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ON A THEOREM MOUCHTARI AND ŠERSTNEV

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Abstract. *In this note we use a duality theorem to quickly and easily obtain the unique solution of a functional equation on the space of probability distribution functions which was first studied and solved by D.H. Mouchtari and A.N. Šerstnev.*

Let Δ^+ be the set of all probability distribution functions of non--negative random variables, i.e.,

$$\Delta^+ = \{ F \mid F : [-\infty,\infty] \longrightarrow [0,1],\ F(0) = 0, F(\infty) = 1,\ F \text{ is non-decreasing and left-continuous on } [-\infty,\infty) \},$$

and let ε_0 be the distribution function in Δ^+ defined by $\varepsilon_0(x) = 0$ for $x \leqq 0$ and $\varepsilon_0(x) = 1$ for $x > 0$. A mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ is a *triangle function* if for all F, G, H, K in Δ^+,

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(a) $\tau(F, \varepsilon_0) = F$,

(b) $\tau(F,G) \leqq \tau(H,K)$ whenever $F \leqq H$, $G \leqq K$,

(c) $\tau(F,G) = \tau(G,F)$,

(d) $\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$.

Thus convolution and the mapping τ_M defined by

(1) $\tau_M(F,G)(x) = \sup_{u+v=x} \mathrm{Min}(F(u),G(v))$

are triangle functions.

If j denotes the identity function on $[-\infty,\infty]$, then for any F in Δ^+ and any $a > 0$, the distribution function in Δ^+ whose value for any $x \geqq 0$ is $F(x/a)$ may be conveniently denoted by $F(j/a)$.

In [2] D.M. Mouchtari and A.N. Šerstnev showed that if τ is a triangle function then the equality

(2) $$\tau(F(j/a),F(j/b)) = F(j/a + b)$$

holds for all F in Δ^+ and all $a,b > 0$ if and only if $\tau = \tau_M$. Thus τ_M is the unique triangle function which satisfies the functional equation (2). The purpose of this note is to show that the duality theorem established in [1] yields a very simple proof of this fact. To this end we recall that for any F in Δ^+ the left-continuous *quasi-inverse* of F is the function $F^\wedge$ from $[0,1]$ into $[0,\infty]$ defined by

(3) $$F^\wedge(y) = \begin{cases} 0, & y = 0, \\ \sup\{x \mid F(x) < y\}, & 0 < y \leqq 1. \end{cases}$$

In particular,

(4) $[F(j/a)]^{\wedge} = aF^{\wedge}$;

and if $F^{\wedge} = G^{\wedge}$ then $F = G$. We denote the space of quasi-inverses of elements of Δ^+ by $(\Delta^+)^{\wedge}$.

It follows from the duality theorem of [1] that

(5) $$[\tau_M(F,G)]^{\wedge} = F^{\wedge} + G^{\wedge},$$

whence,

(6) $$\begin{aligned}[\tau_M(F(j/a),F(j/b))]^{\wedge} &= aF^{\wedge} + bF^{\wedge} \\ &= (a+b)F^{\wedge} = [F(j/a+b)]^{\wedge}.\end{aligned}$$

Thus τ_M is a solution of (2).

Turning to the converse, for any triangle function τ let $\tau^{\wedge}$ be the binary operation induced on $(\Delta^+)^{\wedge}$ by

(7) $$\tau^{\wedge}(F^{\wedge},G^{\wedge}) = [\tau(F,G)]^{\wedge}.$$

Then (2) is equivalent to

(8) $$\tau^{\wedge}(aF^{\wedge},bF^{\wedge}) = (a+b)F^{\wedge}.$$

Next, for any F, G in Δ^+ and any $a,b > 0$, let $U^{\wedge}$ and $V^{\wedge}$ be the functions defined by

(9) $$U^{\wedge} = \mathrm{Min}\left(\frac{1}{a}F^{\wedge}, \frac{1}{b}G^{\wedge}\right) \quad \text{and} \quad V^{\wedge} = \mathrm{Max}\left(\frac{1}{a}F^{\wedge}, \frac{1}{b}G^{\wedge}\right).$$

Then

(10) $$aU^{\wedge} \leqq F^{\wedge} \leqq aV^{\wedge} \quad \text{and} \quad bU^{\wedge} \leqq G^{\wedge} \leqq bV^{\wedge}.$$

Since $\tau^\wedge$ is non-decreasing on $(\Delta^+)^\wedge$, it follows that

(11) $$\tau^\wedge(aU^\wedge,bU^\wedge) \leqq \tau^\wedge(F^\wedge,G^\wedge) \leqq \tau^\wedge(aV^\wedge,bV^\wedge).$$

Suppose that τ satisfies (2). Then combining (8) and (11) we have that for all $a,b>0$,

(12) $$(a+b)U^\wedge \leqq \tau^\wedge(F^\wedge,G^\wedge) \leqq (a+b)V^\wedge.$$

To show that (12) implies that $\tau = \tau_M$, we choose x such that $0<x<1$ and consider the following three cases:

Case 1. $F^\wedge(x) \neq 0$ and $G^\wedge(x) \neq 0$. Then setting $a = F^\wedge(x)$ and $b = G^\wedge(x)$ in (9) yields $U^\wedge(x) = V^\wedge(x) = 1$, and using (12) we have at once that:

(13) $$\tau^\wedge(F^\wedge,G^\wedge)(x) = F^\wedge(x) + G^\wedge(x).$$

Case 2. $F^\wedge(x) = G^\wedge(x) = 0$. Then setting $a = b = 1$ in (9) yields $U^\wedge(x) = V^\wedge(x) = 0$, whence by (12) we have $\tau^\wedge(F^\wedge,G^\wedge)(x) = 0$ and (13) is again valid.

Case 3. $F^\wedge(x) = 0$ and $G^\wedge(x) \neq 0$. Then setting $a = \varepsilon > 0$ and $b = G^\wedge(x)$ in (9) yields $U^\wedge(x)=0$ and $V^\wedge(x) = 1$, whence it follows that $\tau^\wedge(F^\wedge,G^\wedge)(x) \leqq G^\wedge(x) + \varepsilon$. Since ε is arbitrary, we have $\tau^\wedge(F^\wedge,G^\wedge)(x) \leqq G^\wedge(x)$. But (a) and (b) imply $\tau(F,G) \leqq G$, whence $G^\wedge(x) \leqq \tau^\wedge(F^\wedge,G^\wedge)(x)$, and again (13) holds. The same conclusion follows if $F^\wedge(x) \neq 0$ and $G^\wedge(x) = 0$.

Thus (13) holds for all x in $(0,1)$ whence, using (5) and (7), we have $\tau = \tau_M$.

We conclude with several remarks.

1. Note that neither the commutativity nor the associativity of τ was used in the above argument.

2. The above argument also shows that $\tau(F(j/a),F(j/b)) \geqq F(j/a+b)$ (resp., $\leqq F(j/a+b)$) if and only if $\tau \leqq \tau_M$ (resp., $\tau \geqq \tau_M$).

3. If L is a suitable binary operation on $[0,\infty]$ then $\tau(F(j/a),F(j/b)) = F(j/L(a,b))$ if and only if $\tau = \tau_{M,L}$ (see [1], Theorem 4.8. and [3], Section 7.7).

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