Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **36** (2016) suppl. 1, 51–63. doi:10.1285/i15900932v36suppl1p51

The influence of arrangement of subgroups on the group structure

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Received: 21.04.2016; accepted: 05.05.2016.

Abstract. Investigation of groups satisfying certain related to arrangement of subgroups conditions allows algebraists to introduce and describe many important classes of groups. Most of these conditions are based on the fundamental notion of normality and built with the help of this concept different subgroup chains (series). Some of important results obtained on this way we will discuss in the current survey.

Keywords: Arrangement of subgroups, abnormal subgroups, pronormal subgroups, criteria of nilpotency, transitivity of normality.

MSC 2010 classification: 20F19, 20F 22, 20E15, 20E25, 20E34, 20F14, 20F18

Investigation of groups satisfying certain arrangements of subgroups allows algebraists to introduce and describe many important classes of groups. Most of the conditions related to the arrangement of subgroups in a group are based on the fundamental notion of normality and different subgroup chains (series).

Let G be a group. For every of its subgroup H there exists the following naturally defined ascending series

$$\langle 1\rangle = H_0 \leq H_1 \leq \ldots \leq H_i \leq H_{i+1} \leq \ldots H_\alpha \leq H_{\alpha+1} \leq \ldots H_\gamma \leq H_{\gamma+1} = G,$$

$H_1 = H$, $H_2 = N_G(H)$, $H_{\beta+1} = N_G(H_\beta)$, $H_\alpha = \cap_{\beta<\alpha} H_\beta$ for a limit ordinal $\alpha \leq \gamma$, and $N_G(H_\gamma) = H_\gamma$. This chain we call *the upper normalizer chain.* At once, we can consider here the following two cases. If $H_\gamma = G$, then the subgroup H is called *ascendant.* If $H_\gamma = H$ (that is $N_G(H) = H$), then a subgroup H is called *selfnormalizing.* So, we can see that every subgroup is naturally associated with two types of subgroups that are ascendant and selfnormalizing subgroups. In this connection, note that the groups, whose subgroups are either ascendant or selfnormalizing, have been considered by L.A. Kurdachenko, J.

Otal, A. Russo, and G. Vincenzi in the paper [15]. Here are some results of this article.

Theorem 1. (L.A. Kurdachenko, J. Otal, A. Russo, G. Vincenzi [15]). *Let G be a locally finite group, whose subgroups are either ascendant or selfnormalizing. If G is not locally nilpotent, then $G = A\lambda P$ where $P = \langle x \rangle$ is a cyclic p-subgroup for some prime p, $A = [G, G]$ is a normal nilpotent p'-subgroup, $C_P(A) = \langle x^p \rangle$, $C_G(P) = P$. Conversely, if G is a group having this structure, then every subgroup of G either is ascendant or selfnormalizing.*

Let $\mathcal{X}$ be a class of groups. We recall that a group G is said to be a ***hyper-$\mathcal{X}$-group*** if G has an ascending series of normal subgroups whose factors belong to the class $\mathcal{X}$.

Theorem 2. (L.A. Kurdachenko,J. Otal, A. Russo, G. Vincenzi [15]). *Let G be a hyperabelian group, whose subgroups are either ascendant or selfnormalizing. If G is locally nilpotent, then every subgroup of G is ascendant. If G is not periodic, then every subgroup of G is ascendant.*

Consider again the upper normalizer chain of a subgroup H. If K is a subgroup of G such that $H_\gamma \le K$, then H_γ is selfnormalizing in K. However, the subgroup K itself could not be *selfnormalizing*. Therefore we come to the following important type of subgroups.

Let G be a group. A subgroup H is called ***weakly abnormal in* G** if every subgroup, containing H (and H itself) is selfnormalizing.

Using the concept of the normal closure, the following characteristic of the weakly abnormal subgroups was obtained.

Theorem 3. (M.S. Ba and Z.I. Borevich [1]). *Let G be a group and H a subgroup of G. Then H is weakly abnormal in G if and only if $x \in H^{\langle x \rangle}$ for each element $x \in G$.*

If H is a maximal subgroup of G such that H is not normal, then $H = N_G(H)$. Thus if $x \notin H$, then $G = \langle H, H^x \rangle$. In particular, $x \in \langle H, H^x \rangle$ and we come to the following type of subgroups.

