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The influence of minimal subgroups on saturated fusion systems

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Abstract.
Let $\mathcal{F}$ be a saturated fusion system over a p-group S. In this paper, we investigate the influence of the minimal subgroups in $\mathfrak{foc}(\mathcal{F})$, the focal subgroup of $\mathcal{F}$. Our main result is that if for each cyclic subgroup $P \leq \mathfrak{foc}(\mathcal{F})$ of order p (of order 2 and 4 if $p = 2$) and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \mathfrak{foc}(\mathcal{F}))$, φ extends to an automorphism of S, then S is normal in $\mathcal{F}$. We also give several applications of this result.

Keywords: saturated fusion system; focal subgroup; resistant group

MSC 2010 classification: 20D15

1 Introduction and statements of results

All groups considered in this paper are finite. Through out this paper, p is always understood to be a fixed prime.

Let G be a group. Recall that minimal subgroups of G is are cyclic subgroups of prime order. It is known that in some cases the way G acting by conjugaction on its cyclic subgroups of order p (of order 2 and 4 if $p = 2$) has a strong influence on the structure of G. For instance, N.Itô prove that if every cyclic group of order p (of order 2 and 4 if $p = 2$) lies in the center of G, then G is p-nilpotent. In fact, many results in this line can be proved by applied the following well known theorem:

Theorem 1.1 ([4, Satz IV.5.12])**.** Let S be a p-group. Suppose that K is a p'-group that acts trivially on every cyclic subgroup of S of order p (of order 2 and 4 if $p = 2$). Then K acts trivially on S.

A saturated fusion system over a p-group S is a category whose objects are subgroups of S, and whose morphisms satisfy certain axioms mimicking the action by conjugation of the elements of some group G with S as its Sylow p-subgroup. The concept of saturated fusion system was introduced Puig in early

1990s, who used the term "Frobenius categories".

Inspired by theorem 1.1, we investigate the connections between the structure of a saturated fusion system $\mathcal{F}$ and the way the morphisms in $\mathcal{F}$ acting on cyclic groups of S of order p (of order 2 and 4 if $p = 2$). The main result of this paper is the following (the notion $\mathfrak{foc}(\mathcal{F})$ appear in the following theorem denotes the focal subgroup of $\mathcal{F}$, see [1, Part I, Defintion 7.1]).

Theorem A. Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Assume that for each cyclic subgroup $P \leq \mathfrak{foc}(\mathcal{F})$ of order p (of order 2 and 4 if $p = 2$) and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \mathfrak{foc}(\mathcal{F}))$, φ extends to an automorphism of S. Then S is normal in $\mathcal{F}$.

In the following, we give several applications of theorem A.

Let S be a p-group. Following [8, Definition 5.1], S is called *resistant* if for any saturate fusion system $\mathcal{F}$ over S, S is normal in $\mathcal{F}$. Burnside's fusion theorem (see [1, Theorem A.8]) states that if S is abelian, then for any finite group G such that $S \in \mathrm{Syl}_p(G)$, S is normal in $\mathcal{F}_S(G)$, here $\mathcal{F}_S(G)$ denotes the fusion category of G over S (see [1, Part I, Definition 1.1]). As the first application of theorem A, we extends Burnside's fusion theorem.

Theorem B. Let S be a p-group and let $\mathcal{F}$ be a saturated fusion system over S. If every cyclic subgroup of $\mathfrak{foc}(\mathcal{F})$ of order p (of order 2 and 4 if $p = 2$) is normal in S, then S is normal in $\mathcal{F}$. In particular, S is resistant if every cyclic subgroup of S of order p (of order 2 and 4 if $p = 2$) is normal in S.

Remark 1.2. It has been proved in [7] that for almost all p-group S, every cyclic subgroup of S of order p is in the center of S. Combining this fact with theorem B, we know that almost all p-groups of odd order are resistant.

