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Submanifolds of (para-)quaternionic Kähler manifolds

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Abstract. After a concise introduction on (para-)quaternionic geometry, we report on some recent results concerning para-quaternionic Hermitian and Kähler manifolds and their special submanifolds. The second part of the paper is devoted to treat in a unified way some basic matters on (para-)complex submanifolds of (para-)quaternionic manifolds.

Keywords: (para-)quaternionic Kähler manifolds - (para-)complex submanifolds

MSC 2000 classification: primary 53C40, secondary 53C26, 53C15

1 Introduction

This paper is aimed to report on some recent work on special submanifolds of quaternionic and para-quaternionic manifolds.

We start by giving a concise introduction to basic structures, such as complex and para-complex, having to do with (para)-quaternionic geometry and focus in particular on para-quaternionic structures, whose theory was developed more recently.

A first systematic study of submanifolds of a para-quaternionic manifold was made in [27]. In particular, the invariant subspaces of a para-quaternionic (Hermitian) vector space were classified and described in detail.

Passing to submanifolds here we consider in particular almost (para-)complex submanifolds and try to give a unified presentation of known results by dealing simultaneously with all possible interesting cases. A special attention is given to questions of integrability of an almost (para-)complex structure on a submanifold and some problems are pointed out.

Also some basic result on minimality of almost (para-)complex submanifolds of a (para-)quaternionic Kähler manifold are stated, by extending the work previously done for almost complex submanifolds of a quaternionic Kähler manifold.

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2 (Para-)quaternionic structures on a manifold M

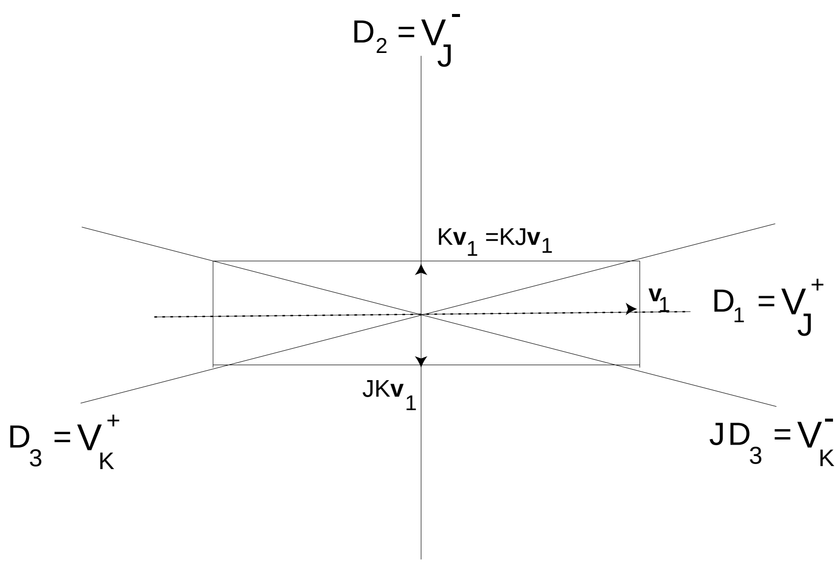
Let M be a differentiable manifold. Let consider the following structures on a real vector space $V \equiv T_xM$, $x \in M$, where one assumes that I, J, K are given endomorphisms:

COMPLEX	**PARA-COMPLEX**
$J^2 = -Id$	$J^2 = Id$
$V \otimes \mathbb{C} = V_J^+ \oplus V_J^-$	$V = V_J^+ \oplus V_J^-$
$V_J^\pm = \{\mathbf{u} \in V \otimes \mathbb{C}, J\mathbf{u} = \pm i\mathbf{u}\}$	$V_J^\pm = \{\mathbf{u} \in V, J\mathbf{u} = \pm\mathbf{u}\}$
	$\dim V_J^+ = \dim V_J^-$

Remark: A para-complex structure is a particular **product structure**, consisting in the symmetry with respect to V_J^+ parallel to V_J^-.

HYPERCOMPLEX	**PARA-HYPERCOMPLEX**
$H = (I, J, K)$	$\widetilde{H} = (I, J, K)$
$I^2 = J^2 = K^2 = -Id$	$I^2 = -Id \quad , \quad J^2 = K^2 = Id$
$IJ = -JI = K$	$IJ = -JI = K$
$(JK = -KJ = I,$	$(JK = -KJ = -I,)$
$KI = -IK = J$	$KI = -IK = J)$

A para-hypercomplex structure $\widetilde{H} = (I, J, K)$ corresponds to a *3-web* structure (D_1, D_2, D_3) on V, [21]:



The analytic point of view is summarized as follows in terms of

real Clifford algebras:

$\mathbb{C} \equiv C(0,1)$	$\widetilde{\mathbb{C}} \equiv C(1,0)$
complex numbers	**para-complex numbers**
$z = x + iy \quad , \quad i^2 = -1$	$z = x + iy \quad , \quad i^2 = 1$
$\lvert z\rvert^2 = x^2 + y^2$	$\lvert z\rvert^2 = x^2 - y^2$
$\mathbb{H} \equiv C(0,2)$	$\widetilde{\mathbb{H}} \equiv C(2,0) = C(1,1)$
quaternions	**para-quaternions**
$q = q_0 + iq_1 + jq_2 + kq_3$	$q = q_0 + iq_1 + jq_2 + kq_3$
$i^2 = j^2 = k^2 = -1$	$i^2 = -1 \quad , \quad j^2 = k^2 = 1$
$\lvert q\rvert^2 \equiv q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$	$q = q_0^2 + q_1^2 - q_2^2 - q_3^2$
no zero divisors	existence of zero divisors
$\mathbb{H} \cong \mathbb{C}^2$	$\widetilde{\mathbb{H}} \cong \mathbb{C}^2$
$q \equiv z_1(q) + jz_2(q)$	$q \equiv z_1(q) + jz_2(q)$

Examples of hypercomplex structures

- $V = \mathbb{H}^n \equiv \mathbb{R}^{4n} \quad , \quad (i\cdot, j\cdot, k\cdot)$ left multiplications
- $V = \mathbb{C}^{2n} \equiv \mathbb{C}^n \oplus \mathbb{C}^n \quad , \quad (I, J, K)$:

$I(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, -\mathbf{u})\,,\ J(\mathbf{u}, \mathbf{v}) = (-i\mathbf{u}, i\mathbf{v})\,,\ K(\mathbf{u}, \mathbf{v}) = (i\mathbf{v}, i\mathbf{u})$

(This example applies to $V = T_xM$ where $M = TN$ and on N it is given an almost complex structure $\mathbb{J} \equiv i\cdot$ which is left invariant by a linear connection ∇ : then, for $X \in TN$ one has the decomposition $T_XTN \equiv T^h_XTN \oplus T^v_XTN = T_{\pi(X)}TN \oplus T_{\pi(X)}TN$ into horizontal and vertical part, [23], Theor. 2.2).

Examples of para-hypercomplex structures

- $V = \widetilde{\mathbb{H}}^n \equiv \mathbb{R}^{4n} \quad , \quad (i\cdot, j\cdot, k\cdot)$ left multiplications
- $V = \mathbb{R}^{2n} \equiv \mathbb{R}^n \oplus \mathbb{R}^n \quad , \quad (I, J, K)$:

$I(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, -\mathbf{u}) \quad , \quad J(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u}) \quad , \quad K(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, -\mathbf{v})$

(This example applies to $V = T_xM$ where $M = TN$)

- $(V = U \oplus U \quad , \quad \{I' : U \to U \mid (I')^2 = Id\}) \quad , \quad (I, J, K)$:

$I(\mathbf{u}, \mathbf{v}) = (I'\mathbf{v}, -I'\mathbf{u})\,,\ J(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})\,,\ K(\mathbf{u}, \mathbf{v}) = (I'\mathbf{u}, -I'\mathbf{v})$

(This example applies to T_xM where $M = TN$ and on N it is given an almost para-complex structure $\mathbb{I}'$ which is left invariant by a linear connection ∇, [19], analogously as for example of [23] mentioned above).