Let G be a group. A subgroup H is called *abnormal* in G if $g \in \langle H, H^g \rangle$ for each element g of G.

The abnormal subgroups have appeared in the paper [11] due to P. Hall, the term “an abnormal subgroup” also belongs to P. Hall, even though it appeared first in the article of R. Carter [6].

Since $H^{\langle x \rangle}$ contains $\langle H, H^x \rangle$, every abnormal subgroup is weakly abnormal. The converse is not true. A corresponding counter example one can find in the paper [1].

By their nature, abnormality is an antagonist to normality: a subgroup H of a group G is *both normal and abnormal only if it coincides with the whole group.* As we saw earlier, every maximal non-normal subgroup is abnormal. More interesting is an example of J. Tits: a subgroup $T(n, F)$ of all triangular matrices is abnormal in a general linear group $GL(n, F)$ over a field F. Finite (soluble) groups provide us with many assorted examples of abnormal subgroups. Among them, we note such an important example as the Carter subgroups (that is a nilpotent self-normalizing subgroup), introduced by R.W. Carter in [6]. In this paper R.W. Carter also obtained the following characterization of abnormal subgroups.

Theorem 4. (R.W. Carter [6]). *Let G be a group and H a subgroup of G. Then H is abnormal in G if and only if the following two conditions hold:*

(i) If K is a subgroup containing H, then K is self-normalizing.

(ii) If K and L are two conjugate subgroups containing H, then $K = L$.

In the case of soluble groups, the condition (ii) in Theorem 4 could be omitted. For finite groups, this fact was noted in the book of B. Huppert [[12], p. 733, Theorem 11.17]. For infinite groups, the most general expansion of this result was obtained in [23].

Recall, that a group G is called an *$\tilde{N}$-group* if G satisfies the following condition: *for every subgroups M, L* the fact that M is maximal in L, implies that M is normal in L.

We observe that the property to be an *$\tilde{N}$-group* is local [[24], §8]. In particular, every locally nilpotent group is an $\tilde{N}$-group, but converse is not true [42].

Theorem 5. (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [23]). *Let G be a hyper-$\tilde{N}$-group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup containing H is self-normalizing. In other words, in hyper-$\tilde{N}$-group every weakly abnormal subgroup is abnormal.*

Recall that a group G is called *radical*, if G has an ascending series of normal subgroups whose factors are locally nilpotent.

Corollary 6. (L.A. Kurdachenko, I.Ya. Subbotin [21]). *Let G be a radical group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup containing H is self-normalizing. In other words, in a radical group every weakly abnormal subgroup is abnormal.*

Corollary 7. (F. de Giovanni, G. Vincenzi [10]). *Let G be a hyperabelian group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup, containing H, is self-normalizing. In other words, in a hyperabelian group every weakly abnormal subgroup is abnormal.*

Corollary 8. *Let G be a soluble group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup, containing H, is self-normalizing. In other words, in hyperabelian group every weakly abnormal subgroup is abnormal.*

Now we consider another connected to a subgroup H natural series, which is in some sense dual to the upper normalizer chain. This is a descending series

$$G = H_0 \geq H^G = H_1 \geq \ldots H_\alpha \geq H_{\alpha+1} \geq \ldots H_\gamma,$$

defined by the following rule: $H_{\alpha+1} = H_\alpha^H$ for every $\alpha < \gamma$, and $H_\gamma = \cap_{\mu<\lambda} H_\mu$ for a limit ordinal λ. This series is called *the normal closure series of* H *in* G. The term H_α of this series is called the α *the normal closure of* H *in* G and will be denoted by $H^{G,\,\alpha}$. The last term H_γ of this series is called *the lower normal closure of* H *in* G and will be denoted by $H^{G,\,\infty}$. Here again we can distinguish two natural types of subgroups. If $H = H^{G,\,\infty}$, then a subgroup H is called *descendant* (*in* G). An important particular case of descendant subgroups are subnormal subgroups. A subnormal subgroup is exactly a descending subgroup having finite normal closure series. These subgroups strongly affect the structure of a group. For example, it is possible to prove that if every subgroup of a locally (soluble-by-finite) group is descendant, then this group is locally nilpotent. If every subgroup of a group G is subnormal, then, by a remarkable result due to W. Mohres [32], G is soluble. Subnormal subgroups have been studied very thoroughly for quite a long period of time. We will not delve into this subject, since it is fairly reflected in the book of J.C. Lennox and S.E. Stonehewer [30] and survey of C. Casolo [4]. Another extreme case leads us to the following type of subgroups. If $G = H^G$, then a subgroup H is called *contranormal* in G.

