Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **36** (2016) no. 1, 123–132. doi:10.1285/i15900932v36n1p123

Completely positive matrices of order 5 with $\widehat{CP}$-graph

Micol Cedolin
Castello 2153, Venezia, Italy
micol.cedolin@gmail.com

Luigi Salce[i]
Dipartimento di Matematica, Via Trieste 63, 35121 Padova, Italy
salce@math.unipd.it

Received: ......; accepted......

Abstract. We characterize completely positive matrices of order five whose associated graph is a $\widehat{CP}$-graph not completely positive, unifying and improving results obtained by Xu. We show by counterexamples that two characterizations obtained by Xu are incorrect.

Keywords: doubly non-negative matrices, completely positive matrices, $\widehat{CP}$-graphs

MSC 2000 classification: Primary: 15B48. Secondary: 15B57

1 Introduction

All matrices considered in this paper are real matrices. We follow notations and terminology used in [2]. In particular, a matrix that is both entry-wise non-negative and positive semidefinite is called *doubly non-negative*, and a matrix of the form WW^T, for W a non-negative matrix, is called *completely positive.* The classes of doubly non-negative and completely positive $n \times n$ matrices are denoted, respectively, by $\mathcal{DNN}_n$ and $\mathcal{CP}_n$. Obviously the inclusion $\mathcal{CP}_n \subseteq \mathcal{DNN}_n$ holds. It is well known that this inclusion is actually an equality for $n \leq 4$, and that for $n \geq 5$ the inclusion is proper. We refer to the monograph [2] for all information on these classes of matrices.

An essential tool in the investigation of completely positive matrices is the graph theory. The vertex set of a graph G of order n is identified with $\{1, 2, \cdots, n\}$. Given a symmetric non-negative matrix $A = [a_{ij}]$ of order n, the (undirected) graph $G(A)$ associated with A has n vertices and edges (i, j) for those $i \neq j$ such that $a_{ij} > 0$ (it is understood that there are no loops in $G(A)$).

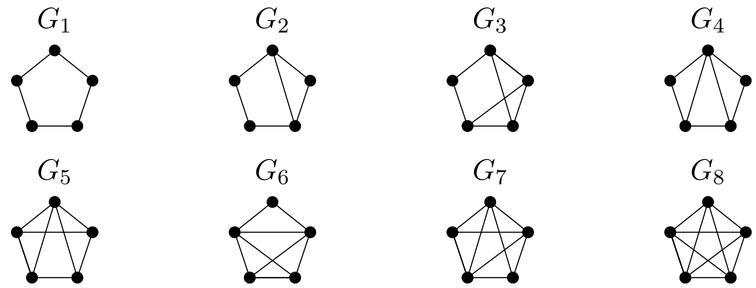
A graph G is *completely positive* if every matrix $A \in \mathcal{DNN}_n$ such that $G(A) \cong G$ is completely positive. The following characterization of completely

[i]Research supported by "Progetti di Eccellenza 2011/12" of Fondazione CARIPARO.

positive graphs settles one of the so-called *qualitative problem* for completely positive matrices, after the contributions by many authors (see [2]).

Theorem 1.1 (Kogan-Berman [3]). A graph G is *completely positive* if and only if it does not contain cycles of odd length > 3.

It follows that a graph G with 5 vertices is not completely positive exactly if it contains a cycle of length 5. Up to isomorphisms, there are 8 different types of graphs with this property. We follow the notation in [4], where these graphs are denoted by G_i $(1 \leq i \leq 8)$. Their patterns, displayed in [4], are, up to isomorphisms, the following:



It is understood that the vertices in the above graphs are numbered clockwise, starting with the top vertex by 1. A graph G is a $\widehat{CP}$-graph if there is a vertex j, that we call *hat-vertex*, such that there are exactly two vertices adjacent to j, not adjacent to each other, and $G \setminus \{j\}$ is completely positive. $\widehat{CP}$-graphs have been introduced under the original name *excellent graphs* by Barioli [1], who characterized completely positive matrices with excellent graphs among the doubly non-negative ones. Among the eight non-CP graphs of order 5 illustrated above, only the first three are $\widehat{CP}$-graphs.

Lemma 1.2. A graph G with five vertices which is not completely positive is a $\widehat{CP}$-graph if and only if it is isomorphic to one of the graphs G_1, G_2, G_3.