Let S be a Sylow p-subgroup of a group G and H be a subgroup of G containing S. Let Q be a subgroup of S. H is said to control fusion of Q in S if for any $g \in G$ such that $Q^g \leq S$, there exists $h \in H$ such that $h = cg$ for some $c \in C_G(Q)$. H is said to controls fusion in S if H control fusion of each subgroup of S in S. Let $\mathcal{F}_S(G)$ be the fusion category of G over S. It is clear that S is normal in $\mathcal{F}_S(G)$ if and only if $N_G(S)$ controls fusions in S. Hence the following corollary is a consequence of theorem A by applying it to $\mathcal{F}_S(G)$.

Corollary 1.3. Let S be a Sylow p-subgroup of a p-constrained group G. Assume that for each cyclic subgroup P of $S \cap G'$ of order p (of order 2 and 4 if $p = 2$), $N_G(S)$ controls fusion of P in S. Then $N_G(S)$ controls fusion in S.

Let S be a Sylow p-subgroup of a finite group G. It is well known that if S controls fusion in S, then G is p-nilpotent, that is, $G = S\,\mathrm{O}_{p'}(G)$. It is natural to ask if an analogue result holds when $N_G(S)$ controls fusion in S, that is, can we obtain that $G = N_G(S)\,\mathrm{O}_{p'}(G)$ provided that $N_G(S)$ controls fusion in S? Unfortunately, this is not true in general (see example 1.5). In the following, we show that when G is p-constrained (G is called p-constrained if $\mathrm{O}_p(G/\mathrm{O}_{p'}(G))$ contains its centralizer in $G/\mathrm{O}_{p'}(G)$), then $N_G(S)$ controlling fusion in S does imply that $G = N_G(S)\,\mathrm{O}_{p'}(G)$.

Proposition 1.4. Let S be a Sylow p-subgroup of a finite group G. Then $G = N_G(S)\,\mathrm{O}_{p'}(G)$ if and only if G is p-constrained and $N_G(S)$ controls fusion in S.

Example 1.5. Let $G = A_5$, the alternating group of degree 5, and let S be a Sylow 5-group of G. Clearly $N_G(S)$ controls fusion in S since S is abelian. But $G \neq N_G(S)\,\mathrm{O}_{p'}(G)$.

Combining corollary 1.3 with proposition 1.4, we have,

Corollary 1.6. Let S be a Sylow p-subgroup of a p-constrained finite group G. Assume that for each cyclic subgroup P of $S \cap G'$ of order p (of order 2 and 4 if $p = 2$), $N_G(S)$ controls fusion of P in S. Then $G = N_G(S)\,\mathrm{O}_{p'}(G)$.

Combining theorem B with proposition 1.4, we have,

Corollary 1.7. Let S be a Sylow p-subgroup of a p-constrained finite group G. If every cyclic subgroup of $S \cap G'$ of order p (of order 2 and 4 if $p = 2$) is normal in S, then $G = N_G(S)\,\mathrm{O}_{p'}(G)$.

Let S be a Sylow p-subgroup of a finite group G. Clearly $G = N_G(S)\,\mathrm{O}_{p'}(G)$ if and only if G is p-solvable with p-length at most 1. Recall that a Hamiltonian group is one all of whose subgroups are normal. Hence the following result is a direct consequence of corollary 1.7

Corollary 1.8 ([3, Corollary 1])**.** If G is p-solvable and a Sylow p-subgroup of G is Hamiltonian, then the p-length of G is at most 1.

Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Recall that $\mathcal{F}$ is nilpotent if $\mathcal{F} = \mathcal{F}_S(S)$, that is, all morphisms of $\mathcal{F}$ are induced by conjugations in S. As an application of theorem A, we prove that $\mathcal{F}$ is nilpotent if for each cyclic subgroup $P \leq \mathfrak{foc}(\mathcal{F})$ of order p or 4, and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \mathfrak{foc}(\mathcal{F}))$, φ is induced by conjugation in S.

