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Groups with identities

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Abstract. This is a survey of a still evolving subject. The purpose is to develop a theory of prounipotent (respectively pro-p) groups satisfying a prounipotent (respectively pro-p) identity that is parallel to the theory of PI-algebras. [1]

Keywords: the Burnside problem, pro-p groups, PI-algebras, Lie algebras

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1 An overview of relevant parts of PI-theory

Historically, the theory of PI-algebras has focused on (i) Burnside-type problems, such as the Kurosh Problem, (ii) the Specht Problem, and (iii) Structure theory. There are other important PI subjects, such as growth of codimensions, geometry of prime ideals, etc., but we don't yet know how to relate them to Group Theory.

1.1 Burnside-type problems

In 1902 W. Burnside [6] formulated his famous problems that became known as the General Burnside Problem and the Burnside Problem.

The General Burnside Problem: Let G be a finitely generated torsion group, i.e. for an arbitrary element $a \in G$ there exists an integer $n(a) \geq 1$ such that $a^{n(a)} = 1$. Does this imply that G is finite?

The Burnside Problem: Let G be a finitely generated group that is torsion of bounded degree, i.e. there exists $n \geq 1$ such that $a^n = 1$ for an arbitrary element $a \in G$. Does this imply that G is finite?

R. Bruck put it in more general terms: what makes a group finite?

In 1941 A. G. Kurosh formulated a Burnside-type problem for algebras [24]. Let A be an associative algebra over a field K. An element $a \in A$ is said to be nilpotent if $a^{n(a)} = 0$ for some $n(a) \geq 1$. An algebra A is nil if every element of A is nilpotent.

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The Kurosh Problem: Is it true that a finitely generated nil algebra is nilpotent?

In 1964 E. S. Golod [13] constructed counterexamples to the Kurosh Problem and used them to construct counterexamples to the General Burnside Problem as well. Since then, new rich and important classes of finitely generated torsion groups were found among self-similar groups (R. I. Grigorchuk [14], N. Gupta-S. Sidki [16], V. Sushchansky [42], et al.).

P. S. Novikov and S. I. Adjan [30, 31, 32], S. Ivanov [18], and I. Lysenok [28] constructed counterexamples to the Burnside Problem for all sufficiently large n.

In contrast, since Golod's work [13] in 1964, only one new class of finitely generated nonnilpotent nil algebras has been constructed. This was done in 2007 by T. Lenagan and A. Smoktunowicz [26]. Both Golod and Lenagan-Smoktunowicz algebras are defined by generators and relations.

The Kurosh Problem, however, has a positive solution in the class of algebras satisfying a polynomial identity (PI-algebras).

Let $f(x_1, x_2, \cdots, x_m)$ be a nonzero element of the free associative K-algebra. We say that an algebra A satisfies the polynomial identity $f = 0$ if $f(a_1, \cdots, a_m) = 0$ for arbitrary elements $a_1, \cdots, a_m \in A$.

One of the high points of the theory of PI-algebras was a solution of the Kurosh Problem (I. Kaplansky [20], J. Levitzki [25], A. I. Shirshov [39]) in the following form:

Let A be an associative PI-algebra generated by elements $a_1, \cdots, a_m$. Let S be the multiplicative semigroup generated by the elements $a_1, \cdots, a_m$. Suppose that an arbitrary element of S is nilpotent. Then the algebra A is nilpotent.

1.2 Burnside-type problems for Lie algebras

Let L be a Lie algebra over a filed K. As above, for a nonzero element $f(x_1, \cdots, x_m)$ of the free Lie algebra we say that L satisfies the identity $f = 0$ if $f(a_1, \cdots, a_m) = 0$ for arbitrary elements $a_1, \cdots, a_m \in A$, See [5].

An element $a \in L$ is said to be ad-nilpotent if the linear operator

$$\mathrm{ad}(a) : L \to L, \ x \to [a, x]$$

is nilpotent.

A subset $S \subset L$ is called a Lie set if, for arbitrary elements $a, b \in S$, we have $[a, b] \in S$. For a subset $X \subset L$, the Lie set generated by X is the smallest Lie set $S\langle X\rangle$ containing X. It consists of X and of all iterated commutators in elements from X.

