Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **37** (2017) suppl. 1, 87–92. doi:10.1285/i15900932v37suppl1p87

G_2 fibre bundle structure on an oriented 3-dimensional manifold

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Received: 01-08-2016; accepted: 09-11-2016.

Abstract. By using algebraic structures of Clifford algebras and octonions, we show that there exists the G_2 principal fibre bundle structure on any oriented 3-dimensional C^∞ Riemannian manifold.

Keywords: Clifford algebras, Octonions

MSC 2010 classification: Primary 53C30; Secondary 53C15.

Introduction

The purpose of this paper is to construct the G_2 principal fibre bundle of any oriented 3-dimensional C^∞ Riemannian manifold, by using Clifford algebras and octonions.

1 Clifford algebra

We give a brief review of Clifford algebra. Let V be an n-dimensional vector space with the fixed positive definite inner product $\langle\ ,\ \rangle$. The tensor algebra $\mathcal{T}(V)$ is given by

$$\mathcal{T}(V) = \mathbf{R}\cdot 1 \oplus V \oplus (V\otimes V) \oplus (V\otimes V\otimes V) \oplus \cdots \oplus (\otimes^k V)\cdots .$$

We define the two sided ideal $\mathcal{I}$ as follows

$$\mathcal{I} = \text{ the two sided ideal of the set } \{x\otimes x + \langle x,\ x\rangle\cdot 1 | x\in V\}.$$

The Clifford algebra$Cl_n = Cl(V)$ is defined as a factor ring

$$Cl_n = Cl(V) = \mathcal{T}(V)/\mathcal{I}.$$

[i]This work is partially supported by JSPS KAKENHI Grant Number 15K04860.

In particular, we explain Clifford algebra Cl_3 of degree 3. Let V be a 3-dimensional vector space over $\mathbf{R}$ and let $(e_1\ e_2\ e_3)$ be an orthonormal frame of V. An element $x \in Cl_3$, is represented by

$$x = \alpha_0\ 1 + \alpha_1\ e_1 + \alpha_2\ e_2 + \alpha_{12}\ e_{12} + \alpha_3\ e_3 + \alpha_{13}\ e_{13} + \alpha_{23}\ e_{23} + \alpha_{123}\ e_{123}, \tag{1}$$

where $\alpha_* \in \mathbf{R}$ and $e_{ij} = e_i \cdot e_j$ and $e_{123} = e_1 \cdot e_2 \cdot e_3$. Here the symbol $\cdot$ is the multiplication of Clifford algebra. The multiplication table is given by

	1	e_1	e_2	e_{12}	e_3	e_{13}	e_{23}	e_{123}
1	1	e_1	e_2	e_{12}	e_3	e_{13}	e_{23}	e_{123}
e_1	e_1	-1	e_{12}	$-e_2$	e_{13}	$-e_3$	e_{123}	$-e_{23}$
e_2	e_2	$-e_{12}$	-1	e_1	e_{23}	$-e_{123}$	$-e_3$	e_{13}
e_{12}	e_{12}	e_2	$-e_1$	-1	e_{123}	e_{23}	$-e_{13}$	$-e_3$
e_3	e_3	$-e_{13}$	$-e_{23}$	e_{123}	-1	e_1	e_2	$-e_{12}$
e_{13}	e_{13}	e_3	$-e_{123}$	$-e_{23}$	$-e_1$	-1	e_{12}	e_2
e_{23}	e_{23}	e_{123}	e_3	e_{13}	$-e_2$	$-e_{12}$	-1	$-e_1$
e_{123}	e_{123}	$-e_{23}$	e_{13}	$-e_3$	$-e_{12}$	e_2	$-e_1$	1

Table 1. the multiplication table of Cl_3

We note the algebraic properties of Clifford algebra. The Clifford algebra is an associative algebra.

$$Cl_1 \simeq \mathbf{C}\ \text{(Complex)}, \quad Cl_2 \simeq \mathbf{H}\ \text{(Quaternions)}.$$