Theorem C. Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then $\mathcal{F}$ is nilpotent if and only if for each cyclic subgroup P of $\mathfrak{foc}(\mathcal{F})$ of order p (of order 2 and 4 if $p = 2$) and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \mathfrak{foc}(\mathcal{F}))$, there is $\bar{\varphi} \in \mathrm{Inn}(S)$ such that $\bar{\varphi}|_P = \varphi$.

Suppose that S is a Sylow p-subgroup of a group G. It is clear that G is a p-nilpotent group if and only if $\mathcal{F}_S(G)$ is a nilpotent fusion system. Hence by applying theorem C to $\mathcal{F}_S(G)$, we have

Corollary 1.9 ([5, Main Theorem]). Let S be a Sylow p-subgroup of a finite group G. Then G is p-nilpotent if and only if for each cyclic subgroup P of S of order p (of order 2 and 4 if $p = 2$), S controls fusion of P in S.

Let S be a Sylow p-subgroup of a finite group G. Adolfo and Guo proved that if all cyclic subgroups of $S \cap G'$ of order p (of order 2 and 4 if $p = 2$) are in the center of $N_G(S)$, then G is p-nilpotent ([2, Theorem 1]). As another consequence of theorem C, we extend this result from finite groups to saturated fusion systems.

Corollary 1.10. Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Assume that $\mathrm{Aut}_{\mathcal{F}}(S)$ acts trivially on P for each cyclic subgroup P of $\mathfrak{foc}(\mathcal{F})$ of order p (of order 2 and 4 if $p = 2$). Then $\mathcal{F}$ is nilpotent.

Proof. Let $P \leq \mathfrak{foc}(\mathcal{F})$ be a cyclic group of order p (of order 2 or 4 if $p = 2$) and $\varphi \in \mathrm{Hom}(P, S)$. Since $\mathrm{Inn}(S) \leq \mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(S)$ acts trivially on P, we have $C_S(P) = S$. It follows that φ extends to $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\varphi = \bar{\varphi}|_P = \mathrm{Id}|_P$. Hence $\mathcal{F}$ is nilpotent by theorem C. QED

We will introduce some basic definitions and preliminary results in the second section. The main results of this paper will be proved in the third section.

2 Preliminaries

In this section, we collect some basic concepts and preliminary results that will be needed later.

Definition 2.1. Let $\mathcal{F}$ be a fusion system over a p-group S.

- Two subgroups $P, Q \leq S$ are $\mathcal{F}$-conjugate if they are isomorphic as objects of the category $\mathcal{F}$. Let $P^{\mathcal{F}}$ denote the set of all subgroups of S which are $\mathcal{F}$-conjugate to P.
- A subgroup $P \leq S$ is fully automized in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
- A subgroup $P \leq S$ is fully receptive in $\mathcal{F}$ if it has the following property: for each $Q \leq S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$, if we set

$$N_\varphi = \{g \in N_S(Q) | \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(P)\},$$

  then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|Q = \varphi$ (here c_g denotes the the automorphism of Q induced by g through conjugation).

- A subgroup $P \leq S$ is fully centralized in $\mathcal{F}$ if $|C_S(P)| \geq C_S(Q)$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is fully normalized in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is $\mathcal{F}$-centric if P is fully centralized in $\mathcal{F}$ and $C_S(P) = Z(P)$.
- A subgroup $P \leq S$ is $\mathcal{F}$-radical if $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ is p-reduced, ie, if $\mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. We say P is $\mathcal{F}$-centric-radical if it is $\mathcal{F}$-centric and $\mathcal{F}$-radical.
- A subgroup $P \leq S$ is normal in $\mathcal{F}$ (denoted $P \trianglelefteq \mathcal{F}$) if $P \trianglelefteq S$, and for all $Q, R \leq S$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, φ extends to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\bar{\varphi}(P) = P$.