In [52] (see also [50, 51], where the result was announced) we proved an analog of the Kaplansky-Levitzki-Shirshov theorem above.

Theorem 1.2.1. *Let L be a Lie algebra satisfying a polynomial identity and generated by elements $a_1, \cdots, a_m$. If an arbitrary element $s \in S\langle a_1, \cdots, a_m\rangle$ is ad-nilpotent then the Lie algebra L is nilpotent.*

1.3 The Specht Problem

Let A be an associative or Lie K-algebra. Let $K\langle X\rangle$ be the corresponding (associative of Lie) free algebra on countably many free generators. Consider

$$T(A) = \{f \in K\langle X\rangle \mid f = 0 \text{ is an identity of the algebra } A\}.$$

Clearly, $T(A)$ is an ideal of the algebra $K\langle X\rangle$. Moreover, $T(A)$ is closed with respect to substitutions: if $f(x_1, \cdots, x_m) \in T(A)$ and $g_1(x), \cdots, g_m(x) \in K\langle X\rangle$ then $f(g_1(x), \cdots, g_m(x)) \in T(A)$. We call such ideals T-ideals.

The following problem has been a leading light in PI-theory for many years.

The Specht Problem ([41]). Let K be a field of characteristic 0 and let A be an associative PI-algebra over K. Is it true that $T(A)$ is finitely generated as a T-ideal?

In other words, is it true that $T(A)$ contains a finite collection of elements $f_1, \cdots, f_s$ such that every element of $T(A)$ can be obtained from $f_1, \cdots, f_s$ via usual ideal operations and substitutions?

This problem was solved in the affirmative in 1987 by A. Kemer [23]. For further developments extending to positive characteristic, see [21].

Now we will introduce an important construction: generic matrices. Let R be a commutative ring. Let $R[[X]]$ be the algebra of infinite series in commuting variables $X = \{x_{ij}^{(k)} \mid 1 \leq i, j \leq n,\ 1 \leq k \leq m\}$ with coefficients from R. Consider the so called generic matrices

$$x_k = (x_{ij}^{(k)})_{1 \leq i,j \leq n} \in M_n(R[[X]]),$$

$1 \leq k \leq m$. The R-algebra $g_m(n, R)$ generated by $x_1, \cdots, x_m$ is referred to as the algebra of generic $n \times n$ matrices over R. It is easy to see that $g_m(n, R)$ is a free algebra in the variety generated by the algebra $M_n(R)$ of $n \times n$ matrices over R.

S. A. Amitsur and C. Procesi [4] showed that if R is a domain then the R-algebra $g_m(n, R)$ is a domain as well. S. A. Amitsur later used $g_m(n, R)$ to construct the first example of a finite dimensional division algebra that is not a crossed product [3].

Every finitely generated PI-algebra over an infinite field K satisfies all identities of $M_n(K)$ (for a sufficiently large n) and therefore is a homomorphic image of $g_m(n,K)$ (A. R. Kemer, [22]).

An R-submodule of $g_m(n,R)$ is called a T-subspace (over R) if it is invariant with respect to all substitutions $x_i \to a_i \in g_m(n,R),\ 1 \le i \le m$. In the case of $R = \mathbb{Z}$ it is more natural to talk about T-additive subgroups.

A. V. Grishin [15] proved that for a field K of zero characteristic, every T-subspace of $g_m(n,K)$ is finitely generated as a T-subspace, i.e. finitely generated with respect to substitutions.

Speaking of Lie algebras, A. V. Il'tyakov [17] proved that identities of a finite dimensional Lie algebra L over a field of zero characteristic are finitely based, i.e. the T-ideal $T(L)$ is finitely generated as a T-ideal.

M. Vaughan-Lee [45] and V. Drenski [9] showed that, for finite dimensional Lie algebras over a field of positive characteristic, this result is no longer true.

1.4 Structure Theory.

We call an algebra linear (over a commutative algebra R) if it is embeddable in some matrix algebra $M_n(R)$. S. A. Amitsur and J. Levitzki [2] proved that the algebra $M_n(R)$ satisfies the identity $\sum\limits_{\sigma \in S_{2n}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)} = 0$. This implies that every linear algebra is PI.

