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Geometric characterization of the rotation centers of a particle in a flow

Blas Herrera[i]
Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili
blas.herrera@urv.cat

Jordi Pallares[ii]
Departament d'Enginyeria Mecànica, Universitat Rovira i Virgili
jordi.pallares@urv.cat

Agustí Reventós
Departament de Matemàtiques, Universitat Autònoma de Barcelona
agusti@mat.uab.cat

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Abstract. We provide a geometrical characterization of the instantaneous rotation centers $\overrightarrow{O}(p,t)$ of a particle in a flow $\mathcal{F}$ over time t. Specifically, we will prove that: a) at a specific instant t, the point $\overrightarrow{O}(p,t)$ is the center of curvature at the vertex of the parabola which best fits the path-particle line $\gamma(t)$ on its Darboux plane at p, and b) over time t, the geometrical locus of $\overrightarrow{O}(p,t)$ is the line of striction of the principal normal surface generated by $\gamma(t)$.

Keywords: Geometry of flows, structure of flows

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1 Introduction

It is well known that the path-particle lines of flows are important objects in Fluid Mechanics, since these lines make up vortices and also determine the movements of any object immersed in the fluid. In this paper we present novel geometric results that help to understand the geometry of the path-particle lines and the geometry of the instantaneous rotation centers of the particles. These results –deriving from a theory which has been applied to the study of vortices in [10] and [15]– relate the fluid with the curvatures of certain curves and surfaces which are intrinsically linked to the geometry of the path-particle lines. Many issues related with the fluid/fluid and fluid/solid interfaces are intimately associated with the curvature of the path-particle lines on such interfaces (see

[i]Corresponding Author

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for example [1], [4], [6, 7, 8, 9], [12, 13], [16, 17, 18, 19]). Therefore, the results present in this paper, concerning vortical flow structures, could become useful.

The literature on Fluid Mechanics describes several techniques to identify vortical flow structures. Some of these techniques use a local analysis of the flow and they are based on the velocity gradient tensor –interested readers can find details in the reference section of [10]–. Some other techniques use non-local methods which are based on quantities averaged in a certain flow region or over a certain period of time that is linked to the vortical motion of the fluid particles.

A non-local method for two-dimensional flows was proposed in [14] and [5]. This method is based on the calculation of the normalized angular momentum function:

$$f_V(x_p) = \frac{1}{V}\int_{x\in V} \frac{(x-x_p)\times\overrightarrow{v}(x)}{\|x-x_p\|\,\|\overrightarrow{v}(x)\|}dV, \tag{1}$$

where V is a volume around the point x_p, $\overrightarrow{v}(x)$ is the velocity vector at point x, and $\times$ is the cross product. The module of $f_V(x_p)$, $|f_V(x_p)|$, ranges between 0 and 1. In two-dimensional cases, if V tends to be a very small volume, then $|f_V(x_p)|$ tends to a characteristic function that equals zero everywhere except in the vortex center, where its value is 1.

This method is simple and robust and it allows the identification of vortical structure cores in two-dimensional flows (see examples in [14]). But extrapolating this normalized angular momentum method to three-dimensional flows is not straightforward. The function $|f_V(x_p)|$ is the integral of $\sin(\theta_x)$, where $\theta_x = \measuredangle(x-x_p, \overrightarrow{v}(x))$ is the angle between the velocity vector $\overrightarrow{v}(x)$ and the radius vector $x-x_p$. In planar cases, the velocity vector of points x that lie on a vortex core tends to be orthogonal to the vortex center direction; i.e., if the point x_p is near to the vortex center, then $\sin(\theta_{x_p}) \approx 1$. In two-dimensional flows there is only one direction which is orthogonal to the velocity vector, but in non-planar and non-axisymmetric cases there are infinite directions which are orthogonal to the velocity vector. Therefore, in three-dimensional cases the condition $\sin(\theta_x) = 1$ is of little use to find vortical structures.

