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Stability and the first Betti number

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Abstract. This short, expository note focuses on the index of compact, minimal hypersurfaces of a sphere. After reviewing the main facts, we announce, without proof, a comparison theorem between the spectrum of the stability operator of such immersions and that of the Laplacian on 1-forms. The geometric consequence is a lower bound of the index by the first Betti number of the hypersurface which implies that, if the first Betti number is large, then the immersion is highly unstable. Proofs will appear elsewhere.

Keywords: minimal hypersurfaces, index, Hodge Laplacian, first Betti number

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1 General facts on minimality

1.1 Minimal immersions and the Jacobi operator

It is known that minimal hypersurfaces of a Riemannian manifold are critical points of the volume functional. Precisely, let M be a compact, orientable Riemannian manifold of dimension n immersed in the $(n+1)$-dimensional manifold $\bar{M}$ and let N be a unit normal vector field along M. To each $u \in C^\infty(M)$ is associated a 1-parameter variation of M as follows:

$$M_t = \{\exp_x(tu(x)N) : x \in M\}.$$

The *shape operator* of M applied to the tangent vector $X \in TM$ is defined by $S(X) = -\bar{\nabla}_X N$ where $\bar{\nabla}$ is the Levi-Civita connection of the ambient manifold $\bar{M}$. Let $H = \frac{1}{n}\mathrm{tr}S$ denote the *mean curvature* of the immersion. It turns out that:

$$\frac{d}{dt}\mathrm{Vol}(M_t)|_{t=0} = -n\int_M uH.$$

Thus M is critical for the volume functional if and only if its mean curvature vanishes identically on M, that is, if and only if M is a minimal hypersurface.

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If we want more information about the nature of these critical points we need to consider the second variation of the volume. So assume that M is minimal. A well-known computation shows that then:

$$Q(u) \doteq \frac{d^2}{dt^2}\mathrm{Vol}(M_t)|_{t=0}= \int_M |\nabla u|^2 - (Ric_{\bar{M}}(N,N) + |S|^2)u^2, \qquad (1)$$

where $\bar{R}$ denotes the Ricci tensor of the ambient manifold $\bar{M}$ and $|S|^2$ the squared norm of the shape operator. The quadratic form Q is called the *index form,* and the associated operator:

$$J = \Delta - (Ric_{\bar{M}}(N,N) + |S|^2),$$

acting on $C^\infty(M)$, is the *Jacobi (or stability) operator* of the immersion. This means that for all $u \in C^\infty(M)$ one has:

$$Q(u) = \int_M uJu.$$

As J is self-adjoint and elliptic, it admits a sequence of eigenvalues diverging to infinity:

$$\lambda_1(J) \le \lambda_2(J) \le \dots \le \lambda_k(J) \le \dots$$

(actually, $\lambda_1(J)$ is simple: $\lambda_2(J) > \lambda_1(J)$).

We define the *index* of the minimal immersion as the number of negative eigenvalues of its Jacobi operator J.

The index is a non-negative integer which measure how far M is from being a local minimum. In fact, assume that the index is zero so that all eigenvalues are non-negative. Then, for all $u \in C^\infty(M)$ one has $Q(u) \ge 0$ and this means that M is a local minimum for the volume functional. We say in that case that M is *stable.* On the other hand, if λ is a negative eigenvalue of J associated to the eigenfunction u, then by (1):

$$\frac{d^2}{dt^2}\mathrm{Vol}(M_t)|_{t=0} = \lambda \int_M u^2 < 0,$$

and the volume is decreasing under the deformation associated to u. In conclusion, the index is the maximal number of linearly independent deformations along which the volume is decreasing, so if the index is very large then the immersion is "highly unstable".

Finally, we point out the following immediate consequence of the definition: if the ambient manifold $\bar{M}$ has positive Ricci curvature, then any minimal hypersurface of $\bar{M}$ is unstable, that is, can be deformed into one of smaller volume. For the proof, just consider the variation associated to the constant function $u = 1$, for which the index form is negative.

1.2 Index of minimal spherical immersions

This case will be the focus of the present note, and we would like to recall some previously known facts. So let M^n be a (compact) minimal hypersurface of $\mathbf{S}^{n+1}$. The Jacobi operator takes the form:

$$J = \Delta - (|S|^2 + n).$$

In particular, if M^n is totally geodesic (i.e. an equator of $\mathbf{S}^{n+1}$) then $J = \Delta - n$ and so J has eigenvalues $-n$ (counted once), then 0 (counted $n+1$ times) etc. It follows that its index is one. Simons proves in [S] that:

Any minimal spherical hypersurface has index at least one. Moreover, if the index is one then the hypersurface is totally geodesic.

Actually it is known that:

If M^n is not totally geodesic then its index is at least $n+3$.

Let us sketch the proof. Compose the given immersion $\phi : M^n \to \mathbf{S}^{n+1}$ with the canonical immersion of $\mathbf{S}^{n+1}$ into $\mathbf{R}^{n+2}$, and pick any parallel vector field $\bar{V}$ on $\mathbf{R}^{n+2}$. Consider the function $u = \langle \bar{V}, N\rangle$ on M. Then an easy calculation shows that u is an eigenfunction of the Jacobi operator associated to the eigenvalue $-n$; as the dimension of the space of parallel vector fields on $\mathbf{R}^{n+2}$ is $n+2$ we obtain in this way $n+2$ linearly independent eigenfunctions associated to $-n$ (they are linearly independent because otherwise M^n would be contained in a hyperplane, hence it would be totally geodesic). Now, the first eigenvalue of J is simple, hence less than $-n$: one concludes that there are at least $n+3$ negative eigenvalues, as asserted.