At least at the level of finitely generated algebras, the main objects of PI-theory are linear algebras over commutative algebras. And the main reason for existence the of PI-theory is the fact that homomorphic images of linear algebras may not be linear (L. Small, [40]). Still, these homomorphic images are PI.

Structure theorems assert that under certain conditions PI-algebras are linear and, in fact, reduce to some basic examples. The following theorem is due to L. H. Rowen [38] (based on the work of E. Formanek [12] and Y. P. Razmyslov [37] on the existence of central polynomials):

Theorem 1.4.1. *Let A be a prime associative algebra satisfying a polynomial identity. Then the center $Z = Z(A)$ is nonzero and the algebra of fractions $(Z \setminus \{0\})^{-1}A$ is a simple finite dimensional algebra over the field $(Z \setminus \{0\})^{-1}Z$.*

2 Groups with prounipotent and pro-p identities.

2.1 Free prounipotent and pro-p groups.

Let K be a field of zero characteristic. Consider a group $UT(n,K)$ of unitriangular $n \times n$ matrices over K. For an arbitrary scalar $\alpha \in K$ and an arbitrary element $g \in UT(n,K)$ it makes sense to talk about the power g^α. Indeed, if

$g = 1 + a,\ a^n = 0$, then $g^\alpha = 1 + \sum_{i=1}^{n-1} \binom{\alpha}{i} a^i$. In other words, $UT(n, K)$ is a K-group in the sense of [27].

By a unipotent group (over K) we mean a K-subgroup of a unitriangular group $UT(n, K)$. A prounipotent group is an inverse limit of unipotent groups. Natural filtrations on groups $UT(n, K)$ induce a topology on inverse limits, so prounipotent groups are topological groups.

Let $K\langle\langle X\rangle\rangle$ be the algebra of infinite series in noncommuting variables $X = \{x_1, \cdots, x_m\}$. Let $\mathrm{id}(X)$ be the ideal that consists of series with zero constant term. Then

$$1 + \mathrm{id}(X)$$

is a prounipotent group, which is equipped with the degree topology. The closed subgroup generated by $y_1 = 1 + x_1, \cdots, y_m = 1 + x_m$ is called the free prounipotent group on free generators $y_1, \cdots, y_m$. We will denote it as $(F_m)_{\hat{u}}$. An arbitrary mapping of free generators in an arbitrary prounipotent group uniquely extends to a continuous homomorphism.

Let $\rho_1, \rho_2, \cdots$ be basic commutators in free generators $y_1, y_1, \cdots, y_m$ (see [29]). An arbitrary element from $(F_m)_{\hat{u}}$ can be uniquely represented as a converging product

$$\rho_1^{\alpha_1} \rho_2^{\alpha_2} \cdots,$$

where $\alpha_1 \alpha_2, \cdots \in K$.

Definition 2.1.1. Let $w(y_1, \cdots, y_m)$ be a nonidentity element of the free prounipotent group. We say that a prounipotent group G satisfies the identity $w = 1$ if $w(g_1, \cdots, g_m) = 1$ for arbitrary elements $g_1, \cdots, g_m \in G$.

Now let p be a prime number. We will review the corresponding definitions for pro-p groups.

A group G is said to be residually-p if there exists a family of homomorphisms $\phi_i : G \to G_i$ into finite p-groups G_i such that $\bigcap_i \mathrm{Ker}(\phi_i) = (1)$. The system of normal subgroups of finite index $\mathrm{Ker}(\phi_i)$ can be viewed as a basis of neighborhoods of 1, thus making G a topological group. If this topology is complete then we say that G is a pro-p group. In this case it can be represented as an inverse limit of finite p-groups.

In any case we can embed G into its completion, which is called the pro-p completion of G and is denoted as $G_{\hat{p}}$.

The free group F_m on m free generators $y_1, \cdots, y_m$ is residually-p for any prime p. The pro-p completion $(F_m)_{\hat{p}}$ is called the free pro-p group.

An arbitrary mapping of free generators $y_1, \cdots, y_m$ into an arbitrary pro-p group uniquely extends to a continuous homomorphism.