Another non-local method to detect vortex cores was proposed in [3]. Let us consider any pair of particles (a, b) in a fluid, where the word particle means the position of a point which satisfies the equation of motion $\frac{d\overrightarrow{x}}{dt} = \overrightarrow{v}(x,t)$. Being $\overrightarrow{v}_a$, $\overrightarrow{v}_b$ the velocities of the pair of particles; the authors in [3] introduced the following ratio:

$$R(x,t) = \frac{\left|\int_0^t \overrightarrow{v}_a(\tau)d\tau - \int_0^t \overrightarrow{v}_b(\tau)d\tau\right|}{\int_0^t |\overrightarrow{v}_a(\tau) - \overrightarrow{v}_b(\tau)|\,d\tau}. \tag{2}$$

More details and a discussion about the advantages and disadvantages of this method can be found in [3] and [10].

In the following subsection we will define the purpose of our study and we will also specify a notation.

1.1 Another non-local method to identify vortical structures

In reference [10] a new non-local method to identify vortex cores in large-scale vortical structures in three-dimensional flows is presented. Rather than an alternative, this method should be considered as a complement to the existing local and non-local techniques. Next, we will summarize the main features of this identification method. A complete analysis and examples can be found in [10].

The underlying idea of this method is based on the answer to the following question: Given a particle and a specific instant, around what and how is it rotating? In order to answer this question, we presented vector field $\overrightarrow{B}(p,t) = \left(\overrightarrow{\Omega}(p,t), \overrightarrow{O}(p,t)\right)$ –the vector field of the instantaneous rotation of a particle around a center–, where $\overrightarrow{O}(p,t)$ is the instantaneous rotation center of the particle of the flow $\mathcal{F}$ at the point p and in the instant t; and $\overrightarrow{\Omega}(p,t)$ is a generalization of the angular velocity vector for this particle. As a summary:

Let $\mathcal{F}$ be a flow in $\mathbb{A}^3$ (oriented Euclidean affine space of dimension three), then we consider the trio $(\overrightarrow{v}, \overrightarrow{\omega}, D)$ formed by $\overrightarrow{v}$: the smooth velocity vector field of $\mathcal{F}$ at a given time t; $\overrightarrow{\omega}$: its vorticity field $\overrightarrow{\omega}$ =curl($\overrightarrow{v}$); and D: its 2-covariant rate-of-strain tensor field. The vectors and the tensors at point p are noted as $\overrightarrow{v}(p,t)$, $\overrightarrow{\omega}(p,t)$ and $D_{(p,t)}$. But the vorticity vector field $\overrightarrow{\omega}$ is not related to a particle's rotation around a center. It is more precise to say that the vorticity vector field is related to the rotation of a particle around itself. In fact, it is well known that $\frac{1}{2}\overrightarrow{\omega}(p,t) \cdot \overrightarrow{m}$ at any point p is the mean value of the angular velocity of two orthogonal line segments which pass through that point and also are orthogonal to $\overrightarrow{m}$, where $\overrightarrow{m}$ is any unit vector and $\cdot$ is the scalar product in $\mathbb{A}^3$.

Next, we construct a vector field $\overrightarrow{B}(p,t) = \left(\overrightarrow{\Omega}(p,t), \overrightarrow{O}(p,t)\right)$ which is intrinsically linked to the rotation of a particle around a center. In order to define and construct this vector field $\overrightarrow{B}$, we apply differential geometry concepts to the path-particle line $\gamma(t)$.

Let $\{\gamma(t); \overrightarrow{t}(t), \overrightarrow{n}(t), \overrightarrow{b}(t)\}$ be the Frenet-Serret frame of γ (see for example [2] or [11]), where $\overrightarrow{t}(t)$ is the unit tangent vector to γ at $p = \gamma(t)$; $\overrightarrow{n}(t)$ is the unit normal vector which points towards the center of curvature of γ at p (center of the osculatrix circumference); and $\overrightarrow{b}(t)$ is the binormal vector defined by $\overrightarrow{b}(t) = \overrightarrow{t}(t) \times \overrightarrow{n}(t)$.

Let $q = \|\overrightarrow{v}(\gamma(t), t)\|$ be the velocity module.

