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On a generalization of geodesic and magnetic curves

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Abstract. The paper deals with a generalization of the notions of geodesic and magnetic curve, namely (F, H)-geodesic, on a manifold endowed with a linear connection and two (1,1)-tensor fields F and H.

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To the memory of Professor Dan Schwarz

Introduction

The topic of geodesics with respect to a linear connection on a manifold is interesting for differential equations, differential geometry, theory of relativity and other fields. In classical mechanics, geodesics are seen as trajectories of free particles in a manifold. Magnetic curves, which generalize geodesics, represent the trajectories around which a charged particle spirals under the action of a magnetic field F.

A new notion, introduced in [3], generalizes both the geodesics and the magnetic curves. These curves, called F-geodesics, are defined on a manifold endowed with a linear connection and an arbitrary (1,1)-tensor field (which can be in particular the electro-magnetic field or the Lorentz force).

The notion of F-geodesic is slightly different from that of F-planar curve (see [4], [5]). In [3], several examples and characterizations are given, and the F-geodesics with respect to Vranceanu connections or adapted connections on almost contact manifolds are studied. Also, the classical projective transformation, holomorphic-projective transformation and C-projective transformation are generalized by considering a pair of symmetric connections which have the same F-geodesics and then the transformations between such two connections, namely F-planar diffeomorphisms (see [6, 7]), are studied.

In the present paper, we go further and consider a manifold M, endowed with a linear connection as well as two given forces described by two (1,1)-tensor fields. We define here (F, H)-geodesics, give some examples and establish the relation between two symmetric connections having the same system of (F, H)-geodesics.

1 (F, H)-geodesics

The main geometric objects used in the present note are provided by the following:

Notations 1.1. By (M, F, H, ∇) we mean a manifold M endowed with the $(1, 1)$-tensor fields F and H, as well as with the linear connection ∇.

The following notion generalizes the classical geodesics and it is followed by some examples.

Definition 1.2. We say that a smooth curve $\gamma : I \to M$ on a manifold (M, F, H, ∇) is an (F, H)-geodesic if the acceleration $\nabla_{\dot\gamma(u)}\dot\gamma(u)$ belongs to the space generated by $F\dot\gamma(u)$ and $H\dot\gamma(u)$. That is, there exist some smooth functions $\alpha, \beta : I \to \mathbb{R}$, such that

$$\nabla_{\dot\gamma(u)}\dot\gamma(u) = \alpha(u)F\dot\gamma(u) + \beta(u)H\dot\gamma(u), \tag{1}$$

where I is a real interval.

A physical interpretation for the particle $\gamma(u)$ which satisfies (1) is that it is moving in a space under the action of the external forces F and H.

By using local coordinates $(x^1, \ldots, x^m)$ on the m-dimensional manifold M, we can write the ordinary differential equation (1) by using the summation convention as:

$$\frac{d^2\gamma^i}{du^2} + \Gamma^i_{jk}\frac{d\gamma^j}{du}\frac{d\gamma^k}{du} = \alpha(u)F^i_j\frac{d\gamma^j}{du} + \beta(u)H^i_k\frac{d\gamma^k}{du}, \tag{2}$$

where $\gamma^i = x^i \circ \gamma(u)$, and Γ^i_{jk} are the Christoffel symbols of the connection ∇.

The mathematical meaning of (2) is that the covariant derivative with respect to ∇ of the velocity field $\dot\gamma(u) = \frac{d\gamma}{du}$ along $\gamma(u)$ remains in span $\{F\dot\gamma(u), H\dot\gamma(u)\}$ and we note that this space may be of dimension $2, 1$ or 0.

Remark 1.3. (a) If t is another parameter for the same curve $\gamma(u)$, then the relation (1) becomes:

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = u(t)\dot\gamma(t) + v(t)F\dot\gamma(t) + w(t)H\dot\gamma(t), \tag{3}$$

where u, v and w are some smooth functions along the curve $\gamma(t)$.

(b) A curve $\gamma(t)$ satisfying the relation (3) describes an (F, H)-geodesic up to a reparametrization.

(c) From geometrical point of view, an (F, H)-geodesic up to a reparametrization is defined as a curve $\gamma(t)$ such that the parallel transport along the curve preserves the linear subspace of dimension $1, 2$ or 3 spanned by $\dot\gamma(t), F\dot\gamma(t)$ and $H\dot\gamma(t)$.

Examples of (F, H)-geodesics

(i) When F is the identity endomorphism up to a multiplicative function, and H is identically zero, then an (F, H)-geodesic is a geodesic up to a reparametrization.

(ii) If both F and H are identically zero, then an (F, H)-geodesic becomes a classical geodesic and moreover an (F, H)-geodesic up to a reparametrization becomes a geodesic up to a reparametrization.

