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Congruences for (2, 3)-regular partition with designated summands

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Abstract. Let $PD_{2,3}(n)$ count the number of partitions of n with designated summands in which parts are not multiples of 2 or 3. In this work, we establish congruences modulo powers of 2 and 3 for $PD_{2,3}(n)$. For example, for each $n \geq 0$ and $\alpha \geq 0$ $PD_{2,3}(6\cdot 4^{\alpha+2}n+5\cdot 4^{\alpha+2}) \equiv 0 \pmod{2^4}$ and $PD_{2,3}(4\cdot 3^{\alpha+3}n+10\cdot 3^{\alpha+2}) \equiv 0 \pmod{3}$.

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1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. A partition is (2, 3)-regular partition of n if none of the parts are divisible by 2 or 3.

Andrews, Lewis and Lovejoy [1] have investigated a new class of partition with designated summands are constructed by taking ordinary partitions and tagging exactly one of each part size. The total number of partitions of n with designated summands is denoted by $PD(n)$. Hence $PD(4) = 10$, namely

$4'$, $3'+1'$, $2'+2$, $2+2'$, $2'+1'+1$, $2'+1+1'$, $1'+1+1+1$, $1+1'+1+1$, $1+1+1'+1$, $1+1+1+1'$.

Andrews et al. [1] have derived the following generating function of $PD(n)$, namely

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}, \tag{1}$$

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where

$$f_n := \prod_{j=1}^{\infty}(1-q^{nj}), n \geq 1. \tag{2}$$

Andrews et al. [1] and N. D. Baruah and K. K. Ojah [3] have also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd. The generating function of $PDO(n)$ is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \tag{3}$$

Mahadeva Naika et al. [12] have studied $PD_3(n)$, the number of partitions of n with designated summands whose parts not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}. \tag{4}$$

In [13] Mahadeva Naika et al. have established many congruences for $PD_2(n)$, the number of bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}. \tag{5}$$

Motivated by the above works, in this paper, we defined $PD_{2,3}(n)$, the number of partitions of n with designated summands in which parts are not multiples of 2 or 3. For example $PD_{2,3}(4) = 4$, namely

$1' + 1 + 1 + 1, \quad 1 + 1' + 1 + 1, \quad 1 + 1 + 1' + 1, \quad 1 + 1 + 1 + 1'.$

The generating function of $PD_{2,3}(n)$ is given by

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}. \tag{6}$$

Following Ramanujan, for $|ab| < 1$, we define his general theta function $f(a, b)$ as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{7}$$

The important special cases of $f(a, b)$ are

$$\varphi(q) := f(q, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{8}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1} \tag{9}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1, \tag{10}$$

where the product representations arise from famous Jacobi's triple product identity [5, p. 35, Entry 19]

$$f(a, b) = (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty. \tag{11}$$

In this paper, we list few formulas which helps to prove our main results in section 2. In section 3, we obtain several congruences modulo powers of 2 and congruences modulo 3 in section 4.

2 Preliminary results

We list few dissection formulas to prove our main results.

Lemma 1. *[14, p. 212] We have the following* 5*-dissection*

$$f_1 = f_{25}\left(a(q^5) - q - q^2/a(q^5)\right), \tag{12}$$

where

$$a := a(q) := \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}. \tag{13}$$

Lemma 2. *The following* 2*-dissection holds:*

$$\frac{f_9}{f_1} = \frac{f_{18}f_{12}^3}{f_{36}f_6f_2^2} + q\frac{f_{36}f_6f_4^2}{f_{12}f_2^3}. \tag{14}$$

Identity (2) *is nothing but Lemma 3.5 in [16].*

Lemma 3. *The following* 3*-dissection holds:*

$$\frac{f_4}{f_1} = \frac{f_{12}f_{18}^4}{f_3^3f_{36}^2} + q\frac{f_6^2f_9^3f_{36}}{f_3^4f_{18}^2} + 2q^2\frac{f_6f_{18}f_{36}}{f_3^3}. \tag{15}$$

Identity (3) *is nothing but Lemma 2.6 in [3].*

Lemma 4. *The following* 3*-dissection holds:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}. \tag{16}$$

Equation (16) was proved by Hirschhorn and Sellers [10].

Lemma 5. *[5, p. 49] We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \tag{17}$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{18}$$

Lemma 6. *The following* 2*-dissections holds:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{19}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{20}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{21}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{22}$$

Lemma (6) is a consequence of dissection formulas of Ramanujan, collected in Berndt's book [5, p. 40, Entry 25].

Lemma 7. *The following* 2*-dissection holds:*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{23}$$

Xia and Yao [18] gave a proof of Lemma (7).