A (para-)hypercomplex structure on V generates respectively a structure

QUATERNIONIC	**PARA-QUATERNIONIC**
generated by $H = (I, J, K)$:	generated by $\widetilde{H} = (I, J, K)$:
$Q \equiv \langle H \rangle = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$	$\widetilde{Q} \equiv \langle \widetilde{H} \rangle = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$
(I, J, K) determined up to	(I, J, K) determined up to
$A \in SO(3)$	$A \in SO(2,1)$
$S(Q) = \{L \in Q \mid \|L\|^2 = 1\}$	$S(\widetilde{Q}) = S^+(\widetilde{Q}) \cup S^-(\widetilde{Q})$
	$S^+(\widetilde{Q}) = \{L \in \widetilde{Q} \mid L^2 = \mathrm{Id}\}$,
	$S^-(\widetilde{Q}) = \{L \in \widetilde{Q} \mid L^2 = -\mathrm{Id}\}$.

It is well known that a quaternionic structure on V is a tensor product structure (see [26]).

Para-quaternionic structure as a tensor product structure:

$$V = \mathbf{H} \otimes \mathbf{E} \qquad \mathbf{H} = \mathbf{H}^2\,,\ \mathbf{E} = \mathbf{E}^n$$

with structure group $\quad G = GL(2,\mathbb{R}) \otimes GL(n,\mathbb{R}) \cong SL(2,\mathbb{R}) \otimes GL(n,\mathbb{R}) \quad$.

Given a symplectic basis $(\mathbf{h}_1, \mathbf{h}_2)$ in $\mathbf{H}^2$, that is $\mathbf{H}^2 = \langle \mathbf{h}_1, \mathbf{h}_2 \rangle$, let take into account the identifications and isomorphisms of Lie algebras:

$$\widetilde{\mathbb{H}} = \mathbb{R}1 + \mathrm{Im}\widetilde{\mathbb{H}} \cong \mathfrak{gl}_2(\mathbb{R})\,,\ \mathrm{Im}\widetilde{\mathbb{H}} \equiv \mathfrak{su}(1,1) \cong \mathfrak{so}(2,1) \cong \mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{sp}_1(\mathbb{R})$$

By denoting

$$\mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \mathcal{K} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

one has the identifications

$$I(\mathbf{h} \otimes \mathbf{e}) = \mathcal{I}(\mathbf{h}) \otimes \mathbf{e}\,,\ J(\mathbf{h} \otimes \mathbf{e}) = \mathcal{J}(\mathbf{h}) \otimes \mathbf{e}\,,\ K(\mathbf{h} \otimes \mathbf{e}) = \mathcal{K}(\mathbf{h}) \otimes \mathbf{e}$$

that is

$$\begin{aligned} I(\mathbf{h}_1 \otimes \mathbf{e} + \mathbf{h}_2 \otimes \mathbf{e}') &= -\mathbf{h}_1 \otimes \mathbf{e}' + \mathbf{h}_2 \otimes \mathbf{e} \quad , \\ J(\mathbf{h}_1 \otimes \mathbf{e} + \mathbf{h}_2 \otimes \mathbf{e}') &= \mathbf{h}_1 \otimes \mathbf{e}' + \mathbf{h}_2 \otimes \mathbf{e} \quad , \\ K(\mathbf{h}_1 \otimes \mathbf{e} + \mathbf{h}_2 \otimes \mathbf{e}') &= -\mathbf{h}_1 \otimes \mathbf{e} + \mathbf{h}_2 \otimes \mathbf{e}' \,. \end{aligned} \tag{1}$$

Prototype of quaternionic manifold: the quaternionic projective space $\mathbb{H}P^n = (\mathbb{H}^{n+1} - \{\mathbf{0}\})/\mathbb{H}^*$

Prototype of para-quaternionic manifold: the para-quaternionic projective space $\hat{H}P^n = \hat{\widetilde{\mathbb{H}}}^{n+1}/\hat{\mathbb{H}}$ where $\hat{\widetilde{\mathbb{H}}}^{n+1}$ is the space of non-singular vectors in $\widetilde{\mathbb{H}}^{n+1}$ and $\hat{\mathbb{H}} = \{q \in \widetilde{\mathbb{H}} \quad | \quad |q|^2 \neq 0\}$, [8].

It has to be noted that from the complex point of view the two notions of quaternionic and para-quaternionic structure coincide and are included in the notion of *quaternionic conformal structure*, see [10].

In the following two paragraphs we will focus mainly on para-type structures, by assuming that they are less familiar.

3 Hermitian structures

3.1 Hermitian para-(hyper)complex structures

The following definitions and results were stated in [27].

1 Proposition. *([27]). Let g be a (non degenerate) scalar product on V. The following notions are equivalent:*

1. A **para-hermitian structure** (g, K) *on V: that is*
- K is a para-complex structure
and moreover
- K is a skew-symmetric endomorphism w.r.t. g,
i.e. $g(KX, Y) + g(X, KY) = 0$ *equiv.,* $g(KX, KY) + g(X, Y) = 0$.

2. A **totally isotropic decomposition** $V = V^+ \oplus V^-$ *w.r.t. g, i.e. a decomposition of V into two n-dimensional totally isotropic subspaces.*

3. A **Lagrangian decomposition** $V = V^+ \oplus V^-$ *w.r.t. a symplectic form ω, that is V^+, V^- are n-dimensional Lagrangian subspaces w.r.t. ω i.e.*

$$\omega(V^\pm, V^\pm) = 0\,.$$

In these last two cases

$$V^+ = V^+_K \quad , \quad V^- = V^-_K \quad .$$

In the first two cases, $\omega = g(K\cdot, \cdot)$.

The proof of the proposition bases on the

2 Lemma (Basic Lemma)**.** *Let g be a pseudo-Euclidean metric on V:*

g has signature (n, n) if and only if there exists a decomposition

$$V = V^+ \oplus V^-$$

where $V^\pm$ are n-dimensional totally isotropic subspaces, i.e. the restriction

$$g_{|V^\pm} = 0\,.$$

3 Definition. A **para-hypercomplex Hermitian structure** on V is a pair $(g, \widetilde{H} = (I, J, K)) \equiv (g, (J_1, J_2, J_3))$ where g is a scalar product and $\widetilde{H}$ is a para-hypercomplex structure such that

- g has signature $(2n, 2n)$
- $g(J_\alpha X, Y) + g(X, J_\alpha Y) = 0 \quad , \quad \alpha = 1, 2, 3$

i.e. (g, J_1) is a Hermitian complex structure, $(g, J_2), (g, J_3)$ are Hermitian para-complex structures.

Standard example (to keep in mind): $\widetilde{\mathbb{H}}^n \cong \mathbb{C}^{2n} \equiv \mathbb{R}^{2n,2n}$

$$\begin{aligned} \langle \mathbf{h}, \mathbf{h}' \rangle : \quad &= \mathrm{Re}(\mathbf{h}\overline{\mathbf{h}}') = \mathrm{Re}(h_1\overline{h_1}' + \cdots + h_n\overline{h_n}') \\ &= \textstyle\sum_{i=1}^n \left(\mathrm{Re}(z_1(h_i)\overline{z_1(h_i')} - z_2(h_i)\overline{z_2(h_i')}\right) \quad . \end{aligned}$$

$g \equiv \langle \, , \rangle$: Hermitian scalar product of real signature $(2n, 2n)$ on $\mathbb{C}^{2n}$

$i\cdot \equiv J_1$ isometry ; $j\cdot \equiv J_2$, $k\cdot \equiv J_3$ anti-isometries.