(iii) Another example of an (F, H)-geodesic can be taken from the Lagrangian mechanics, where the trajectory of a particle is described by the Euler-Lagrange equations, with a particular Lagrangian function.

(iv) We provide now another example of (F, H)-geodesic, by using Lorentz force defined on a (semi-)Riemannian manifold of arbitrary dimension.

For this purpose, we recall now the following notions for which we refer to [1].

Definition 1.4. On a (semi-)Riemannian manifold (M, g), a closed 2-form Ω is called a magnetic field if it is associated by the following relation to the Lorentz force Φ, defined as a skew-symmetric (with respect to g) endomorphism field on M :

$$g(\Phi(X), Y) = \Omega(X, Y), \ \ \forall X, Y \in \Gamma(TM).$$

The Lorentz force Φ is a divergence-free (1,1)-tensor field (i.e. $\mathrm{div}\Phi = 0$).

Let ∇ be the Levi-Civita connection of g and let q be the charge of a particle, describing a smooth trajectory γ on M. Then the curve $\gamma(t)$ where the speed $\dot\gamma(t)$ satisfies the Lorentz equation

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = q\Phi\dot\gamma(t),$$

is known in literature as a magnetic curve.

According to Definition 1.4, the above Lorentz equation describes an (F,H)-geodesic on M, where F is defined by

$$FX = q\Phi(X), \ \ \forall X \in \Gamma(TM),$$

and H vanishes identically.

Therefore, any magnetic curve is a particular case of an (F,H)-geodesic.

Moreover, if on a (semi-)Riemannian manifold (M,g) one has a pair of magnetic fields Ω_1, Ω_2 having the associated Lorentz forces Φ_1 and Φ_2 defined as above, then according to Definition 1.4, a curve $\gamma(t)$ which satisfies the bi-Lorentz equation

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = q_1\Phi_1(\dot{\gamma}(t)) + q_2\Phi_2(\dot{\gamma}(t)),$$

is an (F,H)-geodesic on M, where $q_1, q_2 \in \mathbb{R}$,

$$F(X) = q_1\Phi_1(X) \text{ and } H(X) = q_2\Phi_2(X), \ \ \forall X \in \Gamma(TM).$$

(v) In [3], the first author and Druta-Romaniuc introduced and studied F-geodesics, which are examples of (F,H)-geodesics, when H vanishes identically.

From the Riemannian context, we recall the existence and uniqueness of the solution of a second order differential equation with initial data, which gives the existence and uniqueness of a geodesic passing through a given point p, having a given velocity $X_p \in T_pM$. The above properties were extended in [2] to magnetic curves corresponding to an arbitrary magnetic field and then in [3] to F-geodesics. One question arising on a triple (M,F,H,∇) is about the existence of the (F,H)-geodesics. The answer is given by the theory of differential systems with Cauchy condition, which leads to the following generalization of the above result.

Proposition 1.5. Let (M,F,H,∇) be a manifold considered as in Notation 1.1. Then for any point $p \in M$ and any vector $X_p \in T_pM$, there exists a unique maximal (F,H)-geodesic passing through p and having the velocity X_p.

2 (F,H)-projective transformation

Another question which naturally occurs on a manifold (M,F,H) endowed with a couple of $(1,1)$-tensor fields, would be how are related two linear connections having the same (F,H)-geodesics. For this purpose we introduce the following:

Definition 2.1. Let (M, F, H) be a manifold with a couple of forces given by the $(1,1)$-tensor fields F and H. Then two linear connection $\overline{\nabla}$ and ∇ are called (F, H)-projectively related to each other, if they have the same system of (F, H)-geodesics up to a reparametrization.

Notations 2.2. (i) If ∇ and $\overline{\nabla}$ are two torsion-free linear connections on a manifold M, then we define the deformation tensor field S as the symmetric $(1,2)$-tensor field given by

$$S(X,Y) = \overline{\nabla}_X Y - \nabla_X Y, \forall X, Y \in \Gamma(TM).$$

Obviously, for any common (F, H)-geodesic up to a reparametrization $\gamma(t)$ of $\overline{\nabla}$ and ∇, one has:

$$\begin{aligned} S(\dot\gamma(t), \dot\gamma(t)) &= \overline{\nabla}_{\dot\gamma(t)}\dot\gamma(t) - \nabla_{\dot\gamma(t)}\dot\gamma(t) = \\ &= a(t)\dot\gamma(t) + b(t)F\dot\gamma(t) + c(t)H\dot\gamma(t), \end{aligned} \tag{4}$$

where a, b, c are some smooth functions along the curve $\gamma(t)$.