The above two cases, they are division algebras and $\mathbf{H}$ is a skew field. However

$$Cl_3 \simeq \mathbf{R}^8, Cl_4 \simeq \mathbf{R}^{16}, \ldots, Cl_n \simeq \mathbf{R}^{2^n}, \ldots$$

are not division algebras. In fact, we have

$$(1 + e_{123})(1 - e_{123}) = 1 - e_{123}^2 = 0.$$

In particular, note that $Cl_3 \simeq \mathbf{R}^8$. Cl_3 is a direct sum of two quaternions as follows. We set

$$\begin{aligned}
v_0^+ &= 1/2(1 - e_{123}), & v_0^- &= 1/2(1 + e_{123}), \\
v_1^+ &= 1/2(e_1 + e_{23}), & v_1^- &= 1/2(e_1 - e_{23}), \\
v_2^+ &= 1/2(e_2 + e_{31}), & v_2^- &= 1/2(e_2 - e_{31}), \\
v_3^+ &= 1/2(e_3 + e_{12}), & v_3^- &= 1/2(e_3 - e_{12}).
\end{aligned} \tag{2}$$

Then we have

$$\begin{aligned} Cl_3 &= \mathrm{span}_{\mathbf{R}}\{v_0^+,\ v_1^+,\ v_2^+,\ v_3^+\} \oplus \mathrm{span}_{\mathbf{R}}\{v_0^-,\ v_1^-,\ v_2^-,\ v_3^-\} \\ &\simeq Cl_2 \oplus Cl_2 \simeq \mathbf{H} \oplus \mathbf{H}. \end{aligned} \tag{3}$$

In fact, we easily see that the Clifford multiplication of $\mathrm{span}_{\mathbf{R}}\{v_0^+,\ v_1^+,\ v_2^+,\ v_3^+\}$ (or $\mathrm{span}_{\mathbf{R}}\{v_0^-,\ v_1^-,\ v_2^-,\ v_3^-\}$) coincides with the multiplication of the quaternions.

2 Oriented 3-manifolds

Let M^3 be an oriented 3-dimensional C^∞ Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$. Then there exists a principal $SO(3)$-frame bundle $\mathscr{F}^{SO(3)}$ over M^3, that is

$$\mathscr{F}^{SO(3)} \to M^3$$

with fibre $SO(3)$. For any $p \in \mathscr{F}^{SO(3)}$, p is a pair $p = (m, (\ e_1\ e_2\ e_3\ ))$ where $m \in M$ and $(\ e_1\ e_2\ e_3\ )$ is an oriented orthonormal frame at m. Let U be an open subset of M^3 with local triviality, that is, $\mathscr{F}^{SO(3)}|_U \simeq U \times SO(3)$. Since the tangent space T_mM^3 at $m \in M^3$ is isomorphic to 3-dimensional Euclidean space $\mathbf{R}^3$, we can construct the Clifford algebra $Cl_3(T_mM^3)$. Therefore we obtain Clifford bundle $\mathbf{Cl}_3$ over M^3 with fibre Cl_3 as

$$\mathbf{Cl}_3 = \bigcup_{m \in M^3} Cl_3(T_mM^3).$$

If we take an orthonormal local frame field $(\ \mathfrak{e}_1\ \mathfrak{e}_2\ \mathfrak{e}_3\ )$ on some open subset U. Then, the basis of local sections of $\mathbf{Cl}_3$ are given by

$$(\ 1\ \mathfrak{e}_1\ \mathfrak{e}_2\ \mathfrak{e}_{12}\ \mathfrak{e}_3\ \mathfrak{e}_{13}\ \mathfrak{e}_{23}\ \mathfrak{e}_{123}\ ).$$

We change the order and the sign of this base. Let σ be the new base of $\mathbf{Cl}_3$ define by

$$\sigma(m) = (\ 1\ \mathfrak{e}_1(m)\ \mathfrak{e}_2(m)\ \mathfrak{e}_3(m)\ \mathfrak{e}_{23}(m)\ \mathfrak{e}_{31}(m)\ \mathfrak{e}_{12}(m)\ \mathfrak{e}_{123}(m)\ ).$$

