathrm{Exp}(\tau X), z') = \lim_{\varepsilon\tau \to -\infty} \Phi(g\mathrm{Exp}(\tau X), z') =$$

$$= \lim_{\varepsilon\tau \to -\infty} \Phi(g\mathrm{Exp}(\tau X)g^{-1}, \Phi(g, z')) = \lim_{\varepsilon\tau \to -\infty} \Phi(\mathrm{Exp}(\tau \mathrm{Ad}(g)X), \Phi(g, z')) =$$

$$= \varpi(\Phi(g, z')),$$

where ε is given by the sign of that λ which is in the expression of the genuine homogeneous pregeodesic corresponding to the generator A. Consequently, ϖ is equivariant and thus $S(G(z))$ is an orbit of Φ.

(2) Consider now the smooth coset manifolds $G/G_z,\ G/G_x$ associated with the orbits $G(z),\ G(x)$ and their smooth injective immersions

$$\chi_z : G/G_z \to G(z) \subset M.\ \chi_x : G/G_x \to G(x) \subset M.$$

Then the map defined by

$$\omega = \chi_x \circ \varpi \circ \chi_x^{-1} : G/G_z \to G/G_x$$

is equivariant with respect to the canonical actions of G on these coset manifolds. But a map of a coset manifold to another coset manifold of the same Lie group which is equivariant with respect to the canonical actions of the Lie group on these coset manifolds, is smooth. Thus the map ω is smooth.

(3) The inclusion $G(x) = S(G(z)) \subset \overline{G(z)}$ is a consequence of the definition of stationary points of generators. Observe now that the tangent linear map map

$$T_{z'}\varpi : T_{z'}G(z) \to T_{x'}G(x),$$

where $x' = \varpi(z')$ and the generator A containing z', is an epimorphism. Then

$$T_{z'}A \subset \mathrm{Ker} T_{z'}\varpi$$

holds, and implies the validity of $\dim G(x) < \dim G(z)$. QED

16 Definition. Let $(M, <,>)$ be a Lorentz manifold, $\Phi : G \times M \to M$ isometric action of a connected Lie group and $G(z)$ its orbit of codimension 1. Assume that z has a neighbourhood $V \subset M$ such that $G(x)$ has codimension1 for $x \in V$. Then there are infinitesimal generators $X_1, \ldots, X_{m-1} \in \mathfrak{g}$ of Φ such that

$$X_1(x), \ldots, X_{m-1}(x) \in T_xG(x)$$

yields a base of $T_xG(x)$ for $x \in U$, where $U \subset V$ is a suitable neighbourhood of z. Consider now the vector space $\wedge^{m-1}(T_xM)$ of $(m-1)$-vectors over T_xM which is the vector space of homogeneous elements of degree $m-1$ in the Grassmann algebra over T_xM. Consider also the semi-euclidean inner product

$$<,>_x^{m-1} : \wedge^{m-1}(T_xM) \times \wedge^{m-1}(T_xM) \to \mathbb{R}$$

induced by the Lorentzian inner product $<,>_x$ of T_xM for $x \in U$. Now the smooth function $\omega : U \to \mathbb{R}$ defined by

$$\omega(x) =< X_1(x) \wedge \ldots \wedge X_{m-1}(x), X_1(x) \wedge \ldots \wedge X_{m-1}(x) >_x^{m-1}, \ x \in U$$

is called an *orbit causality characterizing function.* Namely, considering the equality

$$\omega(x) = \begin{vmatrix} < X_1(x), X_1(x) > & \ldots & < X_1(x), X_{m-1}(x) > \\ \vdots & \ddots & \vdots \\ < X_{m-1}(x), X_1(x) > & \ldots & < X_{m-1}(x), X_{m-1}(x) > \end{vmatrix},$$

and the fact that the above one is a Gram determinant, the following holds

$$\omega(x) = \begin{cases} > 0 & \text{if } G(x) \text{ spacelike,} \\ = 0 & \text{if } G(x) \text{ lightlike,} \\ < 0 & \text{if } G(x) \text{ timelike,} \end{cases}$$

as an obvious simple calculation shows.