(ii) We say that the deformation tensor field S satisfies the coefficients linearity (CL) condition, if for any common (F, H)-geodesic in the last relation, the coefficients a, b and c depend linearly on the speed of the curve. Precisely, S satisfies the (CL)-condition, if there exist three 1-forms $A, B, C \in \Gamma(T^*M)$, such that

$$a(t) = A(\dot\gamma(t)), b(t) = B(\dot\gamma(t)), c(t) = C(\dot\gamma(t)), \tag{5}$$

for each common (F, H)-geodesic of $\overline{\nabla}$ and ∇.

Definition 2.3. We say that two symmetric linear connections ∇ and $\overline{\nabla}$ on M are related by an (F, H)-planar diffeomorphism if

$$\begin{aligned} \overline{\nabla}_X Y = \nabla_X Y &+ \omega(Y)X + \omega(X)Y + \theta(X)FY + \\ &+ \theta(Y)FX + \eta(X)HY + \eta(Y)HX, \forall X, Y \in \Gamma(TM), \end{aligned} \tag{6}$$

for some 1-forms ω, θ and η on M.

Two symmetric linear connections related by an (F, H)-planar diffeomorfism are (F, H)-projectively related. More precisely, we have:

Theorem 2.4. Let (M, F, H) be a manifold endowed with two $(1,1)$-tensor fields F and H. Then any two symmetric linear connections are (F, H)-projectively related to each other and their deformation tensor field S satisfies (CL) condition, provided they are related by an (F, H)-planar diffeomorphism.

Proof. Let ∇ and $\overline{\nabla}$ be two symmetric linear connections on M related by an (F, H)-planar diffeomorphism, i.e. (6) is satisfied. If we take $\gamma(t)$ to be a geodesic up to a reparametrization of ∇, then from (3) we obtain:

$$\begin{aligned}\overline{\nabla}_{\dot\gamma(t)}\dot\gamma(t) &= u(t)\dot\gamma(t) + v(t)F\dot\gamma(t) + w(t)H\dot\gamma(t)\\ &+ 2\omega(\dot\gamma(t))\dot\gamma(t) + 2\theta(\dot\gamma(t))F\dot\gamma(t) +\\ &+ 2\eta(\dot\gamma(t))H\dot\gamma(t),\end{aligned}$$

where we have used two notations of (6) and (3).

Hence $\overline{\nabla}_{\dot\gamma(t)}\dot\gamma(t) = \overline{u}(t)\dot\gamma(t) + \overline{v}(t)F\dot\gamma(t) + \overline{w}(t)H\dot\gamma(t)$, where $\overline{u} = u + 2\omega\circ\dot\gamma$, $\overline{v} = v + 2\theta\circ\dot\gamma$, $\overline{w} = w + 2\eta\circ\dot\gamma$, which shows that γ is an (F, H)-geodesic up to a reparametrization of $\overline{\nabla}$. In the same way, it follows that any geodesic up to a reparametrization of $\overline{\nabla}$ is an (F, H)-geodesic up to a reparametrization of ∇ and therefore ∇ and $\overline{\nabla}$ are (F, H)-projectively related to each other.

Any common (F, H)-geodesic up to a reparametrization $\gamma(t)$ of $\overline{\nabla}$ and ∇ satisfies (4). From (6) one has

$$\begin{aligned}S(\dot\gamma(t), \dot\gamma(t)) = \overline{\nabla}_{\dot\gamma(t)}\dot\gamma(t) - \nabla_{\dot\gamma(t)}\dot\gamma(t) &= 2\omega(\dot\gamma(t))\dot\gamma(t) +\\ &+ 2\theta(\dot\gamma(t))F\dot\gamma(t) + 2\eta(\dot\gamma(t))H\dot\gamma(t),\end{aligned}$$

which shows that (CL) condition (5) is satisfied with $A = 2\omega$, $B = 2\theta$ and $C = 2\eta$, which complete the proof.

We recall the following

Definition 2.5. On a manifold M, let F, H, L be three $(1, 1)$-tensor fields such that $L = F\circ H = -H\circ F$. Then the structure (F, H, L) is called

(a) quaternionic (or almost hypercomplex), if $F^2 = H^2 = L^2 = -I$;

(b) almost hyper-para-complex if $F^2 = -H^2 = -L^2 = -I$,

where I is the identity.

The converse of Theorem 2.4 is not true even in very special cases, such as quaternionic (almost hypercomplex) and almost hyper-para-complex structures, where the deformation tensor takes a slightly generalized form with additional terms depending on $F\circ H$.