Lemma 2.2 ([1, Part I, Proposition 2.5])**.** Let $\mathcal{F}$ be a fusion system over a p-group S. Then $\mathcal{F}$ is saturated if and only if the following two conditions hold.

(I) *(Sylow axiom)* Each subgroup $P \leq S$ which is fully normalized in $\mathcal{F}$ is also fully centralized and fully automized in $\mathcal{F}$.

(II) (Extension axiom) Each subgroup $P \leq S$ which is fully centralized in $\mathcal{F}$ is also receptive in $\mathcal{F}$.

Lemma 2.3 ([1, Part I, Lemma 2.6])**.** Let $\mathcal{F}$ be a fusion system over a p-group S. Assume $P \leq S$ is fully automized and receptive. Then for each $Q \in P^{\mathcal{F}}$, there is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ such that $\varphi(Q) = P$.

Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Recall that $\mathcal{F}$ is constrained if there is a normal subgroup $Q \trianglelefteq \mathcal{F}$ which is $\mathcal{F}$-centric. If $\mathcal{F}$ is constrained, a model for $\mathcal{F}$ is a finite group G such that $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G) = \mathcal{F}$ and $C_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$.

Lemma 2.4 ([1, Part II, Lemma 4.4])**.** Suppose that $\mathcal{F}$ is constrained saturated fusion system over a p-group S. If G_1 and G_2 are models for $\mathcal{F}$, then there exists an isomorphism φ: $G_1 \to G_2$ which is the identity on S.

Theorem 2.5 (Alperin's Fusion Theorem)**.** Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(R) : R \in \mathcal{F}^{frc} \rangle$, where $\mathcal{F}^{frc}$ is the set of fully normalized radical centric subgroups of $\mathcal{F}$.

Recall that a group G is p-closed if a Sylow p-subgroup of G is normal in G. The following lemma follows from theorem 2.5

Lemma 2.6. Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Assume that $\mathrm{Aut}_{\mathcal{F}}(P)$ is p-closed for each $P \in \mathcal{F}^{frc}$. Then $S \trianglelefteq \mathcal{F}$.

Proof. By theorem 2.5, it suffice to prove that the only subgroup of S contained in $\mathcal{F}^{frc}$ is S itself. Assume that $P < S$ and $P \in \mathcal{F}^{frc}$. Then $N_S(P)/C_S(P) \cong \mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ by lemma 2.2, and $\mathrm{Inn}(P) \cong P/Z(P) \cong PC_S(P)/C_S(P) = P/C_S(P)$ since P is $\mathcal{F}$-centric. It follows that $\mathrm{Aut}_S(P) > \mathrm{Inn}(P)$ and thus p divides the order of $\mathrm{Out}_{\mathcal{F}}(P)$. Since $\mathrm{Aut}_{\mathcal{F}}(P)$ is p-closed, $\mathrm{Out}_{\mathcal{F}}(P)$ is also p-closed. Hence $\mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(P)) > 1$, which contradicts that P is $\mathcal{F}$-radical. QED

The following lemma is a refinement of theorem 1.1

Lemma 2.7. A p-group P possesses a characteristic subgroup D of exponent p when p is odd, and of exponent at most 4 when $p = 2$, such that every nontrivial p'-automorphism of P induces a nontrivial automorphism of D.

Proof. When p is odd, it follows from [6, Chapter 5, Theorem 3.13] that P possesses a characteristic subgroup D of exponent p such that every nontrivial p'-automorphism of P induces a nontrivial automorphism of D. Assume that $p = 2$. By [6, Chapter 5, Theorem 3.11], P possess a characteristic subgroup C such that the nilpotent class of D is at most 2, $C/Z(C)$ is elementary abelian, and every nontrivial p'-automorphism of P induces a nontrivial automorphism of C. Let $D = \langle x \in C : x^4 = 1\rangle$. Then D char C char P, and hence D is a characteristic subgroup of P. Since every nontrivial p'-automorphism of P induces a nontrivial automorphism of C, and every nontrivial p'-automorphism of C induces a nontrivial automorphism of D by theorem 1.1, we have that every nontrivial p'-automorphism of P induces a nontrivial automorphism of D.