Lemma 8. *The following* 2*-dissections holds:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{24}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}. \tag{25}$$

Xia and Yao[17] proved (24) by employing an addition formula for theta functions. Replacing q by $-q$ in (20) and then using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1 f_4}$, we obtain (25).

Lemma 9. *The following* 2*-dissections holds:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{26}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{27}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{28}$$

Hirschhorn, Garvan and Borwein [8] proved (26) and (27). For proof of (28), see [4].

Lemma 10. *The following* 2*-dissections holds:*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}} \tag{29}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{30}$$

Equation (29) was proved by Baruah and Ojah [3]. Replacing q by $-q$ in (29) and using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1 f_4}$, we get (30).

Lemma 11. *The following* 3*-dissection holds:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{31}$$

One can see this identity in [9].

Lemma 12. *(Cui and Gu [7, Theorem 2.2]). For any prime* $p \geq 5$*,*

$$f_1 = \sum_{\substack{k=\frac{1-p}{2} \\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}, \tag{32}$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, for $\frac{-(p-1)}{2} \leq k \leq \frac{p-1}{2}$ *and* $k \neq \frac{(\pm p-1)}{6}$*,*

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

3 Congruences Modulo Powers of 2.

Theorem 1. *For $n \geq 1$ and $\alpha \geq 0$, then*

$$PD_{2,3}(18n) \equiv 0 \pmod 4, \tag{33}$$

$$PD_{2,3}(2 \cdot 3^{\alpha+3} n) \equiv 0 \pmod 4. \tag{34}$$

Proof. We have

$$\sum_{n=0}^{\infty} PD_{2,3}(n) q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}. \tag{35}$$

Substituting (14) into (35), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(n) q^n = \frac{f_4 f_6 f_{12}}{f_2^2 f_{18}} + q \frac{f_4^3 f_6^3 f_{36}^2}{f_2^3 f_{12}^3 f_{18}^2}. \tag{36}$$

Extracting the even terms in the above equation

$$\sum_{n=0}^{\infty} PD_{2,3}(2n) q^n = \frac{f_2 f_3 f_6}{f_1^2 f_9}. \tag{37}$$

Substituting (16) into (37), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(2n) q^n = \frac{f_6^5 f_9^5}{f_3^7 f_{18}^3} + 2q \frac{f_6^4 f_9^2}{f_3^6} + 4q^2 \frac{f_6^3 f_{18}^3}{f_3^5 f_9}. \tag{38}$$

Extracting the terms involving q^{3n} from both sides of (38) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n) q^n = \frac{f_2^5 f_3^5}{f_1^7 f_6^3}. \tag{39}$$

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod 4 \tag{40}$$

and

$$f_{2k}^{4m} \equiv f_k^{8m} \pmod 8. \tag{41}$$

Invoking (41) into (39), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(6n) q^n \equiv \frac{f_1 f_2 f_3^5}{f_6^3} \pmod 8. \tag{42}$$

Employing (31) into (42), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_3^4 f_9^4}{f_6^2 f_{18}^2} - q\frac{f_3^5 f_9 f_{18}}{f_6^3} - 2q^2\frac{f_3^6 f_{18}^4}{f_6^4 f_9^2} \pmod 8. \tag{43}$$

Extracting the terms involving q^{3n} from both sides of (43) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_1^4 f_3^4}{f_2^2 f_6^2} \pmod 8. \tag{44}$$

Congruence (33) follow from (40) and (44).

Equation (44) can be rewritten as

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_3^4}{f_6^2}\left(\frac{f_1^2}{f_2}\right)^2 \pmod 8. \tag{45}$$

Replacing q by $-q$ in (17) and using the fact that

$$\phi(-q) = \frac{f_1^2}{f_2}, \tag{46}$$

we find that

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{47}$$

Employing (47) into (45), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_3^4 f_9^4}{f_6^2 f_{18}^2} + 4q^2\frac{f_3^6 f_{18}^4}{f_6^4 f_9^2} - 4q\frac{f_3^5 f_9 f_{18}}{f_6^3} \pmod 8. \tag{48}$$

Extracting the terms involving q^{3n} from both sides of (48) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(54n)q^n \equiv \frac{f_1^4 f_3^4}{f_2^2 f_6^2} \pmod 8. \tag{49}$$