3.2 Hermitian (para-)quaternionic structures on V^{4n}

QUATERNIONIC HERMITIAN	PARA – QUATERNIONIC HERMITIAN
$(Q, g) \equiv (\langle H \rangle, g)$	$(Q, g) \equiv (\langle \widetilde{H} \rangle, g)$
where	*where*
(H, g) hypercomplex Hermitian	$(\widetilde{H}, g)$ para-hypercomplex Hermitian

A Hermitian para-quaternionic structure can be interpreted in terms of tensor product:

$$V = \mathbf{H} \otimes \mathbf{E} \quad , \quad g = \omega^{\mathbf{H}} \otimes \omega^{\mathbf{E}} \quad , \quad \widetilde{Q} = \mathfrak{sp}_{\omega^{\mathbf{H}}}(\mathbf{H})$$

where $\omega^{\mathbf{H}}, \omega^{\mathbf{E}}$ are symplectic forms on the real vector spaces $\mathbf{H} = \mathbf{H}^2, \mathbf{E} = \mathbf{E}^n$.

We will refer to such an interpretation as the *Grassmann model* and if $(\mathbf{h}_1, \mathbf{h}_2)$ is a basis of $\mathbf{H}$ the para-hypercomplex structure defined by (1) will be referred as the *standard* (I, J, K).

4 Invariant subspaces in a para-quaternionic (Hermitian) V^{4n} and their Grassmann models

Let V be a 4n-dimensional vector space endowed with a para-quaternionic Hermitian structure $(\widetilde{Q}, g)$. Let think of a subspace U of V as the tangent space

$U \equiv T_xN$ at a point x of a submanifold N of M.

4 Definition. A subspace $U \subset V$ is called a

- **complex subspace** if it exists an endomorphism $I' \in S^-(\widetilde{Q})$ leaving U invariant, $I'U \subset U$,

- **Hermitian subspace** if moreover $g_{|}U$ is non degenerate.

Example: $\mathbb{C}^k \equiv \mathbb{C}^k \oplus \{\mathbf{0}\} \subset \mathbb{C}^n \oplus \mathbb{C}^n \equiv \widetilde{\mathbb{H}}^n$.

Interpretation w.r.t. Grassmann model, [27]:

$$V = \mathbf{H} \otimes \mathbf{E}\,,\ g = \omega^{\mathbf{H}} \otimes \omega^{\mathbf{E}} \text{ and } (I, J, K) \text{ standard.}$$

Assume $(\mathbf{h}_1, \mathbf{h}_2)$ to be a symplectic basis for $(\mathbf{H}, \omega^H)$ and

$$I(\mathbf{h}_1 \otimes \mathbf{e} + \mathbf{h}_2 \otimes \mathbf{e}') = -\mathbf{h}_1 \otimes \mathbf{e}' + \mathbf{h}_2 \otimes \mathbf{e}.$$

For any pair $(\mathbf{F}, L)$ where $\mathbf{F}$ is a subspace of $\mathbf{E}$ and L is a complex structure of $\mathbf{F}$, i.e.

$$\mathbf{F} \subset \mathbf{E} \quad , \quad L \in \mathrm{End}(\mathbf{F}) \quad , \quad L^2 = -\mathrm{Id} \quad ,$$

let consider the corresponding subspace $U^{\mathbf{F},L}$ of $V = \mathbf{H} \otimes \mathbf{E}$ given by

$$U^{F,L} = \{X = \mathbf{h}_1 \otimes \mathbf{f} + \mathbf{h}_2 \otimes L\mathbf{f},\, \mathbf{f} \in F\} \quad ,$$

the endomorphism $I^{F,L}$ of $U^{F,L}$ given by

$$I^{F,L}(\mathbf{h}_1 \otimes \mathbf{f} + \mathbf{h}_2 \otimes L\mathbf{f}) = -\mathbf{h}_1 \otimes L\mathbf{f} + \mathbf{h}_2 \otimes \mathbf{f}$$

i.e. $I^{F,L} = I_{|U^{F,L}}$, and the Hermitian scalar product g^L on $(\mathbf{F}, L)$ given by

$$\begin{aligned} g^L(\mathbf{f}, \mathbf{f}') :\ &= g(\mathbf{h}_1 \otimes \mathbf{f} + \mathbf{h}_2 \otimes Lf, \mathbf{h}_1 \otimes \mathbf{f}' + \mathbf{h}_2 \otimes Lf') \\ &= \omega^E(\mathbf{f}, L\mathbf{f}') - \omega^E(L\mathbf{f}, \mathbf{f}') \end{aligned}$$

The signature $(2p, 2q)$ of $g_{|U}$ equals the signature of g^L.

The maximal dimension of $U^{F,L}$ is 2n, when $F = E$.

Proposition [27]: *A subspace $U \subset V \equiv \mathbf{H} \otimes \mathbf{E}$ is a complex (respectively, complex Hermitian) subspace if and only $U = U' \oplus U''$ where U' is the maximal $\widetilde{Q}$-invariant subspace contained in U and $U = U^{\mathbf{F},L}$ with respect to a basis (resp. symplectic basis)* $(\mathbf{h}_1, \mathbf{h}_2)$ *of* $\mathbf{H}$.

5 Definition. A subspace $U \subset V$ is called a

- **para-complex subspace** if it exists an endomorphism $K' \in S^+(\widetilde{Q})$ leaving U invariant, $K'U \subset U$, and moreover it is called a

- **para-Hermitian subspace** if $g_{|U}$ is non degenerate.

Interpretation w.r.t. Grassmann model, [27]:

A para-hermitian subspace $U \subset V$ has the form

$$U = (\mathbf{h}_1 \otimes \mathbf{E}_1) \oplus (\mathbf{h}_2 \otimes \mathbf{E}_2)$$

where E_1, E_2 are (not necessarily transversal) subspaces of E
$(\mathbf{h}_1 \otimes \mathbf{E}_1, \mathbf{h}_2 \otimes \mathbf{E}_2$ totally isotropic in $U \Rightarrow \dim(\mathbf{E}_1) = \dim(\mathbf{E}_2) = r)$

Also

$$U = \mathbf{H} \otimes \mathbf{E}_0 \oplus \mathbf{h}_1 \otimes \mathbf{E}'_1 \oplus \mathbf{h}_2 \otimes \mathbf{E}'_2 = U_0 \oplus U_1 \oplus U_2$$

where U_0 is $\widetilde{Q}$-invariant and U_1, U_2 totally isotropic and, moreover, the non degeneracy conditions is fulfilled.

6 Definition. A subspace $U \subset V$ is called a **para-quaternionic subspace** if

- $\forall \quad (I, J, K)$ of $\widetilde{Q} \qquad \Rightarrow \qquad IU \subset U, JU \subset U, KU \subset U$

and

- $g_{|U}$ is non degenerate.

(In particular, U is both Hermitian and para-Hermitian).

Grassmann interpretation for a para-quaternionic subspace, [27]:

$$U = \mathbf{H} \otimes \mathbf{E}'$$

where $\mathbf{E}' = (\mathbf{E}')^{2k}$ is a 2k-dimensional subspace of $\mathbf{E}$,

- $\omega^{\mathbf{E}}_{|\mathbf{E}'}$ is non degenerate and
- $\forall X = \mathbf{h}_1 \otimes \mathbf{e} + \mathbf{h}_2 \otimes \mathbf{e}', Y = \mathbf{h}_1 \otimes \mathbf{f} + \mathbf{h}_2 \otimes \mathbf{f}' \quad \in U$

$$g(X, Y) = g(\mathbf{h}_1 \otimes \mathbf{e} + \mathbf{h}_2 \otimes \mathbf{e}', \mathbf{h}_1 \otimes \mathbf{f} + \mathbf{h}_2 \otimes \mathbf{f}' = \omega^{\mathbf{E}}(\mathbf{e}, \mathbf{f}') - \omega^{\mathbf{E}}(\mathbf{e}', \mathbf{f})\,.$$

Also **totally (para-)complex** and **totally real** subspaces, with their respective Grassmann interpretation, were considered in [27].