We now show that the exponent of D is at most 4. Let x, y be two elements of D of order at most 4. Clear the nilpotent class of D is at most 2 since the nilpotent class of C is at most 2. It follows that $(xy)^4 = x^4y^4[y, x]^6 = [y^6, x]$. Since $D/(D \cap Z(C)) \cong DZ(C)/Z(C) \leq C/Z(C)$ is an elementary abelian group, $\Phi(D) \leq (D \cap Z(C)) \leq Z(D)$. Hence $y^6 \leq \Phi(D) \leq Z(D)$ and $(xy)^4 = [y^6, x] = 1$. Therefore the order of xy is at most 4. Since D is generated by all elements of C of order at most 4, the exponent of D is at most 4. □

3 Proof of main results

Proof of theorem A. Suppose that $Q \in \mathcal{F}^{frc}$. By lemma 2.6, we need to prove that $\mathrm{Aut}_{\mathcal{F}}(Q)$ is p-closed.

First we introduce some notations. Assume that $\phi \in \mathrm{Hom}(P, Q)$ for some $P, Q \leq S$ and $z \in P$, we use the notion z^{ϕ} to denote $\phi(z)$. Denote $A = \mathrm{Aut}_{\mathcal{F}}(Q)$, $R = [Q, A]$. By lemma 2.7, R possesses a characteristic subgroup D of exponent p when p is odd, and of exponent at most 4 when $p = 2$, such that every nontrivial p'-automorphism of R induces a nontrivial automorphism of D. Let

$1 = Z_0(S) \le Z_1(S) \le \cdots \le Z_n(S) = S$ be the upper central series of S and put $Z_i(D) = D \cap Z_i(S)$, $0 \le i \le n$. Denote $C_i(A) = \{\alpha \in A | [Z_i(D), \alpha] \le Z_{i-1}(D)\}$, $1 \le i \le n$. We will prove that A is p-closed through the following steps:

Step 1. $Z_i(D)$ is an A-invariant subgroup of Q, $0 \le i \le n$.

Assume that $Z_i(D)$ is not an A-invariant subgroup for some i. Then there exits $x \in Z_i(D)$ and $\varphi \in A$ such that $x^\varphi \notin Z_i(D) = D \cap Z_i(S)$. Since R is an A-invariant subgroup of Q and D is a characteristic subgroup of R, D is an A-invariant subgroup and hence $x^\varphi \in D$. It follows that $x \notin Z_i(S)$. Since $D \le R \le \mathfrak{foc}(\mathcal{F})$ and D is of exponent p when p is odd, of exponent at most 4 when $p = 2$, we have $x^\varphi, x \in \mathfrak{foc}(\mathcal{F})$ and the order of x is p or 4. It then follows from hypotheses that there exists $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $x^\varphi = x^{\bar{\varphi}}$. But $Z_i(S)$ is a characteristic subgroup of S and thus $x^\varphi = x^{\bar{\varphi}} \in Z_i(S)$, a contradiction.

Step 2. $C_i(A)$ is a normal subgroup of A, $1 \le i \le n$.

Let $x \in Z_i(D)$, $\alpha, \beta \in C_i(A)$. First we need to show that $C_i(A)$ is a subgroup of A, that is, to show that $[x, \alpha^{-1}] \in Z_{i-1}(D)$ and $[x, \alpha\beta] \in Z_{i-1}(D)$. From definition, we have $[x, \alpha] \in Z_{i-1}(D)$ and $[x, \beta] \in Z_{i-1}(D)$. Since $Z_{i-1}(D)$ is an A-invariant subgroup of Q by step 1, we have $[x, \alpha^{-1}] = ([x, \alpha]^{-1})^{\alpha^{-1}} \in (Z_{i-1}(D))^{\alpha^{-1}} = Z_{i-1}(D)$ and $[x, \alpha\beta] = [x, \beta][x, \alpha]^\alpha \in Z_{i-1}(D)$, as desired.