In view of the congruences (44) and (49), we get

$$PD_{2,3}(54n) \equiv PD_{2,3}(18n) \pmod 8. \tag{50}$$

Utilizing (50) and by mathematical induction on α, we arrive

$$PD_{2,3}(2 \cdot 3^{\alpha+3}n) \equiv PD_{2,3}(18n) \pmod 8. \tag{51}$$

Using (33) into (51), we get (34). QED

Theorem 2. *For $n \geq 0$ and $\alpha \geq 0$, we have*

$$PD_{2,3}(72n+42) \equiv 0 \pmod 4, \tag{52}$$
$$PD_{2,3}(36n+30) \equiv 0 \pmod 4, \tag{53}$$
$$PD_{2,3}(144n+120) \equiv 0 \pmod 4, \tag{54}$$
$$PD_{2,3}(9 \cdot 4^{\alpha+3}n + 30 \cdot 4^{\alpha+2}) \equiv 0 \pmod 4, \tag{55}$$
$$PD_{2,3}(54n+18) \equiv 4 \cdot PD_{2,3}(18n+6) \pmod 8, \tag{56}$$
$$PD_{2,3}(54n+36) \equiv 2 \cdot PD_{2,3}(18n+12) \pmod 8, \tag{57}$$
$$PD_{2,3}(36n+30) \equiv 2 \cdot PD_{2,3}(72n+60) \pmod 8. \tag{58}$$

Proof. Extracting the terms involving q^{3n+1} from (48), dividing by q and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(54n+18)q^n \equiv -4\frac{f_1^5 f_3 f_6}{f_2^3} \pmod 8. \tag{59}$$

Extracting the terms involving q^{3n+1} from (43), dividing by q and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv -\frac{f_1^5 f_3 f_6}{f_2^3} \pmod 8. \tag{60}$$

From (59) and (60), we arrive at (56).

Extracting the terms involving q^{3n+2} from (48), dividing by q^2 and then replacing q^3 by q, we find

$$\sum_{n=0}^{\infty} PD_{2,3}(54n+36)q^n \equiv 4\frac{f_1^6 f_6^4}{f_2^4 f_3^2} \pmod 8. \tag{61}$$

Extracting the terms involving q^{3n+2} from (43), dividing by q^2 and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv -2\frac{f_1^6 f_6^4}{f_2^4 f_3^2} \pmod 8. \tag{62}$$

In view of the congruences (61) and (62), we get (57).

From (60), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7\frac{f_1^5 f_3 f_6}{f_2^3} \pmod 8. \tag{63}$$

Invoking (41) into (63), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7\frac{f_2f_3f_6}{f_1^3} \pmod{8}. \tag{64}$$

Employing (28) into (64), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7\frac{f_4^6f_6^4}{f_2^8f_{12}^2} + 21q\frac{f_4^2f_6^2f_{12}^2}{f_2^6} \pmod{8}. \tag{65}$$

Extracting the terms involving q^{2n} from (65) and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+6)q^n \equiv 7\frac{f_2^6f_3^4}{f_1^8f_6^2} \pmod{8}. \tag{66}$$

Invoking (40) into (66), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+6)q^n \equiv 3f_2^2 \pmod{4}. \tag{67}$$

Extracting the terms involving q^{2n+1} from (67), we get (52).

Extracting the terms involving q^{2n+1} from (65), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5\frac{f_2^2f_3^2f_6^2}{f_1^6} \pmod{8}. \tag{68}$$

Invoking (41) into (68), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5\frac{f_6^2}{f_2^2}(f_1f_3)^2 \pmod{8}. \tag{69}$$

Employing (30) into (69), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5\frac{f_8^4f_{12}^8}{f_4^4f_{24}^4} + 5q^2\frac{f_4^8f_6^4f_{24}^4}{f_2^4f_8^4f_{12}^4} - 10q\frac{f_4^2f_6^2f_{12}^2}{f_2^2} \pmod{8}. \tag{70}$$

Extracting the terms involving q^{2n+1} from (70), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+60)q^n \equiv 6\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{8}. \tag{71}$$

Invoking (41) into equation (62), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6\frac{f_3^8}{f_1^2 f_3^2} \pmod{8}. \tag{72}$$

Invoking (40) into (72), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 2\frac{f_6^3}{f_2} \pmod{4}. \tag{73}$$

Congruence (53) fellows extracting the terms involving q^{2n+1} from (73).
Which implies that

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+12)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{4}. \tag{74}$$

Substituting (26) into (74), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+12)q^n \equiv 2\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + 2q\frac{f_{12}^3}{f_4} \pmod{4}. \tag{75}$$

Which implies,

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+48)q^n \equiv 2\frac{f_6^3}{f_2} \pmod{4}. \tag{76}$$

Congruence (54) fellows extracting the terms involving q^{2n+1} from (76).
From equation (76) and (73), we have

$$PD_{2,3}(72n+48) \equiv PD_{2,3}(18n+12) \pmod{4}. \tag{77}$$

By mathematical induction on α, we arrive at

$$PD_{2,3}(18\cdot 4^{\alpha+1} + 3\cdot 4^{\alpha+2}) \equiv PD_{2,3}(18n+12) \pmod{4}. \tag{78}$$

Using (54) into (78), we get (55).
Equation (72) can rewritten as

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6\left(\frac{f_3^3}{f_1}\right)^2 \pmod{8}. \tag{79}$$