The term *a contranormal subgroup* has been introduced by J.S. Rose in [40]. Note at once that every subgroup is contranormal in its lower normal closure.

If K is a subgroup of G such that $H \leq K \leq H^{G,\,\infty}$, then K is contranormal in $H^{G,\infty}$. However the subgroup H no longer has to be contranormal in K. Thus we naturally arrive at the following important type of subgroups.

A subgroup H of a group G is said to be *nearly abnormal*, if H is contranormal in every subgroup K containing H [23].

If H is an abnormal subgroup of a group G and K is a subgroup, containing H, then $x \in \langle H, H^x\rangle$ for every element $x \in K$. Taking into account the following obvious embedding $\langle H, H^x\rangle \leq H^{\langle x\rangle}$, we obtain that $H^K = K$. Therefore, every *abnormal subgroup is nearly abnormal.* In particular, every abnormal subgroup of a group G is contranormal in G. However not every contranormal subgroup is abnormal. The following simple example shows this.

Let P be a quasicyclic 2-group. Consider the semidirect product $G = P\lambda\langle d\rangle$ where $\mid d \mid= 2$ and $d^{-1}\, ad = a^{-1}$ for any $a \in P$. Being hypercentral, G satisfies the normalizer condition. It follows that G has no proper abnormal subgroups. But $\langle d\rangle$ is a proper contranormal subgroup of G.

Let now H be a nearly abnormal subgroup of G, and consider an arbitrary subgroup K containing H. Suppose that $N_G(K) = L \neq K$. Then K is normal in L. It follows that $H = H^L \neq L$, so that H is not contranormal in L. This contradiction shows that $N_G(K) = K$. In other words, every nearly abnormal subgroup of G is weakly abnormal. Theorem 5 implies.

Corollary 9. (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [23]). *Let G be a hyper-$\tilde{N}$-group and H be a subgroup of G. Then H is nearly abnormal in G if and only if H is abnormal.*

Conversely, let H be a weakly abnormal subgroup of G and let K be an arbitrary subgroup, containing H. Suppose that $H^K \neq K$. Subgroup H^K is normal in K, so that $N_K(H^K) = K \neq H^K$. Then $N_G(H^K) \neq H^K$, and H^K is selfnormalizing. This contradiction shows that $H^K = K$. In other words, every weakly abnormal subgroup of G is nearly abnormal.

Continuing with the situation where the upper normalizer series is very short, we come to another interesting type of subgroups.

A subgroup H of a group G is called *transitively normal* if H is normal in every subgroup $K \geq H$ in which H is subnormal (L.A. Kurdachenko, I.Ya. Subbotin [22].

In the paper [33], these subgroups have been introduced under a different name. Namely, a subgroup H of a group G is said to satisfy the *subnormalizer condition* in G, if for every subgroup K such that H is normal in K we have $N_G(K) \leq N_G(H)$.

One of the most important types of transitively normal subgroups was introduced by P. Hall as *pronormal subgroups.*

A subgroup H of a group G is said to be ***pronormal in* G** if for every $g \in G$ the subgroups H and H^g are conjugate in the subgroup $\langle H, Hg\rangle$.

From the definition, it is clear that every abnormal subgroup is pronormal. On the other hand, every normal subgroup is also pronormal. Thus, pronormal subgroups have managed to combine these two antagonistic types of subgroups. Such important subgroups of finite soluble groups as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups are pronormal.

Pronormality is tightly connected to the well know T-property, i.e. to the property of transitivity of normal subgroups. The groups having this property are called T-groups. A group G is said to be a $\overline{T}$-group, if every subgroup of G is a T-group.