5 (Para-)quaternionic (Hermitian) manifold (M^{4n}, Q, ∇)

From now on the quaternionic and para-quaternionic case will be treated simultaneously and notation is unified by setting $\eta = -1$ or $\eta = 1$ and $(\epsilon_1, \epsilon_2, \epsilon_3) =$

$(-1,-1,-1)$ or $(\epsilon_1,\epsilon_2,\epsilon_3)=(-1,1,1)$ respectively for hypercomplex or para-hypercomplex structure (J_1,J_2,J_3): then the multiplication table looks

$$J_\alpha^2=\epsilon_\alpha \mathrm{Id} \quad , \quad J_\alpha J_\beta=-J_\beta J_\alpha=\eta\epsilon_\gamma J_\gamma \qquad (\alpha=1,2,3)$$

where (α,β,γ) is any circular permutation of $(1,2,3)$.

7 Definition. An **almost (para-)quaternionic structure** on a differentiable manifold M^{4n} is a rank 3 subbundle $Q\subset \mathrm{End}(TM)$, which is locally spanned by a field $H=(J_1,J_2,J_3)$ of (para-)hypercomplex structures. Such a locally defined triple $H=(J_\alpha),\alpha=1,2,3$, will be called a **(local) admissible basis** of Q.

An **almost (para-)quaternionic connection** is a linear connection ∇ which preserves Q ; equivalently, for any local admissible basis $H=(J_\alpha)$ of Q one has

$$\nabla J_\alpha=-\epsilon_\beta\omega_\gamma\otimes J_\beta+\epsilon_\gamma\omega_\beta\otimes J_\gamma \qquad (P.C.) \tag{2}$$

where (P.C.) means that (α,β,γ) is any circular permutation of (1,2,3).

An almost (para-)quaternionic structure Q is called a **(para-)quaternionic structure** if M admits a **(para-)quaternionic connection**, i.e. a torsion-free almost (para-)quaternionic connection. An **(almost) (para-)quaternionic manifold** (M^{4n},Q) is a manifold M^{4n} endowed with an (almost) (para-)quaternionic structure Q.

An **almost (para-)quaternionic Hermitian manifold** (M,Q,g) is an almost (para-)quaternionic manifold (M,Q) endowed with a (pseudo-)Riemannian metric g which is Q-Hermitian, i.e. any endomorphism of Q is g-skew-symmetric. $(M,Q,g),n>1$, is called a **(para-)quaternionic Kähler manifold** if the Levi-Civita connection of g preserves Q, i.e. ∇^g is a (para-)quaternionic connection.

6 (Para-)quaternionic Kähler manifold (M^{4n},Q,g)

Let (M^{4n},Q,g), $n>1$, be a (para-)quaternionic Kähler manifold and recall some basic results, see for ex. [3],[2].

(M^{4n},g) is an Einstein manifold.

The curvature tensor of g decomposes as:

$$R=\nu R_0+W \tag{3}$$

where

$$R_0(X,Y)=\frac{1}{4}\Big(X\wedge Y-\sum_{\alpha=1}^{3}\epsilon_\alpha J_\alpha X\wedge J_\alpha Y+\sum_{\alpha=1}^{3}2\epsilon_\alpha g(J_\alpha X,Y)J_\alpha\Big)$$

and $\nu = K/4n(n+2)$ is the reduced scalar curvature, W is the **(para-)quaternionic Weyl tensor**, for which

$$[W(X,Y),Q] = 0$$

and all contractions are equal to zero.

The following basic identities hold for the curvature tensor R:

$$[R(X,Y),J_\alpha] = -\eta\epsilon_\alpha\nu\big(F_\gamma(X,Y)J_\beta - F_\beta(X,Y)J_\gamma\big) \qquad (P.C.)$$

One defines a **(para-)quaternionic Kähler manifold of dimension 4** as a (pseudo-)Riemannian manifold (M^{4n},g) endowed with a parallel skew-symmetric (para-)quaternionic structure Q and whose curvature tensor admits a decomposition (3).

From identities (2) for the Levi-Civita connection $\nabla = \nabla^g$, the following integrability conditions hold:

$$-\nu\eta F_\alpha = \epsilon_\alpha(d\omega_\alpha - \epsilon_\alpha\omega_\beta \wedge \omega_\gamma) \qquad (P.C.) \qquad (4)$$

where $F_\alpha = g \circ J_\alpha \equiv g(J_\alpha\cdot,\cdot)$ is the Kähler form of $J_\alpha, \alpha = 1,2,3$.

Symmetric quaternionic Kähler manifolds, [1]:

A Wolf space is a compact, simply connected quaternionic Kähler symmetric space. It has the form $W = G/K$, G compact centerless Lie group and $K = K_1 \cdot Sp(1)$ (local direct product). Main examples:

$$\mathbb{H}P^n = \frac{Sp(n+1)}{Sp(1)\cdot Sp(n)} \quad , \quad G_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S\big(U(2)\times U(n)\big)} \quad ,$$
$$G_4^+(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{S\big(O(4)\times O(n)\big)}$$

+ 5 exceptional spaces: $G_2/SO(4), F_4/Sp(1)\cdot Sp(3)$, etc. and their noncompact duals.

Symmetric para-quaternionic Kähler manifolds, [13],[1]:

$$T^*Q_n \equiv T^*G_2(\mathbb{R}^{n+2}) = \frac{SL(n+2,\mathbb{R})}{S(GL(2,\mathbb{R})\times GL(n,\mathbb{R}))} \quad ,$$

$$G_2^{1,1}(\mathbb{C}^{p+1,q+1}) = \frac{SU_{p+1,q+1}}{S(U_{1,1}\times U_{p,q})} \, , \, G_4^{2,2}(\mathbb{R}^{2+p,2+q}) = \frac{SO_{2+p,2+q}}{SO_{2,2}\times SO_{p,q}},$$

$$T^*G_2(\mathbb{C}^{n+2}) = \frac{SO^*(2n+4)}{SO^*(4)\times SO^*(2n)} = \frac{SL(2(n+2),\mathbb{R})}{S(GL(2n,\mathbb{C})\times GL(4,\mathbb{C}))} \quad ,$$

$$T^*\mathbb{C}P^n = \hat{\mathbb{H}}P^n = \frac{Sp_{n+1}(\mathbb{R})}{Sp_1(\mathbb{R})\cdot Sp_n(\mathbb{R})}$$

+ some exceptional spaces.

Note: As the notation T^* indicates, several of these manifolds happen to be cotangent bundles of Kähler manifolds. One wonder if this is true for all symmetric para-quaternionic Kähler manifolds. Such a result would be the analogous of that one of [12], paragraph 5.2. pag. 100, i.e. every para-Hermitian symmetric space with semisimple group is diffeomorphic to the cotangent bundle of a Riemannian symmetric space of a particular type.

7 Submanifolds of primary interest in $(\overline{M}^{4n}, \overline{Q}, \overline{g})$

7.1 Almost (para-)quaternionic submanifold (M^{4m}, Q)

An **almost (para-)quaternionic submanifold** $(M^{4m}, Q), m \leq n$, of a (para-)quaternionic manifold $(\overline{M}^{4n}, \overline{Q})$ is a submanifold whose tangent bundle is $\overline{Q}$-invariant, $\overline{Q}TM = TM$. It is a classical result of A. Gray that a quaternionic submanifold of a quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ is totally geodesic and hence, endowed with the induced quaternionic Kähler structure $Q = \overline{Q}_{|TM}$ and riemannian metric $g = \overline{g}_{|TM}$, is itself a quaternionic Kähler manifold (M^{4m}, Q, g). The analogous result for a para-quaternionic submanifold of a para-quaternionic Kähler manifold was proved in [27]. In fact, by arguing as in [6], [24], a more general result can be proved.