Employing (26) into (79), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6\frac{f_4^6 f_6^4}{f_2^4 f_{12}^2} + 6q^2\frac{f_{12}^6}{f_4^2} + 12q\frac{f_4^2 f_6^2 f_{12}^2}{f_2^2} \pmod{8}. \tag{80}$$

Extracting the terms involving q^{2n+1} from (80), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+30)q^n \equiv 12\frac{f_2^2 f_3^2 f_6^2}{f_1^2} \pmod{8}. \tag{81}$$

From (71) and (81), we get (58). $\boxed{QED}$

Theorem 3. *For each $n \geq 0$ and $\alpha \geq 0$, we have*

$$PD_{2,3}(72\cdot 25^{\alpha+1}n + 6\cdot 25^{\alpha+1}) \equiv PD_{2,3}(72n+6) \pmod{4}, \tag{82}$$

$$PD_{2,3}(360(5n+i)+150) \equiv 0 \pmod{4}, \tag{83}$$

where $i = 1, 2, 3, 4$.

Proof. From the equation (67), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+6)q^n \equiv 3f_1^2 \pmod{4}. \tag{84}$$

Employing (12) in the above equation, and then extracting the terms containing q^{5n+2}, dividing by q^2 and replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(360n+150)q^n \equiv 3f_5^2 \pmod{4}, \tag{85}$$

which yields

$$\sum_{n=0}^{\infty} PD_{2,3}(1800n+150)q^n \equiv 3f_1^2 \equiv \sum_{n=0}^{\infty} PD_{2,3}(72n+6)q^n \pmod{4}. \tag{86}$$

By induction on α, we obtain (82). The congruence (83) follows by extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from both sides of (85). $\boxed{QED}$

Theorem 4. *For each $n \geq 0$ and $\alpha \geq 0$, we have*

$$PD_{2,3}(24n+20) \equiv 0 \pmod{16}, \tag{87}$$

$$PD_{2,3}(6\cdot 4^{\alpha+2}n + 5\cdot 4^{\alpha+2}) \equiv 0 \pmod{16}. \tag{88}$$

Proof. Extracting the terms involving q^{3n+1} from (38), dividing by q and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n = 2\frac{f_2^4 f_3^2}{f_1^6}. \tag{89}$$

Invoking (41) into equation (89), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n = 2(f_1 f_3)^2 \pmod{16}. \tag{90}$$

Substituting (30) into (90), we arrive

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n \equiv 2\frac{f_2^2 f_8^4 f_{12}^8}{f_4^4 f_6^2 f_{24}^4} + 2q^2\frac{f_4^8 f_6^2 f_{24}^4}{f_2^2 f_8^4 f_{12}^4} - 4q f_4^2 f_{12}^2 \pmod{16}. \tag{91}$$

Extracting the terms involving q^{2n+1} from (91), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+8)q^n \equiv 12 f_2^2 f_6^2 \pmod{16}. \tag{92}$$

Extracting the terms involving q^{2n+1} from (92), we get (87).

Extracting the terms involving q^{2n} from (92) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+8)q^n \equiv 12(f_1 f_3)^2 \pmod{16}. \tag{93}$$

In view of the congruences (90) and (93), we get

$$PD_{2,3}(24n+8) \equiv 6 \cdot PD_{2,3}(6n+2) \pmod{16}. \tag{94}$$

Utilizing (94) and by mathematical induction on α, we arrive

$$PD_{2,3}(6 \cdot 4^{\alpha+1} + 2 \cdot 4^{\alpha+1}) \equiv 6^{\alpha+1} \cdot PD_{2,3}(6n+2) \pmod{16}. \tag{95}$$

Using (87) into (95), we arrive (88). QED

Theorem 5. *For each $n \geq 0$ and $\alpha \geq 0$, we have*

$$PD_{2,3}(6 \cdot 4^{\alpha+1} n + 4^{\alpha+2}) \equiv PD_{2,3}(6n+4) \pmod{32}. \tag{96}$$

Proof. Extracting the terms involving q^{3n+2} from (38), dividing by q^2 and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n = 4\frac{f_2^3 f_6^3}{f_1^5 f_3}. \tag{97}$$

Invoking (41) into (97), we arrive

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n \equiv 4\frac{f_1^3 f_6^3}{f_2 f_3} \pmod{32}. \tag{98}$$

Employing (27) into (98), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n \equiv 4\frac{f_4^3 f_6^3}{f_2 f_{12}} - 12q\frac{f_2 f_6 f_{12}^3}{f_4} \pmod{32}. \tag{99}$$

Extracting the terms involving q^{2n} from (99) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+4)q^n \equiv 4\frac{f_2^3 f_3^3}{f_1 f_6} \pmod{32}. \tag{100}$$