Let $\gamma(s)$ be the path-particle line $\gamma(t)$ but parameterized by the arc length s, with $\gamma(0) = p$. We know that $\frac{ds}{dt} = q$. The Darboux vector field $\overrightarrow{D}(s)$ which is defined along $\gamma(s)$ can be written as

$$\overrightarrow{D}(s) = \tau(s)\overrightarrow{t}(s) + \kappa(s)\overrightarrow{b}(s), \tag{3}$$

where $\tau(s)$ and $\kappa(s)$ are the torsion and the curvature of $\gamma(s)$, respectively (see, for example [2] or other books on Differential Geometry of curves and surfaces). The vector field $\overrightarrow{D}(s)$ is such that: for any arbitrary $\overrightarrow{w} = w_1\overrightarrow{t}(s) + w_2\overrightarrow{n}(s) + w_3\overrightarrow{b}(s)$, then $\frac{d}{ds}\overrightarrow{w} = \overrightarrow{D}(s) \times \overrightarrow{w}$.

Vector $\overrightarrow{D}(s)$ is the rotation vector of the Frenet-Serret frame (because the above formula is the generalization of Equation $\overrightarrow{v}(\gamma(t), t) = \overrightarrow{\phi}(t) \times \overrightarrow{r}(t)$, where $\overrightarrow{\phi}(t)$ is the angular velocity vector and $\overrightarrow{r}(t)$ is the position vector of the particle's circular motion) and it is written with the arc length parameter s.

Using time derivatives, we find that the kinematic rotation of the Frenet-Serret frame is $\frac{d}{dt}\overrightarrow{w} = q\overrightarrow{D}(s) \times \overrightarrow{w}$.

Therefore, the angular velocity vector field along γ is $\overrightarrow{\Omega}(p,t) = q\overrightarrow{D}(s)$.

In [10] we find $\overrightarrow{O}(p,t)$ (which is the center of rotation of a particle in a flow $\mathcal{F}$, at the point p, at the instant t) and we show that

$$\overrightarrow{O}(p,t) = \gamma(t) + \frac{\kappa(t)}{\tau^2(t) + \kappa^2(t)}\overrightarrow{n}(t). \tag{4}$$

Therefore, the vector field $\overrightarrow{B}(p,t)$, in contrast with vector field $\overrightarrow{\omega}(p,t)$, is intrinsically linked to the instantaneous rotation of a particle around its center.

1.1.1 Large-scale vortical structures

In [10] a method was provided to detect vortex cores in large-scale vortical structures in three-dimensional flows. This method takes advantage of the fact that the Frenet-Serret frame's normal vector at the points x located in a vortex core is pointing to the vortex center; i.e., when the points x are in the vortex core, then $\frac{\overrightarrow{xp}\cdot\overrightarrow{n}(x,t)}{\|\overrightarrow{xp}\|} = 1$ (where $\overrightarrow{xp} = p - x$) if the point p is close to the vortex center. ($\overrightarrow{n}(x,t)$ is the normal vector of the path-particle line's Frenet-Serret frame at point x at the instant t.)

The scalar field $B_V(p,t)$ is defined by

$$B_V(p,t) = \frac{1}{V}\int_{x\in V}\frac{\overrightarrow{xp}\cdot\overrightarrow{n}(x,t)}{\|\overrightarrow{xp}\|}dV, \tag{5}$$

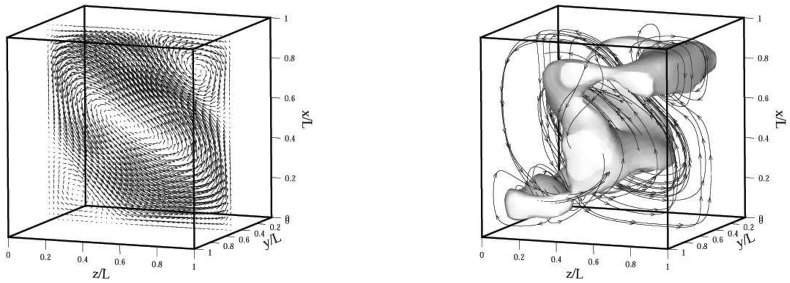

Figure 1. Laminar natural convection flow in a cubical cavity at Rayleigh number $3 \cdot 10^4$ and Prandlt number 0.7. Left: Vector field in the plane y/L=0.5. Right: Three-dimensional view of the isosurface $B/Bmax = 0.75$ and some particle paths.

where V is a volume surrounding point p.