The structure of finite soluble T-groups has been described by W. Gaschtz in [9]. In particular, he proved that ***every finite soluble*$\mathbf{T}$*-group is a* $\overline{\boldsymbol{T}}$*-group***. Observe that ***every finite* $\mathbf{T}$*-group is metabelian***. Such infinite soluble groups have been studied by D.J.S. Robinson in [37]. The main results of this study are the following.

Theorem 10. (D.J. Robinson, [37]) *Let G be a locally soluble $\overline{T}$-group.*

(i) If G is not periodic, then G is abelian.

(ii) *If G is periodic and L is the locally nilpotent residual of G, then the following conditions hold:*

(a) *G/L is a Dedekind group;*

(b) *$\Pi(L) \cap \Pi(G/L) = \emptyset$;*

(c) *$2 \notin \Pi(L)$;*

(d) *Every subgroup of L is G-invariant.*

T.A. Peng [36] has obtained the first characterization of finite $\overline{T}$-groups in terms of pronormal subgroups.
Theorem 11. (T. Peng, [36]) *Let G be a finite group. Then*

(i) *Every subgroup of G is pronormal if and only if G is a T-group.*

(ii) *Every cyclic subgroup of G is pronormal if and only if G is a T-group.*

This result was generalized to infinite groups in [25].
Theorem 12. (N.N. Kuzennyi and I.Ya. Subbotin, [25]) *Let G be a locally soluble group or a periodic locally graded group. Then the following conditions are equivalent:*

(i) *every cyclic subgroup of G is pronormal in G;*

(ii) *G is a soluble $\overline{T}$-group.*

Let G be a group whose subgroups are pronormal and L be a locally nilpotent residual of G.

(i) *If G is periodic and locally graded, then G is a soluble $\overline{T}$-group, in which L complements every Sylow $\Pi(G/L)$-subgroup.*

(ii) *If G is not periodic and locally soluble, then G is abelian.*

In the paper [38], the assertion (ii) has been extended to non-periodic locally graded groups. In fact, in this case, such groups are still abelian.

N.F. Kuzennyi and I.Ya. Subbotin have also completely described the locally graded periodic groups in which all primary subgroups are pronormal [27] and infinite locally soluble groups in which all infinite subgroups are pronormal ([26]).

They proved that *in the infinite case, the class of groups whose subgroups are pronormal is a proper subclass of the class of groups with the transitivity of normality. Moreover, it is also a proper subclass of the class of groups whose primary subgroups are pronormal.*

However, *the pronormality condition for all subgroups can be weakened to the pronormality for only abelian subgroups* ([27]).

We mentioned above some important results on transitivity of normality. Transitivity of such important subgroup properties as pronormality, abnormality and other related to them properties have been studied by L.A. Kurdachenko, I.Ya. Subbotin, and J.Otal (see, [19], [16]).

The groups, *in which pronormality is transitive* are called *TP-groups* and the groups, in which all subgroups are TP-groups, are called *$\overline{T}P$-groups.* For the these groups, the following description has been obtained.

Theorem 13. (L.A. Kurdachenko, I.Ya. Subbotin [19]). *Let G be a locally soluble group. Then G is a $\overline{T}P$-group if and only if G is a $\overline{T}$-group.*

Let G be a periodic soluble group. Then G is a TP-group if and only if $G = A\lambda(B \times P)$ where

(i) A, B are abelian 2'-subgroup and P is a 2-subgroup (if P is non-identity);

(ii) $\mathbf{\Pi}(A) \cap \mathbf{\Pi}(B) = \emptyset$;

(iii) P is a T-group;

(iv) $[G, G] = A \times [P, P]$;

(v) Every subgroup of $[G, G]$ is G-invariant;

(vi) A is a complement to every Sylow $\Pi(B \times P)$-subgroup of G.

Let G be a non-periodic soluble group.

(i) If $C_G([G, G])$ is non-periodic, then G is a TP-group if and only if G is a T-group.

(ii) If $C_G([G, G])$ is periodic, then G is a TP-group if and only if G is a hypercentral T-group.

In [19] the authors list all of types of periodic soluble TP-groups.

The property *to be an abnormal subgroup* is not transitive. However, P. Hall note that if G is a group, H is a normal subgroup of G, D is an abnormal subgroup in DH and DH is abnormal in G, then D is abnormal in G. Here we have the following result.