Employing (26) into (100), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+4)q^n \equiv 4\frac{f_2 f_4^3 f_6}{f_{12}} + 4q\frac{f_2^3 f_{12}^3}{f_4 f_6} \pmod{32}. \tag{101}$$

Extracting the terms involving q^{2n+1} from (101), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+16)q^n \equiv 4\frac{f_1^3 f_6^3}{f_2 f_3} \pmod{32}. \tag{102}$$

In view of the congruences (98) and (102), we obtain

$$PD_{2,3}(24n+16) \equiv PD_{2,3}(6n+4) \pmod{32}. \tag{103}$$

Utilizing (103) and by mathematical induction on α, we get (96). QED

Theorem 6. *For $n \geq 0$, we have*

$$PD_{2,3}(48n+34) \equiv 0 \pmod{8}, \tag{104}$$
$$PD_{2,3}(48n+46) \equiv 0 \pmod{8}, \tag{105}$$
$$PD_{2,3}(96n+52) \equiv 0 \pmod{8}, \tag{106}$$
$$PD_{2,3}(96n+76) \equiv 0 \pmod{8}. \tag{107}$$

Proof. Extracting the terms involving q^{2n+1} from (99), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+10)q^n \equiv 20\frac{f_1 f_3 f_6^3}{f_2} \pmod{32}. \tag{108}$$

Substituting (30) into (108), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+10)q^n \equiv 20\frac{f_6^2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - 20q\frac{f_4^4 f_6^4 f_{24}^2}{f_2^2 f_8^2 f_{12}^2} \pmod{32}. \tag{109}$$

Extracting the terms involving q^{2n} from (109) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 20\frac{f_3^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{32}. \tag{110}$$

Invoking (40) into (110), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 4f_2^2 f_3^2 \pmod{16}. \tag{111}$$

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{2k}^m \equiv f_k^{2m} \pmod{2} \tag{112}$$

Invoking (112) into (111), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 4f_4 f_6 \pmod{8}. \tag{113}$$

Congruences (104) follows that extracting the terms involving q^{2n+1} from (113).

Extracting the terms involving q^{2n+1} from (109), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 12\frac{f_2^4 f_3^4 f_{12}^2}{f_1^2 f_4^2 f_6^2} \pmod{32}. \tag{114}$$

Invoking (40) into (114), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 12\frac{f_3^4 f_6^2}{f_1^2} \pmod{16}. \tag{115}$$

Invoking (112) into (115), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 4\frac{f_6^2 f_{12}}{f_2} \pmod 8. \tag{116}$$

Extracting the terms involving q^{2n+1} from (116), we get (105).

Extracting the terms involving q^{2n} from (101) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+4)q^n \equiv 4\frac{f_1 f_2^3 f_3}{f_6} \pmod{32}. \tag{117}$$

Substituting (30) into (117), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+4)q^n \equiv 4\frac{f_2^4 f_8^2 f_{12}^4}{f_4^2 f_6^2 f_{24}^2} - 4q\frac{f_2^2 f_4^4 f_{24}^2}{f_8^2 f_{12}^2} \pmod{32}. \tag{118}$$

Extracting the terms involving q^{2n} from (118) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+4)q^n \equiv 4\frac{f_1^4 f_4^2 f_6^4}{f_2^2 f_3^2 f_{12}^2} \pmod{32}. \tag{119}$$

Invoking (40) into (119), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+4)q^n \equiv 4\frac{f_4^2}{f_3^2} \pmod{16}. \tag{120}$$

Invoking (112) into (120), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+4)q^n \equiv 4\frac{f_8}{f_6} \pmod 8. \tag{121}$$

Congruences (106) obtained by extracting the term involving q^{2n+1} from (121).

Extracting the terms involving q^{2n+1} from (118), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+28)q^n \equiv 28\frac{f_1^2 f_2^4 f_{12}^2}{f_4^2 f_6^2} \pmod{32}. \tag{122}$$

Invoking (40) into (122), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+28)q^n \equiv 12 f_1^2 f_6^2 \pmod{16}. \tag{123}$$

Invoking (112) into (123), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+28)q^n \equiv 4f_2f_{12} \pmod 8. \tag{124}$$

Extracting the terms involving q^{2n+1} from (124), we get (107). QED

Theorem 7. *For any prime $p \equiv 5$, $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} PD_{2,3}(48p^{2\alpha}n + 10p^{2\alpha})q^n \equiv 4f_2f_3 \pmod 8. \tag{125}$$

Proof. Extracting the terms involving q^{2n} from (113) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+10)q^n \equiv 4f_2f_3 \pmod 8. \tag{126}$$

Define

$$\sum_{n=0}^{\infty} f(n)q^n = f_2f_3 \pmod 8. \tag{127}$$

Combining (126) and (127), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+10)q^n \equiv 4\sum_{n=0}^{\infty} f(n)q^n \pmod 8. \tag{128}$$