Now we show that $C_i(A)$ is normal in A. Suppose $\gamma \in A$ and let $z = [x^\gamma, \alpha] = (x^\gamma)^{-1} x^{\gamma\alpha}$. Then $x^{\gamma\alpha} = x^\gamma z$. Since $Z_i(D)$ is A-invariant, we have $z \in Z_{i-1}(D)$ by definition. It follows that $x^{(\gamma\alpha\gamma^{-1})} = (x^\gamma z)^{\gamma^{-1}} = x z^{\gamma^{-1}}$ and $[x, \gamma\alpha\gamma^{-1}] = x^{-1} x^{(\gamma\alpha\gamma^{-1})} = z^{\gamma^{-1}} \in Z_{i-1}(D)$. Hence $\gamma\alpha\gamma^{-1} \in C_i(A)$ and $C_i(A)$ is normal in A.

Step 3. A is p-closed.

Let $C = \bigcap_{i=1}^n C_i(A)$. We first prove that C is a normal p-subgroup of A. From step 2, we know that C is a normal subgroup of A. Assume that α is a p'-element of C. From the definition of C, it is clear that α stabilizes the normal series $1 = Z_0(D) \le Z_1(D) \le \cdots \le Z_n(D) = D$. Hence α acts trivially on D. It then follows from the choice of D that α acts trivially on R. Therefore α stabilizes the normal series $1 \le R \le Q$ and thus $\alpha = 1$. Hence C is a normal p-subgroup of A.

Now we show that C is a Sylow p-subgroup of A. It is clear from definition that $\mathrm{Aut}_S(Q) \le C_i(A)$ for all $1 \le i \le n$ and thus $\mathrm{Aut}_S(Q) \le C$. Since Q is fully normalized in $\mathcal{F}$, $\mathrm{Aut}_S(Q)$ is a Sylow p-subgroup of A by lemma 2.2. Therefore $C = \mathrm{Aut}_S(Q)$ is a normal Sylow p-subgroup of A. QED

Proof of theorem B. Let P, Q be two cyclic subgroups of $\mathfrak{foc}(\mathcal{F})$ of order p (of order 2 or 4 if $p = 2$). Then $P, Q \trianglelefteq S$ by hypotheses. Let $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$. By theorem A, it suffice to show that φ extends to $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$.

First assume that P, Q are of order p. Then $P, Q \trianglelefteq S$ implies that $P, Q \leq Z(S)$. It then follows that $C_S(P) = C_S(Q) = S$ and Q is fully centralized. Therefore by lemma 2.2 we have that φ extends to $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PC_S(P), S) = \mathrm{Aut}_{\mathcal{F}}(S)$.

Now assume that P, Q are of order 4. Since Q is fully normalized, by lemma 2.3 there is a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(Q)) = \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\phi(P) = Q$. Let $\chi = (\phi|_Q)^{-1}\varphi$. Then $\chi \in \mathrm{Aut}_{\mathcal{F}}(P)$. If $\chi = 1$, then $\varphi = \phi|_Q$ and we are done. Assume that $\chi \neq 1$. Then $\mathrm{Aut}_{\mathcal{F}}(P)$ is a non-trivial subgroup of $\mathrm{Aut}(P)$. Since $\mathrm{Aut}(P)$ is a group of order 2. We have that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}(P)$ is a group of order 2. Note that P is fully normalized in $\mathcal{F}$, $\mathrm{Aut}_S(P)$ is a Sylow subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. It follows that $\chi \in \mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_S(P)$ and thus there exists $\bar{\chi} \in \mathrm{Inn}(S)$ such that $\chi = \bar{\chi}|_P$. Let $\bar{\varphi} = \phi\bar{\chi}$. Then $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\bar{\varphi}|_P = \varphi$, as desired. QED