17 Lemma. *Let $(M, <,>)$ be a Lorentz manifold, $\Phi : G \times M \to M$ isometric action of a connected Lie group, $G(z)$ a lightlike orbit of codimension 1 and $U \subset M$ a neighbourhood of z such that there is an orbit causality characterizing function $\omega : U \to \mathbb{R}$ with*

$$\mathrm{grad}\omega(z) \neq 0_z.$$

Then there is a neighbourhood $W \subset U$ of z such that the following hold:

(1) $W^0 = \omega^{-1}(0) \cap W = G(z) \cap W$.

(2) $W - W^0$ *has two connected components*

$$W' = \omega^{-1}(\mathbb{R}^+) \cap W, \ W'' = \omega^{-1}(\mathbb{R}^-) \cap W$$

such that $G(x)$ is spacelike if $x \in W'$ and timelike if $x \in W''$.

Proof. The *lemma* is an obvious consequence by the assumption

$$\mathrm{grad}\omega(z) \neq 0_z$$

of the preceding observations. $\boxed{QED}$

18 Proposition. *Let $(M, <,>)$ be an m-dimensional Lorentz manifold,*

$$\Phi : G \times M \to M$$

isometric action of a connected Lie group, $G(z)$ a lightlike orbit of codimension 1 and $V \subset M$ a neighbourhood of z such that $G(x)$ has codimension 1 for $x \in V$. Let $X_1, \ldots, X_{m-1} \in \mathfrak{g}$ be infinitesimal generators of Φ such that

(1) *$X_1(z)$ is lightlike;*

(2) *$(X_1(z), \ldots, X_{m-1}(z))$ is an orthogonal base of $T_zG(z)$.*

Let $\omega : U \to \mathbb{R}$ be the orbit causality characterizing function associated with the above system of infinitesimal generators on a suitable neighbourhood $U \subset V$ of z. Then

$$\mathrm{grad}\omega(z) = -2\nabla_{X_1(z)}X_1 < X_2(z), X_2(z) > \cdots < X_{m-1}(z), X_{m-1}(z) >$$

is valid.

Proof. Consider now the smooth vector bundle $\wedge^{m-1}(TM)$ formed by the $(m-1)$-tangent vectors of M, namely

$$\wedge^{m-1}(TM) = \cup\{ \wedge^{m-1}(T_xM) \mid x \in M \}$$

with its canonical smooth vector bundle structure. Consider also the covariant derivation ∇^{m-1} induced by the Levi-Cività covariant derivation ∇ of the Lorentz manifold $(M, <,>)$ on the space $\mathcal{S}(\wedge^{m-1}(TM)$ of smooth sections of the above vector bundle, namely

$$\nabla^{(m-1)} : \mathcal{T}(M) \times \mathcal{S}(\wedge^{(m-1)}(TM)) \to \mathcal{S}(\wedge^{(m-1)}(TM)).$$

Then the following calculations can be made

$$v\omega = 2 < \nabla_v^{(m-1)}(X_1 \wedge \ldots \wedge X_{m-1}), X_1 \wedge \ldots \wedge X_{m-1} >^{(m-1)} =$$

$$= 2\sum_{i=1}^{m-1} < X_1 \wedge \ldots \wedge X_{i-1} \wedge \nabla_v X_i \wedge X_{i+1} \wedge \ldots \wedge X_{m-1}, X_1 \wedge \ldots \wedge X_{m-1} >^{(m-1)} =$$

$$= 2\sum_{i=1}^{m-1}$$

$$\begin{vmatrix} <X_1,X_1> & \dots & <X_{i-1},X_1> & <\nabla_v X_i,X_1> & <X_{i+1},X_1> & \dots & <X_{m-1},X_1> \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ <X_1,X_{m-1}> & \dots & <X_{i-1},X_{m-1}> & <\nabla_v X_i,X_{m-1}> & <X_{i+1},X_{m-1}> & \dots & <X_{m-1},X_{m-1}> \end{vmatrix}$$

$$= 2\begin{vmatrix} <\nabla_v X_1,X_1> & <X_2,X_1> & \dots & <X_{m-1},X_1> \\ \vdots & \vdots & \ddots & \vdots \\ <\nabla_v X_1,X_{m-1}> & <X_2,X_{m-1}> & \dots & <X_{m-1},X_{m-1}> \end{vmatrix} =$$

$$=< \nabla_v X_1, X_1(z) >< X_2(z), X_2(z) > \cdots < X_{m-1}(z), X_{m-1}(z) > .$$