Definition 2.1.2. Let $w(y_1, \cdots, y_m)$ be a nonidentity element of the free pro-p group. We say that a pro-p group G satisfies the identity $w = 1$ if $w(g_1, \cdots, g_m) = 1$ for arbitrary elements $g_1, \cdots, g_m \in G$.

2.2 Torsion groups with pro-p identity

Let us discuss torsion residually-p groups whose pro-p completions satisfy a nontrivial identity. We remark that torsion groups of E. S. Golod [13], R. I. Grigorchuk [14], and N. Gupta and S. Sidki [16] are residually-p.

Let $\mathbb{Z}/p\mathbb{Z}$ be the field of order p and let G be a finitely generated group. Consider the group algebra $(\mathbb{Z}/p\mathbb{Z})[G]$ and its fundamental ideal w spanned by all elements $1-g,\ g \in G$. It is easy to see that the group G is residually-p if and only if $\bigcap\limits_{i \geq 1} w^i = (0)$. The Zassenhaus filtration is defined as

$$G = G_1 > G_2 > \cdots,$$

where $G_i = \{g \in G \mid 1-g \in w^i\}$. Then $[G_i, G_j] \subseteq G_{i+j}$ and each factor G_i/G_{i+1} is an elementary abelian p-group. Hence

$$L_p(G) = \bigoplus_{i \geq 1} G_i/G_{i+1}$$

is a Lie algebra of $\mathbb{Z}/p\mathbb{Z}$.

Definition 2.2.1. We say that the group G is infinitesimally PI if the Lie algebra $L_p(G)$ satisfies a polynomial identity.

Let us show that if the pro-p completion $G_{\hat{p}}$ satisfies a nontrivial pro-p identity $w = 1$ then the group G is infinitesimally PI. Indeed, let $F = (F_m)_{\hat{p}}$ be the free pro-p group. Consider the Zassenhaus filtration $F = F_1 > F_2 > \cdots$. Let $w \in F_n \setminus F_{n+1}$. Then $w = \rho_i^{p^{s_1}} \cdots \rho_r^{p^{s_r}} w'$, where each ρ_i is a left normed group commutator of length $l_i,\ p^{s_i} \cdot l_i = n,\ w' \in F_{n+1}$.

Considering, if necessary, $[w, x_0]$ instead of w, we can assume that n is not a multiple of p and $w = \rho_1 \cdots \rho_r w'$, where all commutators are of length n, $\rho_1, \cdots, \rho_r \in F_n \setminus F_{n+1},\ w' \in F_{n+1}$. Let $\bar{\rho}_i$ be the commutator from the free Lie algebra that mimics the group commutator ρ_i. Then the Lie algebra $L_p(G)$ satisfies the nontrivial polynomial identity $\sum\limits_i \bar{\rho}_i = 0$.

The reverse is not true. Consider the so called Nottingham pro-p group

$$M = \{t + \alpha_2 t^2 + \alpha_3 t^3 + \cdots \mid \alpha_i \in \mathbb{Z}/p\mathbb{Z}\}$$

of infinite series in one variable, t, over $\mathbb{Z}/p\mathbb{Z}$. The group N is infinitesimally PI. On the other hand it contains the free pro-p group $(F_2)_{\hat{p}}$ (R. Camina [7], I. Fesenko [10]) and therefore does not satisfy a nontrivial pro-p identity.

Theorem 1.2.1 from Section 1.2 immediately implies:

Theorem 2.2.2. *Let G be a residually-p finitely generated torsion group that is infinitesimally PI. Then G is a finite group.*

Indeed, since the group G is torsion, it follows (see [46, 51]) that the Lie algebra

$$L_p(G) = \bigoplus_{i \geq 1} G_i/G_{i+1}$$

is graded ad-nil, i.e. an arbitrary homogeneous element from G_i/G_{i+1}, $i \geq 1$, is ad-nilpotent.

Consider the subalgebra L of $L_p(G)$ generated by G_1/G_2. Since the Lie algebra satisfies a polynomial identity, Theorem 1.2.1 implies that the algebra L is nilpotent. Hence the pro-p completion $G_{\hat{p}}$ of the group G is p-adic analytic and therefore linear (see [8]). Now finiteness of G follows from theorems of Burnside and Schur [19].