The three cases of doubly non-negative matrices of order 5 with associated graphs G_1, G_2 and G_3 are treated separately by Xu in [4] and [5]. Recall that the matrices of these three types may be assumed, without loss of generality (after normalization and cogredience), to be in the following forms (**1**), (**2**) and (**3**), respectively (it is understood that an entry a_{ij} denotes a positive real number):

$$(\mathbf{1}) \quad \begin{pmatrix} 1 & a_{12} & 0 & 0 & a_{15} \\ a_{12} & 1 & a_{23} & 0 & 0 \\ 0 & a_{13} & 1 & a_{34} & 0 \\ 0 & 0 & a_{34} & 1 & a_{45} \\ a_{15} & 0 & 0 & a_{45} & 1 \end{pmatrix}; \qquad (\mathbf{2}) \quad \begin{pmatrix} 1 & a_{12} & a_{13} & 0 & a_{15} \\ a_{12} & 1 & a_{23} & 0 & 0 \\ a_{13} & a_{23} & 1 & a_{34} & 0 \\ 0 & 0 & a_{34} & 1 & a_{45} \\ a_{15} & 0 & 0 & a_{45} & 1 \end{pmatrix};$$

$$(\mathbf{3}) \qquad \begin{pmatrix} 1 & a_{12} & a_{13} & 0 & a_{15} \\ a_{12} & 1 & a_{23} & a_{24} & 0 \\ a_{13} & a_{23} & 1 & a_{34} & 0 \\ 0 & a_{24} & a_{34} & 1 & a_{45} \\ a_{15} & 0 & 0 & a_{45} & 1 \end{pmatrix}.$$

Barioli characterized in [1] the CP matrices with associated graphs isomorphic to G_1; the same result was also proved by Xu in [5].

Theorem 1.3 (Barioli [1])**.** For a doubly non-negative matrix A with associated graph G_1 in the form (**1**), the following are equivalent:

(1) A is completely positive;

(2) $\det A \geq 4a_{12}a_{23}a_{34}a_{45}a_{51}$.

We will use the standard notation $A[i_1, \cdots, i_r | j_1, \cdots, j_r]$ to denote the submatrix of A obtained by intersecting the ordered set of rows $\{i_1, \cdots, i_r\}$ and columns $\{j_1, \cdots, j_r\}$. Recall that, if $A = [a_{ij}]$, the *weight* of a cycle $C = \{j_1 \to j_2 \to \cdots \to j_k \to j_1\}$ in $G(A)$ is the product $w_A(C) = (-1)^{k-1} a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_k j_1}$ (see [4]). We follow Xu's notation in [4], where he writes: "By $A_{|n|}$, we mean the algebraic sum of the weights of all cycles of length n of $G(A)$".

Xu characterized in [5] the completely positive matrices with associated graphs isomorphic to G_2 and G_3. We quote his theorems separately as stated in [4] and adapted to our notation.

Theorem 1.4 (Xu, [4] Theorem 2)**.** For a doubly non-negative matrix A with associated graph G_2 in the form (**2**), the following are equivalent:

(1) A is completely positive;

(2) either $\det A[1,2|2,3] \leq 0$, or $\det A[1,2|2,3] > 0$ and $\det A \geq 4A_{|5|}$.

We remark that the conditions in Theorem 1.4 are different from those in Theorem 2 in [4], where $\det A[1,2|2,3] \leq 0$ is replaced by $\det A[1,2|1,3] \geq 0$ and $\det A[1,2|2,3] > 0$ by $\det A[1,2|1,3] < 0$. This difference is due to the permutation of the vertices in Xu's graph G_2 with respect to our graph, as displayed above.

We will show, by means of a counterexample (see Example 3.2), that, when $\det A[1,2|2,3] > 0$, the condition $\det A \geq 4A_{|5|}$ is not necessary for the complete positivity of A; our Theorem 3.1 will provide the right sufficient and necessary condition.

The matrix used by Xu in [4] to prove the next Theorem 1.5 is exactly in our form (**3**). We follow his notation by setting $\mu = \min\{a_{23}, a_{13}/a_{12}, a_{34}/a_{24}\}$.