In order to define the G_2-fibre bundle structure whole on M^3 (in the next section), and to prove the well-definedness of G_2-local frame field, we prepare the following: Let $(\ e_1\ e_2\ e_3\ )$, $(\ e_1'\ e_2'\ e_3'\ )$ be two orthonormal basis of T_mM^3. Then there exists $A \in SO(3)$ such that

$$(\ e_1'\ e_2'\ e_3'\ ) = (\ e_1\ e_2\ e_3\ )A.$$

Then we can show that

$$(\; 1 \; e'_1 \; e'_2 \; e'_3 \; e'_{23} \; e'_{31} \; e'_{12} \; e'_{123} \;) = (\; 1 \; e_1 \; e_2 \; e_3 \; e_{23} \; e_{31} \; e_{12} \; e_{123} \;)\tilde{A} \tag{4}$$

where

$$\tilde{A} = \begin{pmatrix} 1 & 0_{1\times 3} & 0_{1\times 3} & 0 \\ 0_{3\times 1} & A & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 1} & 0_{3\times 3} & A & 0_{3\times 1} \\ 0 & 0_{1\times 3} & 0_{1\times 3} & 1 \end{pmatrix} \in M_{8\times 8}(\mathbf{R}). \tag{5}$$

By (2), we define

$$\begin{aligned} \mathfrak{v}_0^+ &= 1/2(1 - \mathfrak{e}_{123}), & \mathfrak{v}_0^- &= 1/2(1 + \mathfrak{e}_{123}), \\ \mathfrak{v}_1^+ &= 1/2(\mathfrak{e}_1 + \mathfrak{e}_{23}), & \mathfrak{v}_1^- &= 1/2(\mathfrak{e}_1 - \mathfrak{e}_{23}), \\ \mathfrak{v}_2^+ &= 1/2(\mathfrak{e}_2 + \mathfrak{e}_{31}), & \mathfrak{v}_2^- &= 1/2(\mathfrak{e}_2 - \mathfrak{e}_{31}), \\ \mathfrak{v}_3^+ &= 1/2(\mathfrak{e}_3 + \mathfrak{e}_{12}), & \mathfrak{v}_3^- &= 1/2(\mathfrak{e}_3 - \mathfrak{e}_{12}). \end{aligned} \tag{6}$$

We note that vector fields $\mathfrak{v}_0^+$ and $\mathfrak{v}_0^-$, and distributions $\text{span}_{\mathbf{R}}\{\mathfrak{v}_1^+, \mathfrak{v}_2^+, \mathfrak{v}_3^+\}$ $\text{span}_{\mathbf{R}}\{\mathfrak{v}_1^-, \mathfrak{v}_2^-, \mathfrak{v}_3^-\}$, are invariant under the above action of $SO(3)$. Then we obtain the following identification

$$Cl_3(T_m M^3) \simeq \mathbf{H} \oplus \mathbf{H}.$$

By (5), this splitting does not depend on the choice of the orthonormal frame of $T_m M^3$. From this construction, we also obtain the principal $(1 \oplus \Delta SO(3) \oplus 1)$ fibre bundle over M^3, where

$$\Delta SO(3) = \begin{pmatrix} A & 0_{3\times 3} \\ 0_{3\times 3} & A \end{pmatrix}$$

for $A \in SO(3)$.

3 Octonionic structure on an oriented 3-manifold

The octonions (or Cayley algebra) $\mathfrak{C}$ over $\mathbf{R}$ can be considered as a direct sum $\mathbf{H} \oplus \mathbf{H} = \mathfrak{C}$ with the following multiplication

$$(a + b\varepsilon)(c + d\varepsilon) = ac - \bar{d}b + (da + b\bar{c})\varepsilon,$$

where $\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$ and $a, b, c, d \in \mathbf{H}$. The symbol $''-''$ denotes the conjugation of the quaternion. The octonions is a non-commutative, non-associative alternative division algebra. Hence the multiplication of $\mathfrak{C}$ is different from that of Cl_3. The group of automorphisms of the octonions is the exceptional simple Lie group

$$G_2 = \{g \in SO(8) \mid g(uv) = g(u)g(v) \text{ for any } u, v \in \mathfrak{C}\}.$$