Now we are ready to formulate a solution of the General Burnside Problem in the class of residually-p groups with identity.

Theorem 2.2.3. *Let G be a residually-p finitely generated torsion group such that its pro-p completion $G_{\hat{p}}$ satisfies a nontrivial identity. Then G is a finite group.*

Indeed, this theorem is an immediate corollary of Theorem 2.2.2.

The results above significantly extend the positive solution of the Restricted Burnside Problem [47, 48] and the work in [49] on compact torsion groups.

2.3 Identities of linear groups.

Let us start with a discussion of groups that satisfy an identity $w = 1$, where $1 \neq w \in F_m$ is an element of the (discrete) free group.

There are two fundamental general results on bases of identities: the Oats-Powell theorem that a finite group has finite basis of identities [35], and examples due to S. I. Adjan [1], A. Yu. Ol'shanski [34], and M. Vaughan-Lee [44] of varieties of groups that are not finitely based. See also the influential monograph [33].

The class of groups with identity $w = 1,\ w \in F_m$ includes solvable groups and groups of exponent n (including counterexamples to the Burnside Problem). These are two important classes, but I hesitate to say that there exists a unified theory of groups which satisfy an identity $w = 1,\ w \in F_m$, as in the case of PI-algebras.

Why?

As we have already mentioned in Section 1, the basic examples of PI-algebras are linear algebras over commutative rings. In contrast, linear groups usually don't satisfy any identity $w = 1,\ w \in F_m$.

In 1967 V. P. Platonov [36] proved that a linear group that satisfies an identity $w = 1,\ 1 \neq w \in F_m$, is virtually solvable, i.e. contains a solvable subgroup of finite index. Compare this also with the celebrated Tits Alternative: a finitely generated linear group is either virtually solvable or contains F_2 [43].

The situation changes if we allow infinite prounipotent or pro-p identities. Then linear groups are PI.

Example 2.3.1. Let K be a field of zero characteristic and R be a commutative K-algebra with an ideal $I \lhd R$ such that $\bigcap_i I^i = (0)$. Suppose that the algebra R is complete in the I-topology. If $M_n(I)$ is the algebra of $n \times n$ matrices over I then $GL^1(n,R,I) = 1 + M_n(I)$ is a subgroup of $GL(n,R)$. It is easy to see that $GL^1(n,R,I)$ is a prounipotent group.

Theorem 2.3.2.

(1) The group $GL^1(n,R,I)$ satisfies a prounipotent identity.

(2) All prounipotent identities of the group $GL^1(n,R,I)$ are finitely based, i.e. can be obtained from finitely many identities via usual group operations and substitutions.

Consider the algebra $g_m(n,K)$ generated by $n \times n$ generic matrices $X_k = (x_{ij}^{(k)})_{1\leq i,j\leq n},\ 1 \leq k \leq m$ (see Section 1.2). The matrices $1 + X_k,\ 1 \leq k \leq m$ are invertible in the algebra of $n \times n$ matrices over infinite series. The prounipotent group $G_m(n,K)$ generated by $1 + X_k,\ 1 \leq k \leq m$ is a universal n-linear prounipotent group. More precisely, an arbitrary mapping of free generators $1 + X_k,\ 1 \leq k \leq m$ to a prounipotent group of type $GL^1(n,R,I)$ uniquely extends to a continuous homomorphism.

An arbitrary element $a \in GL^1_m(n,K)$ can be represented as a sum of homogeneous components

$$a = 1 + \sum_{i=1}^{\infty} a_i,$$

where a_i is an $n \times n$ matrix having homogeneous polynomials of degree i in X as its entries. Let $\min(a) = a_s$ if $s = \min\{i \geq 1 \mid a_i \neq 0\}$. Arguing as in [53] we can show that $\min(a)$ is always a Lie element in $X_1, \cdots, X_m$. This is a key step towards the proof of Theorem 2.3.2(1).