Theorem 2.6. If (F, H, L) is a quaternionic (resp. almost hyper-para-complex) structure on M, parallel w.r.t. two symmetric connections $\nabla, \overline{\nabla}$ which are (F, H)-projectively related to each other and the (CL)-condition is satisfied, then

$$\begin{aligned}\overline{\nabla}_X Y = \nabla_X Y &+ \omega(Y)X + \omega(X)Y + \theta(X)FY + \theta(Y)FX + \eta(X)HY\\ &+ \eta(Y)HX + \nu(Y)LX + \nu(X)LY, \forall X, Y \in \Gamma(TM),\end{aligned} \tag{7}$$

for some 1-forms $\omega, \theta, \eta, \nu$ on M.

Proof. Let (F, H, L) be a quaternionic structure and let $(\mathcal{U}, x^1, \ldots, x^{4n})$ be a local chart on M. Then

$$\begin{aligned} &F^k_s F^s_r = H^k_s H^s_r = L^k_s L^s_r = -\delta^k_r \\ &F^k_s H^s_r = -H^k_s F^s_r = L^k_r. \end{aligned} \tag{8}$$

Let ∇ and $\overline{\nabla}$ be two symmetric connections whose coefficients are Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$, respectively. Let S denote the deformation tensor:

$$S^h_{ij} = \overline{\Gamma}^h_{ij} - \Gamma^h_{ij}$$

and hence

$$S^h_{ij} = S^h_{ji}. \tag{9}$$

If we assume that ∇ and $\overline{\nabla}$ are (F, H)-projectively related and (CL) condition is satisfied, then at any point of M, one has:

$$S^h_{ij} v^i v^j = a v^h + b F^h_r v^r + c H^h_r v^r, \tag{10}$$

for any vector v (at that point) of components v^r, where a, b, c are real functions.

By multiplying (10) with v^k, one has

$$S^h_{ij} v^i v^j v^k = a v^h v^k + b F^h_r v^r v^k + c H^h_r v^r v^k. \tag{11}$$

Interchanging k and h and substracting the new relation from (11), we obtain a relation in which we replace k by s and multiply by F^k_s, such that from (8) we have:

$$\begin{aligned} &(S^h_{ij} F^k_l - F^k_s S^s_{ij} \delta^h_l) v^i v^j v^l = b(F^h_r v^r)(F^k_s v^s) \\ &+ c(H^h_r v^r)(F^k_s v^s) + b v^k v^h - c L^k_r v^r v^h. \end{aligned} \tag{12}$$

We interchange k with h and obtain a new relation from which we substract (12). After that, we change k with t and multiply by L^k_t. Then we interchange k with h and we substract the last relation from the previous one. Since this new relation holds for any vector v, it follows:

$$\begin{aligned} &S^t_{ij} F^h_l L^k_t - F^h_s S^s_{ij} L^k_l - S^h_{ij} H^k_l + H^k_s S^s_{ij} \delta^h_l - \\ &- S^t_{ij} F^k_l L^h_t + F^k_s S^s_{ij} L^h_l + S^k_{ij} H^h_l - H^h_s S^s_{ij} \delta^k_l = 0. \end{aligned}$$

Now, we make cyclic permutation of (i, j, l) and we add all these three relations. Then we multiply with F^l_m and after that we contract $h = m$. In the new relation we take into account that F, H and L are traces free, i.e. $F_h^h = H_h^h = L_h^h = 0$. We also use the parallelism of F, H, L with respect to ∇ and $\overline{\nabla}$, i.e.:

$$S_{lj}^k F_i^l = S_{ij}^l F_l^k; S_{lj}^k H_i^l = S_{ij}^l H_l^k;\ S_{lj}^k L_i^l = S_{ij}^l L_l^k.$$

Therefore, by using (8) we obtain

$$\begin{aligned}&-2(2n+1)S_{ij}^t L_t^k + S_{jl}^l L_i^k + S_{il}^l L_j^k - \\ &- S_{jl}^h F_h^l H_i^k - S_{li}^h F_h^l H_j^k + S_{jl}^t H_t^l F_i^k + \\ &+ S_{li}^t F_j^k H_t^l - L_s^l S_{jl}^s \delta_i^k - L_s^l S_{li}^s \delta_j^k = 0.\end{aligned}$$

We multiply this relation by L_k^u, we use (8) and then, with the following notations:

$$\begin{aligned}\omega_j &= \frac{1}{2(2n+1)} S_{jl}^l;\ \theta_j = \frac{1}{2(2n+1)} S_{jl}^h F_h^l; \\ \eta_j &= \frac{1}{2(2n+1)} S_{jl}^h H_h^l;\ \nu_j = \frac{1}{2(2n+1)} S_{jl}^h L_h^l,\end{aligned}$$

we obtain the relation (7) written with indices. To prove the case when (F, H, L) is an almost para-hyper-complex structure, we proceed in a similar way and complete the proof.

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