Theorem 1.5 (Xu [4] Theorem 3)**.** For a doubly non-negative matrix A with associated graph G_3 in the form (**3**) the following are equivalent:

(1) A is completely positive;

(2) either $\mu = a_{23}$, or $\mu \neq a_{23}$ and $\det A \geq 4A_{|5|}$.

Also for this case we will show, by means of a counterexample (see Example 3.3), that, when $\mu \neq a_{23}$, the condition $\det A \geq 4A_{|5|}$ is not necessary for the complete positivity of A; our Theorem 3.1 will provide the right condition.

The three cases of doubly non-negative matrices with associated graphs isomorphic to G_1, G_2 and G_3 are covered by Barioli's characterization of completely positive matrices with $\widehat{CP}$-graph in [1]; we recall his result adapting it to our setting and notation.

Theorem 1.6 (Barioli [1])**.** For a doubly non-negative matrix A with associated graph G_1, G_2 or G_3 in the form (**1**), (**2**) or (**3**), the following are equivalent:

(1) A is completely positive;

(2) the matrix A^- obtained by A replacing a_{15} by $-a_{15}$ is positive semidefinite.

The goal of this paper is to unify in a single result Theorems 1.3, 1.4 and 1.5 using Barioli's Theorem 1.6. We will characterize in Section 3 completely positive matrices with associated graph isomorphic to G_1, G_2 or G_3 by means of an inequality relating $\det A$ with the *total weight* of a hat-vertex, a quantity defined in the next section that can be computed from the entries of A just looking at its associate graph $G(A)$. As mentioned above, our characterization of matrices with associated graph isomorphic to G_2 or G_3 corrects those of Xu in [4].

2 The total weight of a hat-vertex

The crucial tool in the proof of our main theorem is the notion of *total weight* of a hat-vertex of the $\widehat{CP}$-graph $G(A)$ associated with a matrix $A = [a_{ij}]_{i,j} \in \mathcal{DNN}_5$. The total weight of such a vertex is defined starting from the notion of *corrected weight* of a cycle.

Definition 2.1. Let $A = [a_{ij}]$ be a doubly non-negative matrix of order n.

(1) Given a cycle $C = \{j_1 \to j_2 \to \cdots \to j_k \to j_1\}$ in $G(A)$, its *corrected weight* is $W_A(C) = w_A(C) \cdot \prod_{i \notin C} a_{ii}$.

(2) The *total weight* $W_A(j)$ of a vertex j is the sum of the corrected weights of the cycles of any length containing j: $W_A(j) = \sum_{j \in C} W_A(C)$.

Example 2.2. Let $A = [a_{ij}] \in \mathcal{DNN}_5$ be such that $G(A) = G_1$. Then the only cycle containing vertex 5 (or any other vertex) is $C = \{1 \to 2 \to 3 \to 4 \to 5 \to$

1}, hence the total weight $W_A(5)$ of the vertex 5 (or any other vertex) coincides with the weight and the corrected weight of C:

$$W_A(5) = a_{12}a_{23}a_{34}a_{45}a_{51} = A_{|5|}.$$

Example 2.3. Let $A = [a_{ij}]_{i,j} \in \mathcal{DNN}_5$ be such that $G(A) = G_2$. Then the cycles containing the hat-vertex 5 (or the other hat-vertex 4) are the cycle C as in Example 2.2, and the cycle $C_1 = \{1 \to 3 \to 4 \to 5 \to 1\}$. The respective corrected weights are

$$\begin{aligned} W_A(C) &= a_{12}a_{23}a_{34}a_{45}a_{51}, \\ W_A(C_1) &= -\ a_{13}a_{34}a_{45}a_{51}a_{22}. \end{aligned}$$

Consequently the total weight of the hat-vertex 5 (and of the hat-vertex 4) is

$$W_A(5) = a_{12}a_{23}a_{34}a_{45}a_{51} - a_{13}a_{34}a_{45}a_{51}a_{22}.$$

Since $A_{|5|} = a_{12}a_{23}a_{34}a_{45}a_{51}$, we have that $W_A(5) = A_{|5|} - a_{13}a_{34}a_{45}a_{51}a_{22}$, hence $W_A(5) < A_{|5|}$.