Function $|B_V|$ is a scalar function and it is bounded between 0 and 1. It provides a way to quantify the topology of the flow around p. For two-dimensional flows, the vortical structure found by B_V is the same as the structure found by f in Equation (1).

A complete description of the properties of the method, the formulae to calculate the Frenet-Serret frame for flows $\mathcal{F}$ and the implementation of the method when the convection velocity is ambiguous or unknown, together with some examples of application, can be found in [10]. In Figure 1 of this paper we show the application of the method to a numerically simulated laminar natural convection flow in a cubical cavity of size L which is heated from below, at Rayleigh number $3 \cdot 10^4$ and Prandlt number 0.7 (see details of this flow in [10] and [20]).

After this reference to [10], in the present paper we will show an in-depth analysis concerning the geometry of the instantaneous rotation centers of a particle. We will provide a geometric characterization of the instantaneous rotation centers of the particle over time. Precisely, we will see that: *a) at a specific instant t, the point $\overrightarrow{O}(p,t)$ is the center of curvature at the vertex of the parabola which best fits the path-particle line $\gamma(t)$ on its Darboux plane at p; and b) over time t, the geometrical locus of $\overrightarrow{O}(p,t)$ is the line of striction of the principal normal surface generated by $\gamma(t)$.*

2 Geometric characterization

2.1 Parabola which best fits the path-particle line

In order to investigate the shape of the path-particle line $\gamma(t)$ in an infinitesimal surrounding of any of its points $p = \gamma(t_0)$, we expand the vector function $\gamma(s)$ according to Taylor's formula in a surrounding of $s = 0$, where s is the arc length parameter of γ and $p = \gamma(s = 0)$.

We consider a Cartesian system of coordinates (x_1, x_2, x_3) with an orthonormal reference system $\mathcal{C} = \{\theta; \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ in $\mathbb{A}^3$, where the origin is $\theta = p$ and the Frenet-Serret frame $\overrightarrow{t}(0)$, $\overrightarrow{n}(0)$, $\overrightarrow{b}(0)$ is $\overrightarrow{e}_1 = \overrightarrow{t}(0)$, $\overrightarrow{e}_2 = \overrightarrow{n}(0)$, $\overrightarrow{e}_3 = \overrightarrow{b}(0)$.

From the Taylor series expansion $\gamma(s) = \gamma(0) + \frac{d\gamma(s)}{ds}\mid_{s=0} s + \frac{d^2\gamma(s)}{ds^2}\mid_{s=0} s^2 + o(s^3)$ and the classic Frenet-Serret formulas $\frac{d\gamma(s)}{ds} = \overrightarrow{t}(s)$, $\frac{dt(s)}{ds} = k(s)\overrightarrow{n}(s)$, $\frac{d\overrightarrow{n}(s)}{ds} = -k(s)\overrightarrow{t}(s) + \tau(s)\overrightarrow{b}(s)$, we obtain the following expression of $\gamma(s)$ in respect of the affine orthonormal reference $\mathcal{C}$:

$$x_1(s) = s - \frac{k^2}{6}s^3 + o(s^3),\ x_2(s) = \frac{k}{2}s^2 + \frac{\dot{k}}{6}s^3 + o(s^3),\ x_3(s) = \frac{k\tau}{6}s^3 + o(s^3), \quad (6)$$

where $k = k(0)$, $\tau = \tau(0)$ and $\dot{k} = \frac{dk(s)}{ds}\mid_{s=0}$. This expression is very well known and it can be found in [2] or in other books on Differential Geometry of curves and surfaces.