8 Proposition. *An almost (para-)quaternionic submanifold (M^{4m}, Q) of a (para-)quaternionic manifold $(\overline{M}^{4n}, \overline{Q})$ is totally geodesic and (para-)quaternionic.*

Proof. For quaternionic case the proof was given in [6], [24]. For para-quaternionic case we observe that the proof in [6] is directly adaptable. QED

Examples of (para-)quaternionic submanifolds: Let $(\overline{N}, \overline{\mathbb{J}}, \nabla')$ be an almost (para-)complex manifold endowed with a linear connection ∇' leaving the almost (para-)complex structure $\overline{\mathbb{J}}$ invariant. Let $(\overline{M} = T\overline{N}, \overline{Q}, \overline{\nabla})$ be the almost (para-)quaternionic manifold whose (para-)quaternionic structure $\overline{Q} = Q_{\overline{\mathbb{J}}}$ is generated by the (para-)hypercomplex structure constructed as indicated in the example of (para-)hypercomplex structure referring to [23],[19] and $\overline{\nabla}$ is the extension of ∇' to $TT\overline{N}$. Let $(N, \mathbb{J})$ be an almost (para-)complex submanifold of $(\overline{N}, \overline{\mathbb{J}})$, i.e. N is a submanifold of $\overline{N}$ which is $\overline{\mathbb{J}}$-invariant, $\overline{\mathbb{J}}(TN) \subset TN$, and $\mathbb{J} = \overline{\mathbb{J}}_{|TN}$. If N is totally geodesic with respect to ∇' then $(TN, Q_{\mathbb{J}})$ is a almost (para-quaternionic) submanifold of $(\overline{N}, Q_{\overline{\mathbb{J}}}, \nabla')$.

Para-quaternionic submanifolds of symmetric para-quaternionic ma-

nifolds:

$$\hat{\mathbb{H}}P^k \subset \hat{\mathbb{H}}P^n \qquad (k \leq n) \quad ,$$

$$G_2^{1,1}(\mathbb{C}^{h+1,k+1}) \subset G_2^{1,1}(\mathbb{C}^{p+1,q+1}) \, , \, G_4^{2,2}(\mathbb{R}^{h+2,k+2}) \subset G_4^{2,2}(\mathbb{R}^{p+2,q+2})$$

$(h \leq p, k \leq q)$ and

$$G_2^{1,1}(\mathbb{C}^{h+1,k+1}) \subset G_4^{2,2}(\mathbb{R}^{2p+1,2q+1}) \qquad , \quad (h \leq p, k \leq q),$$

$$T^*G_2(\mathbb{C}^{h+2}) \subset T^*G_2(\mathbb{C}^{n+2}) \qquad (h \leq n)$$

are very natural totally geodesic para-quaternionic immersions.

7.2 (Almost) (para-)complex submanifolds $(M^{2m}, \overline{J})$

An (almost) (para-)complex submanifold $(M^{2m}, \overline{J})$ is a submanifold $M^{2m} \subset \overline{M}^{4n}$ whose tangent bundle is $\overline{J}$-invariant for some section

$$\overline{J} \in \Gamma(Q_{|M}) \qquad , \qquad \overline{J}^2 = \pm \mathrm{Id}$$

In a para-quaternionic manifold one has to consider simultaneously both types of such submanifolds.

Let consider some examples of (para-)complex submanifolds of a para-quaternionic manifold.

The following simple remark is useful.

Remark. *Let be $(\overline{M}, \overline{Q})$ a para-quaternionic manifold, $(M, Q = \overline{Q}_{|TM})$ an almost para-quaternionic submanifold, $\overline{\mathbb{J}}$ a section of $S^{\epsilon}(\overline{Q})$, $\epsilon = \pm 1$, along M. Then $(M, \mathbb{J} = \overline{\mathbb{J}}_{|TM})$ is an almost complex or para-complex submanifold depending on if $\epsilon = -1$ or $\epsilon = 1$. In particular, if (I, J, K) is a local almost para-hypercomplex base for $\overline{Q}$, then (M, I) and $(M, J), (M, K)$ are respectively almost complex and para-complex, para-complex submanifolds.*

7.3 (Para-)Kähler submanifolds of a (para-)quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$

Let $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ be an almost (para-)quaternionic Hermitian manifold.

9 Definition. An almost (para-)complex Hermitian submanifold $(M^{2m}, \overline{J}, g)$, $m \geq 1$, is a 2m-dimensional g-nondegenerate (para-)complex submanifold $(M^{2m}, \overline{J})$ where $g = \overline{g}_{|M^{2m}}$ is the induced (pseudo-)Riemannian metric.

10 Definition. A (para-)Kähler submanifold $(M^{2m}, \overline{J}, g)$, $m \geq 1$ of the (para-)quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ is an almost (para-)complex Hermitian submanifold such that the section $\overline{J}$ is parallel along M^{2m} with respect to the Levi-Civita connection of $\overline{g}$.

Examples in para-quaternionic Kähler symmetric spaces.

- Submanifolds of $\hat{\mathbb{H}}P^n = \frac{Sp_{n+1}(\mathbb{R})}{Sp_1(\mathbb{R}) \times Sp_n(\mathbb{R})}$:

Kähler – $\mathbb{C}P^h = \frac{SU_{h+1}}{S(U_1 \times U_h)}$

para-Kähler – $\hat{\mathbb{C}}P^h = \frac{SL_{h+1}(\mathbb{R})}{SL_1(\mathbb{R}) \times SL_h(\mathbb{R})}$, $(h \leq n)$

- Submanifolds of $G_2^{1,1}(\mathbb{C}^{p+1,q+1}) = \frac{SU_{p+1,q+1}}{S(U_{p,q} \times U_{1,1})}$:

Kähler – $\mathbb{C}P^h \times \mathbb{C}P^k = \dfrac{SU_{h+1}}{S(U_1 \times U_h)} \times \dfrac{SU_{k+1}}{S(U_1 \times U_k)}$

para-Kähler – $G_2^{1,1}(\mathbb{R}^{h+1,k+1}) = \dfrac{SO_{h+1,k+1}}{S(O_{1,1} \times O_{h,k})}$, $(h \leq p, k \leq q)$

- Submanifolds of $G_4^{2,2}(\mathbb{R}^{p+2,q+2}) = \frac{SO_{p+2,q+2}}{S(O_{2,2} \times O_{p,q})}$:

Kähler – $G_2(\mathbb{C}^{p-h+1}) \times G_2(\mathbb{C}^{q-k+1})$, $Q_h^{h+1,0} \times Q_k^{k+1,0}$

para- Kähler – $G_2^{1,1}(\mathbb{C}^{h+1,k+1})$, $Q_{h,1} \times Q_{k,1}$

- Submanifolds of $M^{8n} = T^*G_2(\mathbb{C}^{n+2})$:

Kähler – $\mathbb{C}P^h$, $G_2(\mathbb{R}^{h+2})$

para-Kähler – $T^*\mathbb{C}P^h \times T^*\hat{\mathbb{C}}P^k$, $T^*G_2(\mathbb{R}^{k+2}) = T^*Q_k$

7.4 Almost (para-)complex submanifolds of a (para-)quaternionic (Hermitian) manifold. Integrability of a compatible almost (para-)complex structure.

A **compatible almost (para-)complex structure** $\overline{J}$ on a (para-)quaternionic manifold $(\overline{M}, \overline{Q})$ is a section of $S^{\pm}(\overline{Q})$ on $\overline{M}$. Integrability of a compatible almost (para-)complex structure $\overline{J}$ or, more generally, of an almost (para-)complex structure $J = \overline{J}_{|TM}$ induced by a compatible almost complex structure $\overline{J}$ on a submanifold M is a natural problem to consider. Several results were obtained in [4],[5] for a quaternionic (Kähler) manifold and some of them were extended to the para-quaternionic case by [27]. A summary of results is the following.