Proof of proposition 1.4. We first prove the necessity. Assume that $G = N_G(S)\,\mathrm{O}_{p'}(G)$. Clearly G is p-constrained. Let P be a subgroup of S and g be an element of G such that $P^g \leq S$. Since $G = N_G(S)\,\mathrm{O}_{p'}(G) = \mathrm{O}_{p'}(G)N_G(S)$, $g = x_1x_2$ for some $x_1 \in \mathrm{O}_{p'}(G)$ and $x_2 \in N_G(S)$. Then $x_2 = x_1^{-1}g$. To show that $N_G(S)$ controls fusion in S, it suffice to show that $x_1^{-1} \in C_G(P)$. Since $P^{x_1x_2} \leq S$ and $x_2 \in N_G(S)$, $P^{x_1} = (P^{x_1x_2})^{x_2^{-1}} \leq S$. Let z be an element of of P. Since $z \in S$ and $z^{x_1} = x_1^{-1}(zx_1z^{-1})z \in S$, $x_1^{-1}(zx_1z^{-1}) = z^{x_1}z^{-1} \in S$. On the other hand, since $x_1 \in \mathrm{O}_{p'}(G)$, $x_1^{-1}(zx_1z^{-1}) \in \mathrm{O}_{p'}(G)$. It follows that $x_1^{-1}(zx_1z^{-1}) = 1$ and $z^{x_1} = z$. Therefore $x_1 \in C_G(P)$ and thus $x_1^{-1} \in C_G(P)$, as desired.

Conversely, assume that G is p-constrained and $N_G(S)$ controls fusion in S. Let $\overline{G} = G/\mathrm{O}_{p'}(G)$, $\overline{S} = S\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$ and $\overline{H} = N_{\overline{G}}(\overline{S})$. We first prove that $N_{\overline{G}}(\overline{S})$ controls fusion in $\overline{S}$. By theorem 2.5, it suffice to show that for any subgroup $\overline{P}$ of $\overline{S}$ and any element $\overline{x}$ of $N_{\overline{G}}(\overline{P})$, there exists $\overline{y} \in N_{\overline{G}}(\overline{S})$ such that $\overline{y} = \overline{xc}$ for some $\overline{c} \in C_{\overline{G}}(\overline{P})$. Clearly there exists some $P \leq S$ and some $x \in N_G(P\,\mathrm{O}_{p'}(G))$ such that $\overline{P} = P\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$ and $\overline{x} = x\,\mathrm{O}_{p'}(G)$. Note that $N_G(P\,\mathrm{O}_{p'}(G)) = N_G(P)\,\mathrm{O}_{p'}(G)$, $x = zl$ for some $z \in N_G(P)$ and $l \in \mathrm{O}_{p'}(G)$. Hence $\overline{x} = \overline{z}$. Now since $z \in N_G(P)$ and $N_G(S)$ controls fusion in S, there exists $y \in N_G(S)$ such that $y = zc$ for some $c \in C_G(P)$. Set $\overline{y} = y\,\mathrm{O}_{p'}(G)$. Then $\overline{y} \in N_{\overline{G}}(\overline{S})$ and $\overline{y} = \overline{zc} = \overline{xc}$, where $\overline{c} = c\,\mathrm{O}_{p'}(G) \in C_{\overline{G}}(\overline{P})$, as desired.