Example 2.4. Let $A = [a_{ij}] \in \mathcal{DNN}_5$ be such that $G(A) = G_3$. The only hat-vertex is 5, and there are four cycles containing 5: $C = \{1 \to 2 \to 3 \to 4 \to 5 \to 1\}$, $C_1 = \{1 \to 3 \to 2 \to 4 \to 5 \to 1\}$, $C_2 = \{1 \to 3 \to 4 \to 5 \to 1\}$, $C_3 = \{1 \to 2 \to 4 \to 5 \to 1\}$. The respective corrected weights are

$$\begin{aligned} W_A(C) &= a_{12}a_{23}a_{34}a_{45}a_{51}, \\ W_A(C_1) &= a_{13}a_{32}a_{24}a_{45}a_{51}, \\ W_A(C_2) &= -\ a_{13}a_{34}a_{45}a_{51}a_{22}, \\ W_A(C_3) &= -\ a_{12}a_{24}a_{45}a_{51}a_{33}. \end{aligned}$$

Consequently the total weight of the only hat-vertex 5 is

$$\begin{aligned} W_A(5) = a_{12}a_{23}a_{34}a_{45}a_{51} &+ a_{13}a_{32}a_{24}a_{45}a_{51} \\ &- a_{13}a_{34}a_{45}a_{51}a_{22} - a_{12}a_{24}a_{45}a_{51}a_{33}. \end{aligned}$$

Since $A_{|5|} = a_{12}a_{23}a_{34}a_{45}a_{51} + a_{13}a_{23}a_{24}a_{45}a_{51}$, we have

$$W_A(5) = A_{|5|} - a_{13}a_{34}a_{45}a_{51}a_{22} - a_{12}a_{24}a_{45}a_{51}a_{33}.$$

Remark 2.5. We note that the weight and the corrected weight of a cycle are identical for normalized matrices in the forms (**1**), (**2**) and (**3**). Non-normalized matrices better reveal the role of the corrected weight and of the total weight, as the proof of the main theorem below will show considering the matrix $\tilde{A}$. A straightforward computation shows that the relationship between the total weight of a vertex j of $G(A)$ of a non-normalized matrix A and its normalized form DAD, where $D = \mathrm{Diag}(d_1, \ldots, d_n)$, is the following: $W_{DAD}(j) = W_A(j) \cdot \prod_i d_i^2$.

3 The main theorem

Our main result collects Theorems 1.3, 1.4 and 1.5 quoted in Section 2 in a single theorem, which characterizes 5×5 completely positive matrices with $\widehat{CP}$-graphs by means of the total weight of a hat-vertex, improving the characterizations appearing in Xu's papers [4] and [5]. Our theorem relies on Barioli's Theorem 1.6.

Theorem 3.1. For a doubly non-negative matrix A with associated graph $G(A)$ isomorphic to G_1, G_2 or G_3, the following conditions are equivalent:

(1) A is completely positive;

(2) $\det A \geq 4W_A(5)$.

Proof. In view of Remark 2.5, we can assume, without loss of generality, that the matrix A is in normalized form (**1**), (**2**) or (**3**), since $W_{DAD}(5) = W_A(5) \cdot \prod_i d_i^2$ and $\det(DAD) = \det(A) \cdot \prod_i d_i^2$.

If $G(A) = G_1$, our result coincides with Theorem 1.3, once it is recalled that $a_{12}a_{23}a_{34}a_{45}a_{51} = W_A(5)$, as seen in Example 2.2. We prove now the theorem separately for the cases $G(A) = G_2$ and $G(A) = G_3$.

CASE $G(A) = G_2$**.** (1) $\Rightarrow$ (2) By Theorem 1.6, $\det(A^-) \geq 0$. So we have:

$$
\begin{aligned}
0 &\leq \det(A^-) = \det A[1,2,3,4|1,2,3,4] \\
&\quad - a_{54} \det \begin{pmatrix} 1 & a_{12} & a_{13} & -a_{15} \\ a_{21} & 1 & a_{23} & 0 \\ a_{31} & a_{32} & 1 & 0 \\ 0 & 0 & a_{43} & a_{45} \end{pmatrix} + a_{51} \det \begin{pmatrix} a_{12} & a_{13} & 0 & -a_{15} \\ 1 & a_{23} & 0 & 0 \\ a_{23} & 1 & a_{34} & 0 \\ 0 & a_{34} & 1 & a_{45} \end{pmatrix} \\
&= \det A[1,2,3,4|1,2,3,4] \\
&\quad - a_{54}\big\{a_{15} \det A[2,3,4|1,2,3] + a_{45} \det A[1,2,3|1,2,3]\big\} \\
&\quad - a_{51}\big\{a_{15} \det A[2,3,4|2,3,4] + a_{45} \det A[1,2,3|2,3,4]\big\}.
\end{aligned}
$$