Let $(\overline{M}^{4n}, \overline{Q}, \overline{\nabla})$ be a (para-)quaternionic manifold (if $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ is (para-)quaternionic Kähler we assume that $\overline{\nabla} = \nabla^{\overline{g}}$) and consider an *almost (para-)-complex submanifold* $(M^{2m}, \overline{J})$, where it is induced the *almost (para-)complex structure* $J = \overline{J}_{|TM}$.

Remark: The case where $M^{2m} \equiv \overline{M}^{4n}$ is included.

We are interested to state conditions under which $J = \overline{J}_{|TM}$ is integrable.

Let (J_1, J_2, J_3) be an admissible basis for $\overline{Q}$, i.e.

$$J_\alpha^2 = \epsilon_\alpha \mathrm{Id} \qquad , \qquad J_\alpha J_\beta = -J_\beta J_\alpha = \eta \epsilon_\gamma J_\gamma \qquad , \qquad (P.C.)$$

where (α, β, γ) is a circular permutation of $(1, 2, 3)$, (note that $\epsilon_\alpha \epsilon_\beta \epsilon_\gamma = -1$), and consider the identities

$$\overline{\nabla} J_\alpha = -\big(\epsilon_\beta \omega_\gamma \otimes J_\beta - \epsilon_\gamma \omega_\beta \otimes J_\gamma\big) \tag{5}$$

It results

$$\eta J_\alpha \overline{\nabla} J_\alpha = \epsilon_\alpha (\omega_\beta \otimes J_\beta + \omega_\gamma \otimes J_\gamma) \qquad (\alpha = 1, 2, 3)$$

and

$$\eta \mathrm{Tr}(\epsilon_\alpha J_\alpha \overline{\nabla} J_\alpha) = \omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma$$

The following local 1-forms play an important role:

$$\begin{aligned} &\theta_\alpha := \omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma \quad , \quad \chi_\alpha := \omega_\beta \circ J_\beta - \omega_\gamma \circ J_\gamma = \theta_\gamma - \theta_\beta \,, \\ &\psi_\alpha := \chi_\alpha \circ J_\beta = \epsilon_\beta \omega_\beta + \eta \epsilon_\alpha \omega_\gamma \circ J_\alpha \qquad (\alpha = 1, 2, 3) \end{aligned} \tag{6}$$

Remark: If the manifold is quaternionic Kähler then θ_α is the *Lie form* of the almost complex structure J_α, [7], being, in general,

$$\theta_\alpha = -\eta \epsilon_\alpha (\delta F_\alpha) \circ J_\alpha \tag{7}$$

(In fact, for a vector field X and a (pseudo-)orthonormal frame $E_i, i = 1, \ldots, 4n$ one has

$$\begin{aligned} \eta(\delta F_\alpha \circ J_\alpha)(X) &= \eta \mathrm{Tr}_g\Big(- g(\overline{\nabla} J_\alpha, J_\alpha X)\Big) \\ &= -\epsilon_\alpha (\omega_\gamma \circ J_\gamma + \omega_\beta \circ J_\beta)(X) = -\epsilon_\alpha \theta_\alpha(X) \end{aligned}$$

and hence $\delta F_\alpha = -\eta \theta_\alpha \circ J_\alpha$.)

Coming back to the general case, let observe also that it results

$$\epsilon_\alpha(\overline{\nabla}_{J_\alpha X}J_\alpha - J_\alpha\overline{\nabla}_X J_\alpha) = -\eta\epsilon_\beta\psi_\alpha(X)J_\beta - \psi_\alpha(J_\alpha X)J_\gamma\,. \tag{8}$$

For $x \in M$ and $X, Y \in T_xM$ the Nijenhuis tensor $\overline{N}_{J_\alpha}$ of J_α is given by

$$4\overline{N}_{J_\alpha}(X,Y) = \big[(\overline{\nabla}_{J_\alpha X}J_\alpha)Y - J_\alpha(\overline{\nabla}_X J_\alpha)Y\big] - \big[(\overline{\nabla}_{J_\alpha Y}J_\alpha)X - J_\alpha(\overline{\nabla}_Y J_\alpha)X\big].$$

Hence

$$\begin{aligned} 4\epsilon_\alpha \quad & \overline{N}_{J_\alpha}(X,Y) = \\ & \eta\epsilon_\beta J_\beta\big[\psi_\alpha(Y)X + \epsilon_\alpha\psi_\alpha(J_\alpha Y)J_\alpha X - \psi_\alpha(X)Y - \epsilon_\alpha\psi_\alpha(J_\alpha X)J_\alpha Y\big]\,. \end{aligned} \tag{9}$$

A local admissible basis (J_1, J_2, J_3) of $\overline{Q}$ defined on a neighborhood U in $\overline{M}^{4n}$ of a point $x \in M^{2m}$ is called an **adapted basis** for the almost (para-)complex submanifold $(M^{2m}, \overline{J})$ if $J_{\alpha|(M\cap U)} = \overline{J}$ for some index $\alpha \in (1, 2, 3)$.

Let now assume that (J_1, J_2, J_3) is an adapted basis for the submanifold $(M, \overline{J})$, being $J_{\alpha|TM} = \overline{J}$. Then the Nijenhuis tensor N_J of J is just given by the restriction of $\overline{N}_{J_\alpha}$ to TM^{2m}.

Hence

11 Proposition. *If $m > 1$, $J \equiv J_{\alpha|TM}$ is integrable if and only if $\psi \equiv \psi_{\alpha|TM} = 0$, i.e. $\chi \equiv (\omega_\beta \circ J_\beta - \omega_\gamma \circ J_\gamma)_{|TM} = 0$.*

Concerning the case of a surface, where the integrability of J always holds, let consider the following definition.

12 Definition. An almost (para-)complex surface $(M^2, \overline{J})$ of $\overline{M}^{4m}$ is **super-(para-)complex** if $\psi = 0$.

From now on let denote $J = J_{\alpha|TM}, \psi = \psi_{\alpha|TM}, \epsilon = \epsilon_\alpha$.

Also, at any point $x \in M^{2m}$ let denote by $\overline{T}_xM$ the **maximal Q-invariant subspace of** T_xM, $\overline{T}_xM = T_xM \cap J_\beta T_xM$.

Let observe that $N_J(X,Y) \in T_xM$, $\forall X, Y \in T_xM$; hence (9) implies that $\forall X, Y \in T_xM$

$$\psi(Y)X + \epsilon\psi(JY)JX - \psi(X)Y - \epsilon\psi(JX)JY \in \overline{T}_xM \tag{10}$$

A rather strong consequence of that remark in the non-integrable case is the following result which holds in full generality.

13 Proposition. *Let $\psi_x \neq 0$ at a point x of the almost (para-)complex submanifold $(M^{2m}, \overline{J})$. Then the following possibilities hold for T_xM:*
1) $T_xM = \overline{T}_xM$
or
2) $T_xM = \overline{T}_xM \oplus \mathcal{D}_x$
where $\mathcal{D}_x$ is a J_x-invariant 2-dimensional subspace of T_xM.