Since G is p-constrained, by [6, Chapter 8, Theorem 1.1] we have $C_{\overline{G}}(\mathrm{O}_p(\overline{G})) \leq \mathrm{O}_p(\overline{G})$ and thus $\overline{G}$ is a model for $\mathcal{F}_{\overline{S}}(\overline{G})$. Since $\overline{S} \trianglelefteq N_{\overline{G}}(\overline{S})$, we have $C_{N_{\overline{G}}(\overline{S})}(\mathrm{O}_p(N_{\overline{G}}(\overline{S})) = C_{N_{\overline{G}}(\overline{S})}(\overline{S}) \leq C_{\overline{G}}(\overline{S}) \leq C_{\overline{G}}(\mathrm{O}_p(\overline{G})) \leq \mathrm{O}_p(\overline{G}) \leq \overline{S} = \mathrm{O}_p(N_{\overline{G}}(\overline{S}))$ and thus $N_{\overline{G}}(\overline{S})$ is a model for $\mathcal{F}_{\overline{S}}(N_{\overline{G}}(\overline{S}))$. On the other hand, Since $N_{\overline{G}}(\overline{S})$ controls p-fusion of $\overline{G}$, $\mathcal{F}_{\overline{S}}(\overline{G}) = \mathcal{F}_{\overline{S}}(N_{\overline{G}}(\overline{S}))$. Clearly $\mathcal{F}_{\overline{S}}(N_{\overline{G}}(\overline{S}))$ is a constrained since $\overline{S} \trianglelefteq \mathcal{F}_{\overline{S}}(N_{\overline{G}}(\overline{S}))$. It then follows from lemma 2.4 that $\overline{G} \cong N_{\overline{G}}(\overline{S})$ and thus

$G/\operatorname{O}_{p'}(G) = \overline{G} = N_{\overline{G}}(\overline{S}) = N_G(S\operatorname{O}_{p'}(G))/\operatorname{O}_{p'}(G)$. Therefore $G = N_G(S\operatorname{O}_{p'}(G)) = N_G(S)\operatorname{O}_{p'}(G)$. QED

Proof of theorem C. The necessity is obviously. We need only to prove the sufficiency. By theorem A, S is normal in $\mathcal{F}$. Hence to prove that $\mathcal{F} = \mathcal{F}_S(S)$, it suffice to show that $\operatorname{Aut}_{\mathcal{F}}(S) = \operatorname{Inn}(S)$, that is, to show that $\operatorname{Aut}_{\mathcal{F}}(S)$ is a p-group. Let φ be a p'-element of $\operatorname{Aut}_{\mathcal{F}}(S)$ and let $\langle x\rangle \leq \mathfrak{foc}(\mathcal{F})$ be a cyclic group of order p or 4. Clearly $x^{\varphi} \leq \mathfrak{foc}(\mathcal{F})$ since now $\mathfrak{foc}(\mathcal{F}) = [S, \operatorname{Aut}_{\mathcal{F}}(S)]$ is $\operatorname{Aut}_{\mathcal{F}}(S)$-invariant. From hypotheses, there exists $\bar{\varphi} \in \operatorname{Inn}(S)$ such that $x^{\bar{\varphi}} = x^{\varphi}$. Hence $[x, \varphi] = [x, \bar{\varphi}] \in [\langle x\rangle, S]$. Suppose that the nilpotent class of S is n and repeat the above procedure for n times,

$$[x, \underbrace{\varphi, \ldots, \varphi}_{n}] \in [\langle x\rangle, \underbrace{S, \ldots, S}_{n}] = 1.$$

It then follows from [6, Chapter 5, Theorem 3.6] that $[x, \varphi] = 1$, and we have proved that φ acts trivially on any cyclic subgroup of $\mathfrak{foc}(\mathcal{F})$ of order p or 4. Hence φ acts trivially on $\mathfrak{foc}(\mathcal{F})$ by theorem 1.1 and therefore stabilizes a normal series $1 \leq \mathfrak{foc}(\mathcal{F}) \leq S$. It then follows from that $\varphi = 1$ and thus $\operatorname{Aut}_{\mathcal{F}}(S)$ is a p-group, as desired. QED

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