Comparing with the analogous expression of $\det A$, we get

$$
\begin{aligned}
\det(A^-) &= \det A - 2a_{54}a_{15} \det A[1,2,3|2,3,4] - 2a_{51}a_{45} \det A[1,2,3|2,3,4] \\
&= \det A - 2a_{54}a_{15}a_{34} \det A[1,2|2,3] - 2a_{51}a_{45}a_{34} \det A[1,2|2,3] \\
&= \det A - 4a_{51}a_{45}a_{34} \det A[1,2|2,3] \\
&= \det A - 4a_{51}a_{45}a_{34}(a_{12}a_{23} - a_{31}) \\
&= \det A - 4(a_{51}a_{45}a_{34}a_{12}a_{23} - a_{31}a_{51}a_{45}a_{34}) \\
&= \det A - 4W_A(5).
\end{aligned}
$$

We deduce that $\det A \geq 4W_A(5)$, as desired.

(2) $\Rightarrow$ (1) For the sake of completeness, we include a sketch of Xu's proof that $\det A[1,2|2,3] \le 0$ implies A completely positive. Let us consider the matrix

$$A' = PAP = \begin{pmatrix} 1 & \mathbf{a}^T \\ \mathbf{a} & A_1 \end{pmatrix}$$

where P is the transposition matrix permuting first and second row of I_5. The Schur complement $A/A[1|1] = A_1 - \mathbf{a}\mathbf{a}^T$ is positive semidefinite, and the hypothesis that $\det A[1,2|2,3] \le 0$ ensures that it is also entry-wise non-negative. Therefore $A/A[1|1] \in \mathcal{DNN}_4 = \mathcal{CP}_4$ and consequently $A/A[1|1] = B_1 B_1^T$ for a non-negative matrix B_1. Then a direct computation shows that $A' = BB^T$, where

$$B = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{a} & B_1 \end{pmatrix}$$

thus A', and consequently A, is completely positive.

Let us assume now that $\det A[1,2|2,3] = a_{12}a_{23} - a_{13} > 0$. Setting $\mu = a_{13}/a_{23}$, we have that $\mu < a_{12} \le 1 \le 1/a_{12}$. Let e_i denote an elementary vector of appropriate dimension whose i-th component is equal to 1, and $E_{ij} = e_i e_j^T$. Now we follow the line of the proof in [5]. Let $S = I - \mu E_{12}$ be the elementary transformation matrix and consider $\tilde{A} = SAS^T$. Then $\tilde{A}$ is obviously positive semidefinite, and the inequalities $\mu < a_{12} \le 1 \le 1/a_{12}$ ensure that $\tilde{A}$ is non-negative. Furthermore, the $(1,3)$-entry of $\tilde{A}$ is 0, as is easy to check, so $G(\tilde{A}) = G_1$. A direct computation shows that

$$\tilde{A}_{|5|} = (a_{12} - \mu)a_{23}a_{34}a_{45}a_{51} = W_A(5)$$

so the hypothesis ensures that $\det \tilde{A} = \det A \ge 4W_A(5) = 4\tilde{A}_{|5|}$. Now Theorem 1.3 implies that $\tilde{A}$ is completely positive. But then also A is completely positive, because $A = S^{-1}\tilde{A}(S^{-1})^T$ and $S^{-1} = I + \mu E_{12}$ is non-negative.