We consider the rotation vector of the Frenet-Serret frame, which is the Darboux vector in (3). Dividing it by its module, we obtain the unit Darboux vector field along the path-particle line γ,

$$\frac{\overrightarrow{D}(s)}{|D|(s)} = \frac{\tau(s)\overrightarrow{t}(s) + k(s)\overrightarrow{b}(s)}{\sqrt{k^2(s) + \tau^2(s)}}. \quad (7)$$

Let $\mathbb{D}$ be the plane which is orthogonal to $\overrightarrow{D} = \frac{\overrightarrow{D}(0)}{|D|(0)} = \frac{\tau\overrightarrow{e}_1 + k\overrightarrow{e}_3}{\sqrt{k^2+\tau^2}}$ and passes through the origin p. This plane is called Darboux plane. The orthogonal projection of the curve $\gamma(s)$ on $\mathbb{D}$ is the curve $\gamma^\perp(s)$ defined by

$$\gamma^\perp(s) = \gamma(s) - \left(\gamma(s) \cdot \overrightarrow{D}\right)\overrightarrow{D}. \quad (8)$$

We find that

$$\gamma(s)\cdot\overrightarrow{D} = \left(\left(s - \frac{k^2}{6}s^3 + o(s^3)\right)\overrightarrow{e}_1 + \left(\frac{k}{2}s^2 + \frac{\dot{k}}{6}s^3 + o(s^3)\right)\overrightarrow{e}_2 \right. \tag{9}$$
$$\left. + \left(\frac{k\tau}{6}s^3 + o(s^3)\right)\overrightarrow{e}_3\right)\cdot\frac{\tau\overrightarrow{e}_1 + k\overrightarrow{e}_3}{\sqrt{k^2+\tau^2}} = \frac{s\tau}{\sqrt{k^2+\tau^2}} + o(s^3).$$

Therefore, using equations (6) to (9), we find that

$$\gamma^{\perp}(s) = \left(s + o(s^3)\right)\overrightarrow{e}_1 + \left(\frac{k}{2}s^2 + o(s^3)\right)\overrightarrow{e}_2 + o(s^3)\overrightarrow{e}_3 \tag{10}$$
$$- \left(\overrightarrow{\gamma}(s)\cdot\overrightarrow{D}\right)\frac{\tau\overrightarrow{e}_1 + k\overrightarrow{e}_3}{\sqrt{k^2+\tau^2}} = \left(\frac{1}{k^2+\tau^2}\left(sk^2 + o(s^3)\right)\overrightarrow{e}_1\right.$$
$$\left. + \left(\frac{k}{2}s^2 + o(s^3)\right)\overrightarrow{e}_2 + \left(-sk\tau + o(s^3)\right)\overrightarrow{e}_3\right);$$

and we can write $\gamma^{\perp}(s) = \left(x_1^{\perp}(s), x_2^{\perp}(s), x_3^{\perp}(s)\right)$, where

$$x_1^{\perp}(s) = \frac{sk^2}{k^2+\tau^2} + o(s^3),\ x_2^{\perp}(s) = \frac{k}{2}s^2 + o(s^3),\ x_3^{\perp}(s) = -\frac{sk\tau}{k^2+\tau^2} + o(s^3). \tag{11}$$

Now we express $\gamma^{\perp}(s)$ with respect to the orthonormal reference system $\mathcal{D} = \left\{p; \overrightarrow{n}, \overrightarrow{D}^{\perp}\right\}$ in $\mathbb{D}$, where $\overrightarrow{n} = \overrightarrow{e}_2 = \overrightarrow{n}(0)$ and

$$\overrightarrow{D}^{\perp} = \frac{1}{\sqrt{k^2+\tau^2}}(-k\overrightarrow{e}_1 + \tau\overrightarrow{e}_3). \tag{12}$$

Therefore $\gamma^{\perp}(s) = x(s)\overrightarrow{n} + y(s)\overrightarrow{D}^{\perp}$, where

$$x(s) = \frac{k}{2}s^2 + \{\text{terms of degree} \geq 3\},$$
$$y(s) = -\frac{k}{\sqrt{k^2+\tau^2}}s + \{\text{terms of degree} \geq 3\},$$

and (x, y) are the coordinates in $\mathbb{D}$ with respect to the Cartesian system $\mathcal{D}$.