Since $\Delta SO(3)$ is a closed subgroup of G_2, by (6), we can identify the basis of the octonions and the above basis (as an orthonormal basis of $\mathfrak{C}$)

$$\begin{aligned} 1 \leftrightarrow 1_{\mathfrak{C}},\ \ \mathfrak{v}_1^+ \leftrightarrow i,\ \ \mathfrak{v}_2^+ \leftrightarrow j,\ \ \mathfrak{v}_3^+ \leftrightarrow k,\\ e_{123} \leftrightarrow \varepsilon,\ \ \mathfrak{v}_1^- \leftrightarrow i\varepsilon,\ \ \mathfrak{v}_2^- \leftrightarrow j\varepsilon,\ \ \mathfrak{v}_3^- \leftrightarrow k\varepsilon. \end{aligned} \tag{7}$$

Here we use the multiplication of $\mathfrak{C}$ only. By using this identification, we define the new octonionic multiplication on $Cl_3(T_mM^3)$. Therefore, we can define the G_2 local frame field as follows

$$\begin{aligned} &u = g(e_{123}),\\ &f_1 = 1/2\{g(\mathfrak{v}_1^+) - \sqrt{-1}g(\mathfrak{v}_1^-)\}, \quad \overline{f_1} = 1/2\{g(\mathfrak{v}_1^+) + \sqrt{-1}g(\mathfrak{v}_1^-)\},\\ &f_2 = 1/2\{g(\mathfrak{v}_2^+) - \sqrt{-1}g(\mathfrak{v}_2^-)\}, \quad \overline{f_2} = 1/2\{g(\mathfrak{v}_2^+) + \sqrt{-1}g(\mathfrak{v}_2^-)\},\\ &f_3 = -1/2\{g(\mathfrak{v}_3^+) - \sqrt{-1}g(\mathfrak{v}_3^-)\}, \quad \overline{f_3} = -1/2\{g(\mathfrak{v}_3^+) + \sqrt{-1}g(\mathfrak{v}_3^-)\}, \end{aligned} \tag{8}$$

for any $g \in G_2$. The multiplication table of the product of the complexfied octonions $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C}$ is given by

	u	f_1	f_2	f_3	$\bar{f}_1$	$\bar{f}_2$	$\bar{f}_3$
u	-1	$-\sqrt{-1}f_1$	$-\sqrt{-1}f_2$	$-\sqrt{-1}f_3$	$\sqrt{-1}\bar{f}_1$	$\sqrt{-1}\bar{f}_2$	$\sqrt{-1}\bar{f}_3$
f_1	$\sqrt{-1}f_1$	0	$-\bar{f}_3$	$\bar{f}_2$	$-\bar{n}$	0	0
f_2	$\sqrt{-1}f_2$	$\bar{f}_3$	0	$-\bar{f}_1$	0	$-\bar{n}$	0
f_3	$\sqrt{-1}f_3$	$-\bar{f}_2$	$\bar{f}_1$	0	0	0	$-\bar{n}$
$\bar{f}_1$	$-\sqrt{-1}\bar{f}_1$	$-n$	0	0	0	$-f_3$	f_2
$\bar{f}_2$	$-\sqrt{-1}\bar{f}_2$	0	$-n$	0	f_3	0	$-f_1$
$\bar{f}_3$	$-\sqrt{-1}\bar{f}_3$	0	0	$-n$	$-f_2$	f_1	0

Table 2. the multiplication table of $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C}$

where $n = 1/2(1 - \sqrt{-1}u)$. We set

$$\mathscr{F}^{G_2} = \{(m, (\ u, f_1, f_2, f_3, \overline{f_1}, \overline{f_2}, \overline{f_3})|_m) \mid \ (\ u, f_i, \overline{f_i})|_m \text{ is a } G_2 \text{ frame at } m\}.$$

We can see that

$$\mathscr{F}^{G_2} \to M^3$$

is a principal fibre bundle on M^3 with fibre G_2.

Theorem 1. *There exists the G_2 fibre bundle structure on any oriented 3-dimensional manifold M^3.*

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