The above inequality is sharp because the Clifford tori:

$$Cl_{k,n} = \mathbf{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbf{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right),$$

for $k = 1, \dots, n-1$, can be minimally immersed in $\mathbf{S}^{n+1}$ and have index equal to $n+3$. In that case $|S|^2 = n$ so that $J = \Delta - 2n$ and the spectrum of the Jacobi operator is completely known and computable.

It is an interesting fact that, in dimension 2, the equality characterizes the (unique) Clifford torus among non-totally geodesic surfaces. In fact one has the following result, due to F. Urbano (see [5]):

1 Theorem. *Let M^2 be a compact, orientable minimal surface in $\mathbf{S}^3$. Then its index is at least 5, and the equality holds if and only if M is the Clifford torus $\mathbf{S}^1\left(\sqrt{\frac{1}{2}}\right) \times \mathbf{S}^1\left(\sqrt{\frac{1}{2}}\right)$.*

The validity of Theorem 1 in dimension $n \geq 3$ is still an open question (to the best of our knowledge).

Finally, some facts on the first eigenvalue $\lambda_1(L)$. J. Simons proved in [4] that, if M^n is not totally geodesic, then

$$\lambda_1(L) \leq -2n, \tag{2}$$

and O. Perdomo showed in [1] that the equality holds if and only if M^n is a Clifford torus.

2 Stability and the first Betti number

The scope of this section is to announce a new lower bound of the index of a spherical minimal immersion by the first Betti number. This shows that, if the de Rham cohomology in degree one is large, then the immersion is highly unstable. The result follows from a general comparison theorem between the spectrum of the Jacobi operator and that of the Laplacian acting on 1-forms.

2.1 The Hodge Laplacian

On a compact Riemannian manifold M, the Laplacian acting on smooth differential p-forms (also known as the *Hodge Laplacian*) is the operator:

$$\Delta_p = d\delta + \delta d,$$

where d is the exterior differential and δ is the formal adjoint of d with respect to the canonical L^2-inner product of p-forms. We list its eigenvalues repeating them according to multiplicity:

$$\lambda_1(\Delta_p) \leq \lambda_2(\Delta_p) \leq \dots$$

One has always $\lambda_1(\Delta_p) \geq 0$. A p-form ω is *harmonic* if $\Delta_p \omega = 0$. A fundamental theorem, due to Hodge and de Rham, asserts that the vector space of harmonic p-forms is isomorphic to the p-th de Rham cohomology space of M (with real coefficients). Hence by our convention,

$$\lambda_j(\Delta_p) = 0 \quad \text{for all } j = 1, \dots, b_p(M)$$

where $b_p(M)$ is the p-th Betti number of M.

2.2 The main estimates

The estimates in this section are proved in [3] and will appear soon. We first give the result in dimension $n \geq 3$.

2 Theorem. *Let M^n be a minimal hypersurface of $\mathbf{S}^{n+1}$ with dimension $n \geq 3$ and Jacobi operator J. Then for all positive integers k one has:*

$$\lambda_k(J) \leq \lambda_{m(k)}(\Delta_1) - 2(n-1),$$

where $m(k) = \binom{n+2}{2}(k-1) + 1$. In particular, if $b_1(M) \geq 1$ then:

$$\lambda_k(J) \leq -2(n-1) \quad \text{for all } k \leq I(n) \doteq \left[\frac{b_1(M) + \binom{n+2}{2} - 1}{\binom{n+2}{2}} \right].$$

where $[x]$ denotes the largest integer which is less than or equal to x.

Thus, we have at least $I(n)$ eigenvalues which are less than or equal to $-2n+2$ (this fact generalizes the estimate at the end of the previous section when the first Betti number is large). As the first Betti number is positive, M can't be totally geodesic and so we have at least $n+2$ eigenvalues which are equal to $-n > -2n+2$. In conclusion, we have at least $I(n)+n+2$ negative eigenvalues, which brings us to the following consequence on the index:

3 Corollary. *Let M^n be a minimal hypersurface of $\mathbf{S}^{n+1}$. Assume that $b_1(M) \geq 1$ and $n \geq 3$. Then:*

$$\text{Index}(M) \geq \left[\frac{b_1(M) + \binom{n+2}{2} - 1}{\binom{n+2}{2}} \right] + n + 2.$$

Thus, the index is bounded below by a linear function of the first Betti number. Note that we get an equality for the (unique) Clifford torus with $b_1(M) = 1$.

In dimension 2 we have a slightly better estimate.

4 Theorem. *Let M^2 be a minimal surface in $\mathbf{S}^3$ with genus $g \geq 1$. Then:*

$$\text{Index}(M) \geq \left[\frac{g+1}{2} \right] + 4.$$

Note that this improves Theorem 1 when $g \geq 5$; in particular, if the index is equal to 6 then the genus g must be less than or equal to 4.

For (compact, orientable, non totally geodesic) minimal surfaces in a flat three torus, A. Ros obtained the lower bound:

$$\text{Index}(M) \geq \frac{2g-3}{3},$$

which is sharp for the Schwarz P-surface (see [2]).

2.3 Final remarks

The idea of the proof is to pair a given eigenform of Δ_1 with some natural families of vector fields on the hypersurface and obtain in this way test-functions for the Jacobi operator. These test-vector fields depend on the dimension of M. Complete details will appear soon, and we refer to [3].

The present methods do not work for degrees $p \neq 1, n-1$ (unless we assume some unnatural lower bound on the scalar curvature). It is then natural to ask whether or not it is possible to have a lower bound of the index by the p-th Betti number, for $p \neq 1, n-1$ and $n \geq 4$, similar to that of Theorem 2.

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