Now, we consider the congruence equation

$$2\cdot\frac{3k^2+k}{2} + 3\cdot\frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod p, \tag{129}$$

which is equivalent to

$$(2\cdot(6k+1))^2 + 6\cdot(6m+1)^2 \equiv 0 \pmod p,$$

where $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$ and p is a prime such that $(\frac{-6}{p}) = -1$. Since $(\frac{-6}{p}) = -1$ for $p \equiv 5 \pmod 6$, the congruence relation (129) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute (32) into (127) and then extracting the terms in which the powers of q are congruent to $5\cdot\frac{p^2-1}{24}$ modulo p and then divide by $q^{5\cdot\frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} f\left(pn + 5\cdot\frac{p^2-1}{24}\right)q^{pn} = f_{2p}f_{3p},$$

which implies

$$\sum_{n=0}^{\infty} f\left(p^2 n + 5\cdot\frac{p^2-1}{24}\right) q^n = f_2 f_3 \tag{130}$$

and for $n \geq 0$,

$$f\left(p^2 n + pi + 5\cdot\frac{p^2-1}{24}\right) = 0, \tag{131}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha} n + 5\cdot\frac{p^{2\alpha}-1}{24}\right) = f(n). \tag{132}$$

Replacing n by $p^{2\alpha}n + 5\cdot\frac{p^{2\alpha}-1}{24}$ in (128), we arrive at (125). QED

Corollary 1. *For each $n \geq 0$ and $\alpha \geq 0$, we have*

$$PD_{2,3}(3\cdot 4^{\alpha+3}n + 34\cdot 4^{\alpha+1}) \equiv 0 \pmod 8, \tag{133}$$
$$PD_{2,3}(3\cdot 4^{\alpha+3}n + 46\cdot 4^{\alpha+1}) \equiv 0 \pmod 8, \tag{134}$$
$$PD_{2,3}(6\cdot 4^{\alpha+3}n + 13\cdot 4^{\alpha+2}) \equiv 0 \pmod 8, \tag{135}$$
$$PD_{2,3}(6\cdot 4^{\alpha+3}n + 19\cdot 4^{\alpha+2}) \equiv 0 \pmod 8. \tag{136}$$

Proof. Corollary (1) follows from the Theorem (5) and Theorem (6). QED

Theorem 8. *For $n \geq 0$, we have*

$$PD_{2,3}(12n+11) \equiv 0 \pmod 4, \tag{137}$$
$$PD_{2,3}(24n+19) \equiv 0 \pmod 4, \tag{138}$$
$$PD_{2,3}(24n+17) \equiv 0 \pmod 4, \tag{139}$$
$$PD_{2,3}(108n+63) \equiv 0 \pmod 4, \tag{140}$$
$$PD_{2,3}(108n+99) \equiv 0 \pmod 4, \tag{141}$$
$$PD_{2,3}(216n+27)q^n \equiv 2\psi(q) \pmod 4, \tag{142}$$
$$PD_{2,3}(72n+6) \equiv PD_{2,3}(36n+3) \pmod 4, \tag{143}$$
$$PD_{2,3}(96n+28) \equiv 2\cdot PD_{2,3}(24n+7) \pmod 4. \tag{144}$$

Proof. Extracting the odd terms in (36), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n = \frac{f_2^3 f_3^3 f_{18}^2}{f_1^3 f_6^3 f_9^2}. \tag{145}$$

Invoking (40) into (145), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_1 f_2 f_3^3 f_9^2}{f_6^3} \pmod{4}. \tag{146}$$

Substituting (31) into (146), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_3^2 f_9^6}{f_6^2 f_{18}^2} - q\frac{f_3^3 f_9^3 f_{18}}{f_6^3} - 2q^2\frac{f_3^4 f_{18}^4}{f_6^4} \pmod{4}. \tag{147}$$

Extracting the terms involving q^{3n} from (147) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_1^2 f_3^6}{f_2^2 f_6^2} \pmod{4}. \tag{148}$$

Invoking (40) into (148), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_3^2}{f_1^2} \pmod{4}. \tag{149}$$

Employing (24) into (149), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \pmod{4}. \tag{150}$$

Extracting the terms involving q^{2n+1} from (150), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+7)q^n \equiv 2\frac{f_2 f_3^2 f_4 f_{12}}{f_1^4 f_6} \pmod{4}. \tag{151}$$

Invoking (112) into (151), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+7)q^n \equiv 2f_2 f_{12} \pmod{4}. \tag{152}$$

Extracting the terms involving q^{2n+1} from (152), we obtain (137).