Therefore, modulo terms of degree three in s and using the orthonormal reference system $\mathcal{D} = \left\{p; \overrightarrow{n}, \overrightarrow{D}^{\perp}\right\}$ in $\mathbb{D}$, the geometric locus of $\gamma^{\perp}(s)$ is the parabola of the following equation:

$$x = \frac{k^2+\tau^2}{2k}y^2. \tag{13}$$

It is well known that the coordinates of the focus of the parabola defined by the equation $x = by^2$ are $(\frac{1}{4b}, 0)$, and the parabola's directrix equation is $x = -\frac{1}{4b}$, and the parabola's curvature at its vertex is $\kappa = 2b$. Therefore, the curvature of $\gamma^{\perp}(s)$ in $s = 0$ is $\kappa = \frac{k^2+\tau^2}{k}$, and the curvature center O of $\gamma^{\perp}(s)$ in $s = 0$ is the point

$$O = \gamma^{\perp}(0) + \frac{k}{k^2+\tau^2}\overrightarrow{n} = \gamma(0) + \frac{k}{k^2+\tau^2}\overrightarrow{n}\,(0)\,. \tag{14}$$

That is, the instantaneous rotation center of $\overrightarrow{\gamma}(0) = p$.

We have proved that:

Theorem 2.2. Let $\mathcal{F}$ be a flow in $\mathbb{A}^3$, let p be a point of a path-particle line $\gamma\,(t)$ in a specific instant t. Let $\gamma^{\perp}(t)$ be the orthogonal projection of $\gamma\,(t)$ onto its Darboux plane at p. Then, the second-order approximation of $\gamma^{\perp}(t)$ is a parabola $\mathcal{P}$ with vertex p and whose curvature center at p is the point $\overrightarrow{O}\,(p,t)$ (the instantaneous rotation center of p).

2.3 The principal normal surface of the path-particle line

Let $\gamma\,(t)$ be a path-particle line of a flow $\mathcal{F}$ in $\mathbb{A}^3$, and let $\gamma\,(s)$ be the same line parameterized by the arc-length parameter s. *The ruled surface N parameterized by*

$$\overrightarrow{x}(s,r) = \gamma(s) + r\,\overrightarrow{n}(s), \tag{15}$$

where $\overrightarrow{n}(s)$ is the Frenet-Serret normal vector of the path-particle line γ, is called the principal normal surface generated by γ. The straight lines $\overrightarrow{x}(c,r)$ with $c = constant$ are the generators of the ruled surface.

Next, we calculate the first and second fundamental forms of $\mathcal{N}$, and its Gauss curvature. These calculations can be found for instance in [2].

We have

$$\begin{aligned} \frac{\partial \overrightarrow{y}}{\partial s} &= \overrightarrow{t}\,(s) + r(-k(s)\,\overrightarrow{t}\,(s) + \tau(s)\,\overrightarrow{b}\,(s)) = (1 - rk(s))\,\overrightarrow{t}\,(s) + r\tau(s)\,\overrightarrow{b}\,(s), \\ \frac{\partial \overrightarrow{y}}{\partial r} &= \overrightarrow{n}(s). \end{aligned} \tag{16}$$

Thus, the coefficients of the first fundamental form I of the surface $\mathcal{N}$ are

$$\begin{aligned} E(s,r) &= \frac{\partial \overrightarrow{y}}{\partial s}\cdot\frac{\partial \overrightarrow{y}}{\partial s} = (1 - rk(s))^2 + r^2\tau^2(s), \\ F(s,r) &= \frac{\partial \overrightarrow{y}}{\partial s}\cdot\frac{\partial \overrightarrow{y}}{\partial r} = 0,\; G(s,r) = \frac{\partial \overrightarrow{y}}{\partial r}\cdot\frac{\partial \overrightarrow{y}}{\partial r} = 1. \end{aligned} \tag{17}$$

The unit normal vector $\overrightarrow{N}$ of the surface $\mathcal{N}$ is

$$\overrightarrow{N}(s,t) = \frac{\frac{\partial \overrightarrow{y}}{\partial s} \times \frac{\partial \overrightarrow{y}}{\partial t}}{\left| \frac{\partial \overrightarrow{y}}{\partial s} \times \frac{\partial \overrightarrow{y}}{\partial t} \right|} = \frac{1}{\sqrt{E(s,r)}} \Big((1 - rk(s)) \overrightarrow{b}(s) - r\tau(s) \overrightarrow{t}(s) \Big). \quad (18)$$