Since X_1 is a Killing field, the corresponding $(1,1)$-tensor field $A_{X_1} = \mathcal{L}_{X_1} - \nabla_{X_1}$ is skew symmetric and therefore

$$< \nabla_v X_1 . X_1(z) >= - < A_{X_1} v, X_1(z) >=< v, A_{X_1} X_1(z) >= - < v, \nabla_{X_1(z)} X_1 >$$

holds. Thus the following is obtained

$$v\omega = -2 < v, \nabla_{X_1(z)} X_1 >< X_2(z), X_2(z) > \cdots < X_{m-1}(z), X_{m-1}(z) > .$$

Therefore

$$\mathrm{grad}\omega(z) = -2 < X_2(z), X_2(z) > \cdots < X_{m-1}(z), X_{m-1}(z) > \nabla_{X_1(z)} X_1.$$

is obtained, since $v \in T_z M$ is arbitrary. QED

19 Corollary. *Let $(M, <,>)$ be a Lorentz manifold, $\Phi : G \times M \to M$ isometric action of a connected Lie group, $G(z)$ a lightlike orbit of codimension 1 and $V \subset M$ a neighbourhood of z such that $G(x)$ has codimension 1 if $x \in V$ and $\omega : U \to \mathbb{R}$ a causality characterizing function defined on a neighbourhood $U \subset V$ as above. Then*

$$\mathrm{grad}\omega(z) \neq 0_z$$

if and only if the integral curve of the infinitesimal generator $X_1 = X \in \mathfrak{g}$ with $X(z)$ lightlike is a genuine homogeneous pregeodesic.

Proof. Assume that $\mathrm{grad}\omega(z) \neq 0_z$ holds. Then by the preceding *proposition* $\nabla_{X(z)} X \neq 0_z$ holds. Moreover, also by the preceding *proposition*

$$0 =< \mathrm{grad}\omega(z), T_z\omega^{-1}(0) >=< \nabla_{X(z)} X, T_z\omega^{-1}(0) >$$

holds, and therefore

$$\nabla_{X(z)} X = \nu X(z)$$

holds with a $\nu \neq 0$. But then the integral curve of X with initial point z is a genuine homogeneous pregeodesic.

Assume conversely that the integral curve of $X = X_1$ with initial point z is a genuine homogeneous pregeodesic. Then by the preceding *proposition* $\mathrm{grad}\omega(z) \neq 0_z$ holds, since $X_2(z), \dots, X_{m-1}(z)$ are non-vanishing spacelike vectors. QED

20 Theorem. *Let* $(M, <,>)$ *be a geodesically complete Lorentz manifold,* $\Phi : G \times M \to M$ *isometric action of a connected Lie group and* $G(z)$ *its lightlike orbit of codimension* 1 *such that the following hold:*

(1) *The generators of* $G(z)$ *are genuine homogeneous pregeodesics.*

(2) $\dim\ G_z \geq 1$ *is valid.*

(3) *There is a neighbourhood* $V \subset M$ *of* z *such that* $G(x)$ *has codimension* ≥ 1 *for* $x \in V$.

Then there is a neighbourhood $W \subset V$ *of* z *such that* $W - (G(z) \cap W)$ *has two connected components* W', W'' *and*

$$G(x)\ is\ a \begin{cases} spacelike\ orbit,\ if\ x \in W' \\ timelike\ orbit,\ if\ x \in W'', \end{cases}$$

which has codimension 1.

Proof. The above *theorem* is a direct consequence of *lemma 17*, *proposition 18* and of the preceding *corollary*. QED

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