Theorem 14. (Kurdachenko, Subbotin, KSu2005). *Let G be a group and H be a normal subgroup of G. Suppose that G/H has no proper abnormal subgroups and H satisfies the normalizer condition. Then abnormality is transitive in G.*

A subgroup H of a group G is called an *NE-subgroup* if $N_G(H) \cap H^G = H$ (Li Shirong, [28]). The following characteristics of finite $\overline{T}$-groups are interesting.

Theorem 15. (Yangming Li, [29]). *If every subgroup of a finite group G is NE-subgroups, then G is a $\overline{T}$-group. If every primary subgroup of a finite group G is NE-subgroups, then G is a $\overline{T}$-group.*

A subgroup H of a group G is called an H-subgroup if $N_G(H) \cap H^g \leq H$ for all elements $g \in G$ (M. Bianchi, A.G.B. Mauri, M. Herzog and L. Verardi, [3]).

Theorem 16. (M. Bianchi, A.G.B. Mauri, M. Herzog and L. Verardi, [3]). *If every subgroup of a finite group G is an H-subgroup, then G is a $\overline{T}$-groups. If every primary subgroup of a finite group G is H-subgroups, then G is a $\overline{T}$-groups.*

Recall that a subgroup H of a group G is called weakly normal in G if for each element g such that $H^g \leq N_G(H)$ we have $g \in N_G(H)$. ([31])

Theorem 17. (A. Ballester-Bolinches and R. Esteban-Romero [2]). *Every pronormal subgroup is weakly normal. If every subgroup of a finite group G is weakly normal in G, then G is a $\overline{T}$-group. If every primary subgroup of a finite group G is weakly normal in G, then G is a $\overline{T}$-group.*

The following result is tightly related to this theme.

Theorem 18. (P. Csorga and M. Herzog, [5]). *Let G be a finite group, whose cyclic subgroup of order 4 and of all prime orders are H-subgroups. Then G is supersoluble.*

Yangming Li has proved a similar result for NE-subgroups. Namely:

Theorem 19. Theorem (Yangming Li, [29]). *Let G be a finite group, whose cyclic subgroups of order 4 and of all prime orders are NE-subgroups. Then G is supersoluble.*

For infinite groups we have the following results.

Theorem 20.(L.A. Kurdachenko, J. Otal [14]). *Let G be a locally finite group. If every cyclic subgroup of prime order or order 4 is transitively normal in G, then G is hypercyclic. Moreover, if L is a locally nilpotent residual of G, then L is an abelian Hall subgroup of G, and every subgroup of L is G-invariant.*

Using this theorem, L.A. Kurdachenko and J. Otal have obtained the next more general characterization of $\overline{T}$-group.

Theorem 21. (L.A. Kurdachenko, J. Otal [14]). *Let G be a locally finite group and L be a locally nilpotent residual of G. If every primary cyclic subgroup of G is transitively normal in G, then G is a $\overline{T}$-groups.*

We note that this result *cannot be extended on arbitrary periodic groups.* A.Yu. Olshanskii constructed his famous example of an infinite p-group G (p is a big enough prime) whose proper subgroups are of order p. Clearly, every subgroup of G is transitively normal. For the non-periodic case, we also need some restrictions. A.Yu. Olshanskii has constructed another sophisticated example of an infinite torsion-free group G, whose proper subgroups are cyclic. Clearly, every subgroup of G is transitively normal.

On the other hand, it seems very possible that in the Grigorchuk's group every pronormal subgroup is normal, while in the Olshanskii group every pronormal subgroup is abnormal.

Recall that a group G is called a *generalized radical*, if G has an ascending series whose factors are locally nilpotent or locally finite.

Theorem 22. (L.A. Kurdachenko, J. Otal [14]). *Let G be a non-periodic locally generalized radical group. If every cyclic subgroup of G is transitively normal in G, then either G is abelian or $G = R\langle b\rangle$ where R is abelian, $b^2 \in R$ and $a^b = a^{-1}$ for each element $a \in R$.*

Moreover, in the second case, the following conditions hold:

(i) if $b^2 = 1$, then the Sylow 2-subgroup D of R is elementary abelian;

(ii) if $b^2 \neq 1$, then either D is elementary abelian or $D = E \times \langle v\rangle$ where E is elementary abelian and $\langle b, v\rangle$ is a quaternion group.