Extracting the terms involving q^{2n} from (152), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+7)q^n \equiv 2f_1 f_6 \pmod{4}. \tag{153}$$

Extracting the terms involving q^{2n} from (124) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(96n+28)q^n \equiv 4f_1f_6 \pmod 8. \tag{154}$$

In view of congruences (154) and (153), we obtain (144).

Extracting the terms involving q^{3n+1} from (147), dividing by q and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+3)q^n \equiv 3\frac{f_1^3f_3^3f_6}{f_2^3} \pmod 4. \tag{155}$$

Invoking (40) into (155), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+3)q^n \equiv 3\frac{f_3^3f_6}{f_1f_2} \pmod 4. \tag{156}$$

Employing (26) into (156), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+3)q^n \equiv 3\frac{f_4^3f_6^3}{f_2^3f_{12}} + 3q\frac{f_6f_{12}^3}{f_2f_4} \pmod 4. \tag{157}$$

Extracting the terms involving q^{2n} from (157) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+3)q^n \equiv 3\frac{f_2^3f_3^3}{f_1^3f_6} \pmod 4. \tag{158}$$

Invoking (40) into (158), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+3)q^n \equiv 3\frac{f_1f_2f_3^3}{f_6} \pmod 4. \tag{159}$$

Substituting (31) into (159), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+3)q^n \equiv 3\frac{f_3^2f_9^4}{f_{18}^2} - 3q\frac{f_3^3f_9f_{18}}{f_6} - 6q^2\frac{f_3^4f_{18}^4}{f_6^2f_9^2} \pmod 4. \tag{160}$$

Extracting the terms involving q^{3n} from (160) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+3)q^n \equiv 3\frac{f_1^2f_3^4}{f_6^2} \pmod 4. \tag{161}$$

Invoking (40) into (161), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+3)q^n \equiv 3f_1^2 \pmod{4}. \tag{162}$$

Extracting the terms involving q^{2n} from (67) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+6)q^n \equiv 3f_1^2 \pmod{4}. \tag{163}$$

In view of congruences (163) and (162), we obtain (143).

Extracting the terms involving q^{3n+2} from (160), dividing by q^2 and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2\frac{f_1^4 f_6^4}{f_2^2 f_3^2} \pmod{4}. \tag{164}$$

Invoking (40) into (164), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2\frac{f_6^4}{f_3^2} \pmod{4}. \tag{165}$$

Congruences (140) and (141) follows extracting the terms involving q^{3n+1} and q^{3n+2} from (165).

Invoking (112) into (165), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2\frac{f_{12}^2}{f_6} \pmod{4}. \tag{166}$$

Extracting the terms involving q^{6n} from (166) and replacing q^6 by q, we get (142).

Extracting the terms involving q^{3n+2} from (147), dividing by q^2 and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+5)q^n \equiv 2\frac{f_1^4 f_6^4}{f_2^4} \pmod{4}. \tag{167}$$

Invoking (40) into (167), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+5)q^n \equiv 2\frac{f_6^4}{f_2^2} \pmod{4}. \tag{168}$$

Congruences (137) follows that extracting the terms involving q^{2n+1} from (168).

Extracting the terms involving q^{2n} from (168) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+5)q^n \equiv 2\frac{f_3^4}{f_1^2} \pmod 4. \tag{169}$$

Substitute (26) and (23) in (169)

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+5)q^n$$
$$\equiv 2\frac{f_4^4 f_6^3 f_{16} f_{24}^2}{f_2^4 f_8 f_{12}^2 f_{48}} + 2q\frac{f_4^3 f_6^3 f_8^2 f_{48}}{f_2^4 f_{12} f_{16} f_{24}} + 2q\frac{f_6 f_{12}^2 f_{16} f_{24}^2}{f_2^2 f_8 f_{48}} + 2q^2\frac{f_6 f_8^2 f_{12}^3 f_{48}}{f_2^2 f_{16} f_{24}} \pmod 4. \tag{170}$$

Extracting the terms involving q^{2n+1} from (170), dividing by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+17)q^n \equiv 2\frac{f_2^3 f_3^3 f_4^2 f_{24}}{f_1^4 f_6 f_8 f_{12}} + 2\frac{f_3 f_6^2 f_8 f_{12}^2}{f_1^2 f_4 f_{24}} \pmod 4. \tag{171}$$

Invoking (112) into (171), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+17)q^n \equiv 2f_2 f_3 f_{12} + 2f_2 f_3 f_{12} \pmod 4, \tag{172}$$

which implies (139). QED

Theorem 9. *For $n \geq 0$, $\alpha \geq 0$*

$$PD_{2,3}(648n+459) \equiv 0 \pmod 4, \tag{173}$$
$$PD_{2,3}(8 \cdot 9^{\alpha+3}n + 51 \cdot 9^{\alpha+2}) \equiv 0 \pmod 4. \tag{174}$$

Proof. Employing (18) into (142), we get

$$PD_{2,3}(216n+27)q^n \equiv 2f(q^3, q^6) + 2q\psi(q^9) \pmod 4. \tag{175}$$

Congruences (173) follows extracting the terms involving q^{3n+2} from (175).