We say that a subspace $V \subset g_m(n,K)$ is a Lie T-subspace if V is invariant under all Lie substitutions $x_i \to a_i,\ 1 \leq i \leq m$, where the elements $a_1, \cdots, a_m$

lie in the Lie K-algebra generated by generic matrices $X_1, \cdots, X_m$. The second part of Theorem 2.3.2 follows from the following generalization of the results of Il'tyakov and Grishin.

Theorem 2.3.3. *Let V be a Lie T-subspace of the algebra $g_m(n, K)$. Then all elements of V can be obtained from finitely many identities by ordinary linear operations and Lie substitutions.*

Example 2.3.4. Let R be a commutative ring and let I be an ideal of R such that $R/I \cong GL(p^k)$; $\bigcap_{i\geq 1} I^i = (0)$ and R is complete in the I-adic topology. Then

$$GL^1(n, R, I) = 1 + M_n(I)$$

is a pro-p group.

Theorem 2.3.5. *There exists a function $f : N \to N$ such that if $p > f(n)$ then*

(1) the pro-p group $GL^1(n, R, I)$ satisfies a pro-p identity;

(2) all pro-p identities of $GL^1(n, R, I)$ follow from finitely many identities.

The first part of the theorem for $n = 2,\ p > 2$ was proved by A. Zubkov [53].

Let $\mathbb{Z}_p$ be the ring of p-adic integers. As above, the pro-p group $G_m(n, \mathbb{Z}_p)$ of generic matrices is a universal n-linear pro-p group: an arbitrary mapping of free generators $1 + X_k,\ 1 \leq k \leq m$ into an arbitrary pro-p group $GL^1(n, R, I)$ of the example above uniquely extends to a continuous homomorphism.

Both parts of Theorem 2.3.5 above follow from a Specht-type property of Lie T-additive subgroups of $G_m(n, \mathbb{Z})$.

It is not true that all Lie T-additive subgroups of $g_m(n, \mathbb{Z})$ are finitely based. We will prove however a weaker version of this fact that will be enough to imply the theorem.

Let $p_1, \cdots, p_s$ be prime numbers. Let $Lie_m(n, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_s}])$ be the Lie $\mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_s}]$-algebra generated by generic matrices $X_1, \cdots, X_m$.

Theorem 2.3.6. *Let V be a Lie T-additive subgroup of $g_m(n, \mathbb{Z})$. There exists a finite collection of prime numbers $p_1, \cdots, p_s$ and a finite collection of elements of V such that an arbitrary element of V can be obtained from these elements via Lie substitutions*

$$x_i \to a_i \in \mathrm{Lie}_m(n, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_s}]),\ 1 \leq i \leq m.$$

2.4 Structure Theory

As has already been mentioned in Section 1.4, structural problems usually sound as follows: under assumptions close to simplicity, prove that a group (algebra, etc.) is close to basic examples.

A prounipotent (respectively pro-p) group can not be simple. Instead we call a prounipotent (respectively pro-p) group *just infinite* if every nonidentical closed normal subgroup is of finite codimension (respectively, finite).

Theorem 2.4.1. *Let G be a just infinite prounipotent group that satisfies all identities of $G_m(n, K)$. Then G is linear.*

It is likely that a stronger assertion is valid: G is close to a group of type $GL^1(n, R, I)$.

Problem 2.4.2. *Let G be a just infinite pro-p group ($p \geq 3$) that satisfies all pro-p identities of $GL^1(2, \mathbb{Z}_p)$. Is G linear and is it close to a group of type $GL^1(2, R, I)$?*

This problem may be related to a famous conjecture in Number Theory. Let S be a finite set of primes, $p \notin S$, K the maximal pro-p extension of $\mathbb{Q}$ unramified outside of S, $G = Gal(K/\mathbb{Q})$.

Fontaine-Mazur Conjecture [11]: For an arbitrary homomorphism $\phi : G \to GL^1(2, R, I)$, the image is finite.

The image group $\phi(G)$ satisfies all identities of 2×2 matrices. If infinite, the group $\phi(G)$ has a just infinite homomorphic image, which (modulo Problem 2.4.2 above) is close to a group of type $GL^1(2, R, I)$. Hence ϕ is close to being an epimorphism.

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