Since $\frac{\partial^2 \overrightarrow{y}}{\partial r^2}(s,r) = 0$, the coefficient $g(s,r) = \frac{\partial^2 \overrightarrow{y}}{\partial r^2}(s,r) \cdot \overrightarrow{N}(s,r)$ of the second fundamental form II is zero, and its determinant is equal to $\det II = e(s,r)g(s,r) - f^2(s,r) = -f^2(s,r)$.

Given that

$$f(s,r) = \frac{\partial^2 \overrightarrow{y}}{\partial r \partial s}(s,r) \cdot \overrightarrow{N}(s,r) = (-k(s)\overrightarrow{t}(s) + \tau(s)\overrightarrow{b}(s)) \cdot \overrightarrow{N}(s,r) = \frac{\tau(s)}{\sqrt{E(s,r)}}, \quad (19)$$

the Gauss curvature of the principal normal surface $\mathcal{N}$ of γ is

$$K(s,r) = \frac{-f^2(s,r)}{E(s,r)} = \frac{-\tau^2(s)}{(1 - rk(s))^2 + r^2\tau^2(s)}. \quad (20)$$

Using this expression *we can prove* the following:

Theorem 2.4. Considering a generator (at the point p) of the principal normal surface $\mathcal{N}$ of γ, the Gauss curvature along this generator reaches its maximum absolute value at the instantaneous rotation center of p.

Proof. If we consider $s = s_0$ and $p = \gamma(s_0)$ and we calculate the Gauss curvature along the corresponding generator, we find that

$$|K(r)| = \frac{a^2}{(1 - rb)^2 + r^2 a}, \quad (21)$$

where a, b are the constants $a^2 = \tau^2(s_0)$, $b = k(s_0)$. If $a \neq 0$ then $a^2 + b^2 > 0$.

If $a \neq 0$, function (21) reaches the maximum when

$$r = \frac{b}{a^2 + b^2} = \frac{k(s_0)}{k^2(s_0) + \tau^2(s_0)}. \quad (22)$$

That is, it reaches the maximum at the instantaneous rotation center of $p = \gamma(s_0)$.

In the special case $a = 0$ and $b = 0$ ($a^2 + b^2 = 0$), we find that $|K(r)| = 0 \forall r$ along the generator. Therefore, the maximum curvature is obtained at point $\overrightarrow{x}(s_0, 0) = \gamma(s_0) = p$ which coincides with its own instantaneous rotation center.□

It is well known that, for any ruled surface S, the geometric locus of the points where the absolute value of the Gauss curvature is maximum along each generator is the striction line $\mathcal{L}$ of that ruled surface; see for example [2].

The striction line $\mathcal{L}$ of a ruled surface $S \equiv \overrightarrow{y}(t,u) = \alpha(t) + u\overrightarrow{w}(t)$ (where $|\overrightarrow{w}(t)| = 1$) is the line $\mathcal{L} \equiv \beta(t) = \alpha(t) + v(t)\overrightarrow{w}(t)$ such that $\frac{d}{dt}\beta(t) \cdot \frac{d}{dt}\overrightarrow{w}(t) = 0$. The point $o_r = \beta \cap r$ (the intersection of β with a generator $r = \overrightarrow{y}(constant, u)$) is called the central point of the generator r because the points of r which are symmetrical with respect to o_r have the same Gauss curvature.

Therefore, *we can establish* the following corollary:

Corollary 2.5. Let $\mathcal{F}$ be a flow in $\mathbb{A}^3$, let $\gamma(t)$ be a path-particle line of $\mathcal{F}$. Let $\mathcal{N}$ be the principal normal surface generated by γ. Then $\mathcal{L}$ (the striction line of $\mathcal{N}$) is the geometrical locus of the points $\overrightarrow{O}(\gamma(t), t)$, which are the instantaneous rotation centers of the points $\gamma(t)$ over time.

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