Extracting the terms involving q^{3n+1} from (175), dividing by q and then replacing q^3 by q, we have

$$PD_{2,3}(648n+243)q^n \equiv 2\psi(q^3) \pmod 4. \tag{176}$$

Extracting the terms involving q^{3n} from (176) and replacing q^3 by q, we obtain

$$PD_{2,3}(1944n+243)q^n \equiv 2\psi(q) \pmod 4. \tag{177}$$

In view of congruences (142) and (177), we have

$$PD_{2,3}(1944n+243) \equiv PD_{2,3}(216n+27) \pmod 4. \tag{178}$$

Utilizing (178) and by mathematical induction on α, we get

$$PD_{2,3}(24\cdot 9^{\alpha+2}n+3\cdot 9^{\alpha+2}) \equiv PD_{2,3}(216n+27) \pmod 4. \tag{179}$$

Using (173) into (179), we obtain (174). QED

4 Congruences Modulo 3.

Theorem 10. *For $n \geq 0$ and $\alpha \geq 0$, then*

$$PD_{2,3}(6n+3) \equiv 0 \pmod 3, \tag{180}$$
$$PD_{2,3}(6n+5) \equiv 0 \pmod 3, \tag{181}$$
$$PD_{2,3}(36n+30) \equiv 0 \pmod 3, \tag{182}$$
$$PD_{2,3}(4\cdot 3^{\alpha+3}n+10\cdot 3^{\alpha+2}) \equiv 0 \pmod 3. \tag{183}$$

Proof. Substituting (15) into (35), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_6^2 f_9 f_{18}^2}{f_3^3 f_{12} f_{36}} + q\frac{f_6^4 f_9^4 f_{36}^2}{f_3^4 f_{12}^2 f_{18}^4} + 2q^2\frac{f_6^3 f_9 f_{36}^2}{f_3^3 f_{12}^2 f_{18}}. \tag{184}$$

Extracting the terms involving q^{3n} from (184) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n = \frac{f_2^2 f_3 f_6^2}{f_1^3 f_4 f_{12}}. \tag{185}$$

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{3k}^m \equiv f_k^{3m} \pmod 3. \tag{186}$$

Invoking (186) into (185), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n \equiv \frac{f_2^8}{f_4^4} \pmod 3. \tag{187}$$

Extracting the terms involving q^{2n} from (187) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^8}{f_2^4} \pmod 3. \tag{188}$$

But

$$\frac{f_1^8}{f_2^4} = \frac{f_1^2 f_3^2}{f_2^4}. \tag{189}$$

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^2 f_3^2}{f_2^4} \pmod 3. \tag{190}$$

Substituting (30) into (190), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_8^4 f_{12}^8}{f_2^2 f_4^4 f_6^2 f_{24}^4} + q^2 \frac{f_4^8 f_6^2 f_{24}^4}{f_2^6 f_8^4 f_{12}^4} - 2q \frac{f_4^2 f_{12}^2}{f_2^4} \pmod 3. \tag{191}$$

Extracting the terms involving q^{2n+1} from (191), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \frac{f_2^2 f_6^2}{f_1^4} \pmod 3. \tag{192}$$

Invoking (186) into (192), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \frac{f_2^2 f_6^2}{f_1 f_3} \pmod 3, \tag{193}$$

which implies

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \psi(q)\psi(q^3) \pmod 3. \tag{194}$$

Employing (18) into (194), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \psi(q^3)f(q^3,q^6) + q\psi(q^3)\psi(q^9) \pmod 3. \tag{195}$$

Congruences (182) follows by extracting the terms involving q^{3n+2} from (195).

Extracting the terms involving q^{3n+1} from (195), dividing by q and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+18)q^n \equiv \psi(q)\psi(q^3) \pmod 3. \tag{196}$$

In view of congruences (194) and (196), we obtain

$$PD_{2,3}(36n+18)q^n \equiv PD_{2,3}(12n+6) \pmod 3. \tag{197}$$

Utilizing (197) and by mathematical induction on α, we get

$$PD_{2,3}(4\cdot 3^{\alpha+2}n + 2\cdot 3^{\alpha+2}) \equiv PD_{2,3}(12n+6) \pmod 3. \tag{198}$$

Using (182) into (198), we get (183).

Invoking (186) into (145), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_6^4}{f_3^4} \pmod 3. \tag{199}$$

Congruences (180) and (181) follows by extracting the terms involving q^{3n+1} and q^{3n+2} from (199). QED

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