Conversely, if a group G has the above structure, then every cyclic subgroup is transitively normal.

Let G be a group. A subgroup H is called *weakly pronormal* in G (or *has Frattini Property*), if for every subgroups K and L such that $H \leq K$ and K is normal in L, we have $L = N_L(H)K$.

Here is some characterization of weakly pronormal subgroups.

Theorem 23. (M.S. Ba, Z.I. Borevich, [1]). *Let G be a group and H be a subgroup of G. Then H is weakly pronormal in G if and only if the subgroups H and H^x conjugate in $H^{\langle x\rangle}$ for each element $x \in G$.*

The inclusion $\langle H, Hx\rangle \leq H^{\langle x\rangle}$ shows that every pronormal subgroup is weakly pronormal. In particular, *every pronormal subgroup has Frattini Property.*

We recall that a group G is called an *N-group* or a *group with the normalizer condition*, if $N_G(H) \neq H$ for every subgroup H of G. A group G is an N-group if and only if every subgroup of G is ascendant.

Theorem 24. (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [16]). *Let G be a hyper-N-group and D be a subgroup of G. Then D is pronormal in G if and only if D has Frattini Property.*

This theorem is a generalization of the following results.

Theorem 25. (F. de Giovanni, G. Vincenzi, [10]). *Let G be a hyperabelian group and D be a subgroup of G. Then D is pronormal in G if and only if D has Frattini Property.*

Theorem 26. *Let G be a soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has Frattini Property.*

J. Rose [39] has introduced the *balanced chain* connecting a subgroup H to a group G, that is, the chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_{n-1} \leq H_n = G$$

such that for each j, $0 \leq j \leq n-1$, either H_j is normal in H_{j+1}, or H_j is abnormal in H_{j+1}. The number n is the length of this chain. In finite groups, every subgroup can be connected to the group by some balanced chain. If the lengths of these chains are 1, then every subgroup is either normal or abnormal in a group.

Such finite groups were studied by Fattahi in [8]. Infinite groups of this kind and some of their generalizations were studied by I. Subbotin in [41], and M. De Falko, L. Kurdachenko and I. Subbotin in [7]. In this setting, the groups whose subgroups are either abnormal or subnormal have been considered.

L.A. Kurdachenko and H. Smith in [18] considered the groups whose subgroups are either self-normalizing or subnormal.

Observe that in a group, in which a normalizer of any subgroup is abnormal, and in a group in which every subgroup is abnormal in its normal closure, the mentioned balanced chain lengths are at most 2.

If G is a soluble $\overline{T}$-group, then every subgroup of G is abnormal in its normal closure. For any pronormal subgroup H of a group G, the normalizer $N_G(H)$ is an abnormal subgroup of G. Thus the property for a subgroup to have an abnormal normalizer is a generalization of pronormality.

Theorem 27. (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [17]).

(i) Let G be a radical group. Then G is a $\overline{T}$-group if and only if every cyclic subgroup of G is abnormal in its normal closure.

(ii) Let G be a periodic soluble group. Then G is a $\overline{T}$-group if and only if it's locally nilpotent residual L is abelian and the normalizer of each cyclic subgroup of G is abnormal in G.

Theorem 28. (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [17]). *Let G be a periodic soluble group. Then every subgroup of G is pronormal if and only if it's locally nilpotent residual L is abelian and a normalizer of every subgroup of G is abnormal in G.*

In the non-periodic case, there exist non-periodic non-abelian groups in which normalizers of all subgroups are abnormal. On the other hand, *the non-periodic locally soluble groups in which all subgroups are pronormal are abelian* (N.Kuzennyi, I. Subbotin [25]).

Theorem 29. (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [17]). *Let G be a non-periodic group with an abelian locally nilpotent residual L. If a*

normalizer of every cyclic subgroup is abnormal and for each prime $p \in \Pi(L)$ the Sylow p-subgroup of L is bounded, then G is abelian.
J.Otal, N. Semko, and N. Semko (Jr.) in [34] published interesting results concerning groups whose transitively normal subgroups are normal or self-normalizing. They show that they are almost locally nilpotent ((locally nilpotent)-by-finite). They also classified these groups and gave a detailed description of their structure.

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