Proof. By (10), for any $X \in T_xM$ we have

$$\begin{cases} \psi(X)Y + \epsilon\psi(JX)JY \equiv \psi(Y)X + \epsilon\psi(JY)JX \\ \\ \psi(JX)Y + \psi(X)JY \equiv \psi(JY)X + \psi(Y)JX \end{cases} (\mathrm{mod}\,\overline{T}_xM), \forall Y \in T_xM \tag{11}$$

a) If there exists X s.t. $\psi_x(X)^2 - \epsilon\psi(JX)^2 \neq 0$, then from (11) it follows that $Y \in \overline{T}_xM + \mathcal{D}'_x$, where $\mathcal{D}'_x = \mathrm{span}\{X, JX\}$, $\forall Y \in T_xM$. b) If $\psi_x(X)^2 - \epsilon\psi(JX)^2 \equiv 0$, we first notice that it must be $\epsilon = 1$, since $\psi_x \neq 0$, and $\psi(JX) = \pm\psi(X), \forall X \in T_xM$. Moreover the first of (11) reduces to the identity

$$\psi(X)(Y \pm JY) \equiv \psi(Y)(X \pm JX)\ (\mathrm{mod}\,\overline{T}_xM),\ \forall X, Y \in T_xM\,. \tag{12}$$

If $\exists\,\overline{X} \in \overline{T}_xM$ s.t. $\psi(\overline{X}) \neq 0$ then $Y \pm JY \in \overline{T}_xM$, $\forall Y \notin T_xM$ and hence $JY = \pm Y\ \forall Y \in \overline{T}_xM \Longrightarrow T_xM = \overline{T}_xM$ (since $J \neq \pm \mathrm{Id}$).

If $\exists\, X' \notin \overline{T}_xM$ (and hence also $JX' \notin \overline{T}_xM$) s.t. $\psi(X') \neq 0$ then $Y \pm JY \in \mathbb{R}(X' \pm JX') + \overline{T}_xM\ \forall Y \in T_xM$, hence $JX' = \mp X'$ and $T_xM = \mathbb{R}X' + \overline{T}_xM$ by dimensionality reasons, and that is contradiction. QED

Let now apply the above result to a submanifold.

14 Proposition. *Let $(M^{2m}, \overline{J})$ be an almost (para-)complex submanifold of $(\overline{M}^{4n}, \overline{Q})$.*

If the codimension of $\overline{T}_xM$ in T_xM is bigger than 2, i.e. dim$\overline{T}_xM < 2(m-1)$,

a) on an open dense set $U \subset M^{2m}$

or

b) in a point x, if (M^{2m}, J) is analytic,

then J is integrable.

As a consequence one has also the following corollary.

15 Corollary. *If dim$(M) = 4k$ and $N(J) \neq 0$ on an open set U dense in M, then M is a totally geodesic (para-)quaternionic submanifold.*

The construction of examples of $2(2k+1)$-dimensional almost (para-)complex submanifolds which are not (para-)complex is an open problem.

From results of [26],[2] it follows that *in a neighborhood of any point x of a (para-)quaternionic Kähler manifold there exists a compatible (para-)complex structure $\overline{J}$*. Equivalently, *in a neighborhood of any point x there exists an admissible basis (J_1, J_2, J_3) such that one of the almost (para-)complex structures J_α is integrable* (in para-quaternionic case both possibilities occur). A rather extensive study of compatible complex structures on a quaternionic Kähler manifold was made in [7], dealing also with the existence of *global* complex structures. An analogous study could be performed for (para)complex structures of a para-quaternionic Kähler manifolds

8 Minimal almost complex submanifolds

In this section, following the lines of [5], we calculate the mean curvature vector of an almost (para-)complex hermitian submanifold of a (para-)quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$.

Let N^k be a (non-degenerate) submanifold of $\overline{M}^{4n}$, $g = \overline{g}_{|N}$ the metric induced by $\overline{g}$ and h the second fundamental form of N^k. We recall that the "mean curvature vector" $H = \frac{1}{k}\mathrm{Tr}_g h$ of N^k at a point x is given by

$$H = \frac{1}{k}\sum_{k} g_i h(E_i, E_i) \tag{13}$$

where $(E_1, \ldots, E_k)$ is a (pseudo-)orthonormal basis of T_xM^k and $g_i = g(E_i, E_i) \in (-1, 1), i = 1, \ldots, k$, [22].

Let $(M^{2m}, \overline{J})$ be an almost (para-)complex submanifold of the (para-)quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$.

Without any loss of validity, we assume that $M^{2m} \subset U \subset M^{4n}$, where U is an open set where it is given a local adapted (para-)hypercomplex frame (J_1, J_2, J_3) such that $\overline{J} = J_{\alpha|M}$.

To handle simultaneously all possible cases of ambient manifold, possibly pseudo-Riemannian, we assume the following **additional hypothesis**:

At any point $x \in M^{2m}$ the tangent space of the submanifold admits a (pseudo-)orthogonal decomposition

$$T_xM = \overline{T}_xM \oplus \mathcal{D}_x \tag{14}$$

where $\overline{T}_xM = T_xM \cap J_\beta T_xM$ is the maximal Q-invariant subspace of T_xM and $\mathcal{D}_x$ is the (possibly zero) $\overline{J}$-invariant orthogonal complement to $\overline{T}_xM$. Equivalently, $\overline{T}_xM$ *is a $\overline{g}$-nondegenerate subspace of* T_xM, $\forall x \in M^{2m}$.

Note that the space $\widetilde{\mathcal{D}}_x = J_\beta \mathcal{D}_x$ does not depend on the adapted basis (J_1, J_2, J_3) and

$$T^Q_xM = \overline{T}_xM \oplus \mathcal{D}_x \oplus \widetilde{\mathcal{D}}_x$$

is a direct sum decomposition of the minimal Q-invariant subspace T^Q_xM of $T_x\overline{M}$ which contains T_xM.

Remark that $\widetilde{\mathcal{D}}_x$ is orthogonal to $\overline{T}_xM$ but in general, if $\dim\mathcal{D}_x > 2$, not orthogonal to $\mathcal{D}_x$. Let recall also that by (13), in case $n > 1$, if $\dim\mathcal{D}_x > 2$ for any $x \in M$ then the almost complex structure $J = \overline{J}_{|M}$ is integrable.

We denote by

$$\mathbf{t}_\alpha = \overline{g}^{-1} \circ \theta_\alpha \in T\overline{M}|_M$$

the (local) vector field along M, dual to the 1-form $\theta_\alpha = \omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma$ with respect to $\overline{g}$.

16 Proposition. *Let $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ be a (para-)quaternionic Kähler manifold and $(M^{2m}, \overline{J})$ be a (para-)complex hermitian submanifold. Moreover let assume that the hypothesis (14) holds at any point $x \in M$. Then, with respect to an adapted frame (J_1, J_2, J_3) such that $\overline{J} = J_\alpha$,*

$$h(X,X) - \epsilon_\alpha h(JX,JX) = -\eta[\epsilon_\beta \theta_\alpha(J_\beta X)J_\beta X + \epsilon_\gamma \theta_\alpha(J_\gamma X)J_\gamma X]^\perp . \qquad (15)$$

where $\perp$ means the projection on $TM^\perp$, and

- the mean curvature vector H of an almost (para-)complex submanifold (M^{2m}, J_α) of $\overline{M}^{4n}$ is given by

$$H = -\frac{\eta}{2m}\big[Pr_{\widetilde{\mathcal{D}}}\mathbf{t}_\alpha\big]^\perp \qquad (16)$$

where, for any $X \in T_x\tilde{M}$, $Pr_{\mathcal{D}'}(X)$ is the orthogonal projection of X onto the subspace $\widetilde{\mathcal{D}}_x$ and $X^\perp$ means the orthogonal projection of X onto $T_x^\perp M$.

If $m = 1$ the formula can be written as

$$H = -\frac{\eta}{2}\Big[(\epsilon_\beta \theta_\alpha(J_\beta X)J_\beta X + \epsilon_\gamma \theta_\alpha(J_\gamma X)J_\gamma X\Big] \qquad (17)$$

where X is any unit vector of TM.

Proof. Let $J = J_{\alpha|TM}$ with respect to an adapted basis (J_1, J_2, J_3). For any vectors $X, Y \in TM$ one has

$$\begin{aligned} -(\epsilon_\beta\omega_\gamma(X)J_\beta Y - \epsilon_\gamma\omega_\beta(X)J_\gamma Y) &= (\overline{\nabla}_X J_\alpha)Y \\ &= (\nabla_X J)Y + h(X,JY) - J_\alpha h(X,Y). \end{aligned}$$

Hence

$$h(X,JY) - J_\alpha h(X,Y) = -[\epsilon_\beta\omega_\gamma(X)J_\beta Y - \epsilon_\gamma\omega_\beta(X)J_\gamma Y]^\perp . \qquad (18)$$

By comparing with the identity where X is exchanged with Y, one gets the identity

$$\begin{aligned} h(X,JY) - h(Y,JX) &= -\epsilon_\beta[\omega_\gamma(X)J_\beta Y - \omega_\gamma(Y)J_\beta X]^\perp \\ &\quad +\epsilon_\gamma[\omega_\beta(X)J_\gamma Y - \omega_\beta(Y)J_\gamma X]^\perp \end{aligned}$$

that is, by exchanging X with JX,

$$\begin{aligned} -\epsilon_\alpha h(X,Y) + h(JX,JY) = &-\epsilon_\beta[\omega_\gamma(J_\alpha X)J_\beta Y + \eta\epsilon_\gamma\omega_\gamma(Y)J_\gamma X]^\perp \\ &+\epsilon_\gamma[\omega_\beta(J_\alpha X)J_\gamma Y - \eta\epsilon_\beta\omega_\beta(Y)J_\beta X]^\perp . \end{aligned} \qquad (19)$$

Let now $(E_1, \ldots, E_m, J_\alpha E_1, \ldots, J_\alpha E_m)$ be a (pseudo-)orthonormal basis of T_xM such that $(E_1, \ldots, E_k, J_\alpha E_1, \ldots, J_\alpha E_k)$ is an orthonormal basis of $\mathcal{D}$ and, hence,

$(E_{k+1}, \dots, E_m, J_\alpha E_{k+1}, \dots, J_\alpha E_m)$ is a (pseudo-)orthonormal basis of $\bar{T}M$. By using the previous identity, we find

$$
\begin{aligned}
&-2mH \\
&= \textstyle\sum_{i=1}^{m}\{(\omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma)(J_\beta E_i)J_\beta E_i + (\omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma)(J_\gamma E_i)J_\gamma E_i\}^\perp \\
&= \textstyle\sum_{i=1}^{k}\{(\omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma)(J_\beta E_i)J_\beta E_i + (\omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma)(J_\gamma E_i)J_\gamma E_i\}^\perp \\
&= -\left[\mathcal{P}r_{J_\beta \mathcal{D}}\mathbf{t}_\alpha\right]^\perp
\end{aligned}
$$

since $(J_\beta E_1, \dots, J_\beta E_k, J_\beta E_1, \dots, J_\beta E_k)$ is a (pseudo-)orthonormal basis of $\widetilde{\mathcal{D}} = J_\beta \mathcal{D}$. QED

17 Corollary. *The almost (para-)complex submanifold $M^{2m} \subset \tilde{M}^{4n}$ is minimal if the 1-form*

$$\theta := \theta_\alpha = (\omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma)_{|TM}$$

vanishes on $\widetilde{\mathcal{D}}$ for some adapted basis (J_1, J_2, J_3).

Since for a (para)Kähler submanifold (M^{2m}, J_α), the 1-forms $\omega_\beta, \omega_\gamma$ vanish on M^{2m}, we have the following corollary (see [4], [5], [27]).

Let recall that

18 Proposition ([5],[2],[27])**.** *The almost (para-)complex submanifold $(M^{2m}, \overline{J})$ of the (para-)quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ is (para-)-Kähler if and only with respect to an adapted frame (J_1, J_2, J_3) where $\overline{J} = J_\alpha$ one has*

$$\omega_{\beta|TM} = \omega_{\gamma|TM} = 0\,.$$

Proof. It is immediate from (5). QED

Then we have also the following corollary of Proposition 16.

19 Corollary ([5],[2],[27])**.** *A (para)Kähler submanifold $(M^{2m}, \overline{J})$ of a (para-)quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ is minimal.*

Moreover, as another corollary we get the following result, see also [5], [27].

20 Corollary. *Let $(M^2, \overline{J})$ be a 2-dimensional (para-)complex submanifold of a 4-dimensional (para-)quaternionic Kähler manifold. Then the following conditions are equivalent:*

1) $(M^2, \overline{J})$ is (para-)Kähler ,

2) $(M^2, \overline{J})$ is minimal and super(para-)complex.

Proof. (M^2, J_α) is super(para-)complex if and only if $(\omega_\beta \circ J_\beta - \omega_\gamma \circ J_\gamma)_{|TM} = 0$ and by corollary (17) of proposition 16 it is minimal if and only if $(\omega_\beta \circ J_\beta + \omega_\gamma \circ J_\gamma)_{|TM} = 0$. These two conditions imply $\omega_{\beta|TM} = \omega_{\gamma|TM} = 0$, i.e. $(M^2, \overline{J})$ is Kähler. The converse statement is clear. QED

By the same proof one gets the following corollary.

21 Corollary. *Let $(M^2, \overline{J})$ be a super(para-)complex surface of a (para-)-quaternionic Kähler manifold $(\overline{M}^{4n}, \overline{Q}, \overline{g})$. Then it is minimal if and only if it is a (para-)Kähler submanifold.*

The following proposition, which was proved in [4], [27] respectively for quaternionic Kähler, (para-)quaternionic Kähler case, gives a characterization of (para-)Kähler submanifolds between almost (para-)complex submanifolds of a (para-)quaternionic manifold.

22 Proposition. ([5],[27]) *Let $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ be a (para-)quaternionic Kähler manifold with non zero scalar curvature and $(M^{2m}, \overline{J})$ an almost complex submanifold of $\overline{M}$ which is not a (para-)quaternionic submanifold. Then $(M^{2m}, \overline{J})$ is a (para-)Kähler submanifold if and only if the shape operators A^ξ verify the condition*

$$A^{\overline{J}\xi} + \overline{J}A^\xi = 0 \qquad \forall\, \xi \in TM^\perp$$

or, equivalently, the second fundamental form h of M satisfies the condition

$$h(X, \overline{J}Y) - \overline{J}h(X, Y) = 0 \qquad \forall X, Y \in TM. \tag{20}$$

23 Definition. An almost (para-)complex submanifold $(M^{2m}, \overline{J})$ of a (para-)quaternionic Kähler manifold $(\tilde{M}^{4n}, \overline{Q}, \overline{g})$ is called **pluriminimal** or $(1,1)$-**geodesic** if one of the following equivalent conditions holds:

i) the second fundamental form h of M satisfies

$$-\epsilon h(X, Y) + h(JX, JY) = 0 \qquad \forall X, Y \in TM; \tag{21}$$

ii) the shape operators A^ξ anticommute with $J = \overline{J}_{|TM}$,

$$A^\xi J + JA^\xi = 0 \qquad \forall\, \xi \in TM^\perp;$$

iv) any J-invariant 2-dimensional submanifold N^2 of M^{2m} is minimal in $\overline{M}^{4n}$.

A pluriminimal almost (para-)complex submanifold $(M^{2m}, \overline{J})$ is minimal.

A (para-)Kähler submanifold $(M^{2m}, \overline{J})$ of a (para-)quaternionic Kähler manifold $(\tilde{M}^{4n}, \overline{Q}, \overline{g})$ is pluriminimal, since the identity (20) implies (21), as it was observed by Y. Ohnita, [23]. We do not know if the converse is also true under general hypothesis. The following proposition is a partial answer to this question, by giving a characterization of *(para-)complex* pluriminimal submanifolds.

24 Proposition ([5], [27]). *A (para-)complex submanifold* $(M^{2m}, \overline{J})$, $m > 1$*, of the (para-)quaternionic Kähler manifold* $(\overline{M}^{4n}, \overline{Q}, \overline{g})$ *with non zero scalar curvature is pluriminimal if and only if it is a (para-)Kähler submanifold or a (para-)quaternionic (hence totally geodesic) submanifold, and these cases cannot happen simultaneously.*

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