CASE $G(A) = G_3$**.** (1) $\Rightarrow$ (2) By Theorem 1.6, $\det(A^-) \ge 0$. So we have:

$$\begin{aligned}
0 &\le \det(A^-) = \det A[1,2,3,4|1,2,3,4] \\
&- a_{54} \det \begin{pmatrix} 1 & a_{12} & a_{13} & -a_{15} \\ a_{21} & 1 & a_{23} & 0 \\ a_{31} & a_{32} & 1 & 0 \\ 0 & a_{42} & a_{43} & a_{45} \end{pmatrix} + a_{51} \det \begin{pmatrix} a_{12} & a_{13} & 0 & -a_{15} \\ 1 & a_{23} & a_{24} & 0 \\ a_{32} & 1 & a_{34} & 0 \\ a_{42} & a_{43} & 1 & a_{45} \end{pmatrix} \\
&= \det A[1,2,3,4|1,2,3,4] \\
&- a_{54}\big\{a_{15} \det A[2,3,4|1,2,3] + a_{45} \det A[1,2,3|1,2,3]\big\} \\
&- a_{51}\big\{a_{15} \det A[2,3,4|2,3,4] + a_{45} \det A[1,2,3|2,3,4]\big\}.
\end{aligned}$$

Comparing with the analogous expression of $\det A$, we get

$$\det(A^-) = \det A - 2a_{54}a_{15} \det A[2,3,4|1,2,3] - 2a_{51}a_{45} \det A[1,2,3|2,3,4].$$

As $A[2,3,4|1,2,3] = A[1,2,3|2,3,4]^T$, we get

$$\begin{aligned}
\det(A^-) &= \det A - 2a_{54}a_{15} \det A[1,2,3|2,3,4] - 2a_{51}a_{45} \det A[1,2,3|2,3,4] \\
&= \det A - 4a_{45}a_{15} \det A[1,2,3|2,3,4] \\
&= \det A - 4a_{45}a_{15}(-a_{24}a_{12} + a_{24}a_{13}a_{32} + a_{34}a_{12}a_{23} - a_{34}a_{13}) \\
&= \det A - 4(-a_{12}a_{24}a_{45}a_{51} + a_{13}a_{32}a_{24}a_{45}a_{51} + a_{12}a_{23}a_{34}a_{45}a_{51} \\
&\quad - a_{13}a_{34}a_{45}a_{51}).
\end{aligned}$$

Comparing with $W_A(5)$ as computed in Example 2.4, and since $a_{22} = a_{33} = 1$, the last expression equals $\det A - 4W_A(5)$ So we have the desired inequality $\det A \geq 4W_A(5)$.

(2) $\Rightarrow$ (1) Also in this case, for the sake of completeness, we include a sketch of Xu's proof that $\mu = a_{23}$ implies A completely positive. Let $\tilde{A} = SAS^T$, where $S = I - \mu E_{23}$. Then $\tilde{A}$ is positive semidefinite, being congruent to A, and the hypotesis $\mu = a_{23}$ ensures that it is non-negative. Furthermore, being the $(2,3)$-entry of $\tilde{A}$ zero, the graph $G(\tilde{A})$ is completely positive, since it does not contain cycles of odd length ≥ 5. Therefore $\tilde{A}$ is completely positive, and such is obviously A.

Let us assume now that $\mu = a_{13}/a_{12}$. Let $\tilde{A} = SAS^T$ as above. Since $\tilde{a}_{13} = a_{13} - a_{12}\mu = 0, \tilde{a}_{23} = a_{23} - \mu > 0, \tilde{a}_{33} = \mu^2 - 2a_{23}\mu + 1 = (a_{23} - \mu)^2 > 0, \tilde{a}_{34} = a_{34} - a_{24}\mu$, and the remaining elements of $\tilde{A}$ are as those in A, the inequalities $\mu \leq a_{12} \leq 1 \leq a_{34}/a_{24}, a_{23}$ ensure that $\tilde{A}$ is non-negative. But $G(\tilde{A})$ is isomorphic to G_2, so, by the preceding case, we can infer that $\tilde{A}$ is completely positive if we can prove that $\det \tilde{A} \geq 4W_{\tilde{A}}(5)$ (the vertex 5 is still a hat-vertex of $G(\tilde{A})$). Obviously $det\tilde{A} = \det A$. Taking care of the fact that $G(\tilde{A})$ is obtained from G_2 by vertex permutation $(1 \to 2 \to 3 \to 4 \to 5 \to 1)$, we get:

$$\begin{aligned}
W_{\tilde{A}}(5) &= \tilde{a}_{12}\tilde{a}_{23}\tilde{a}_{34}\tilde{a}_{45}\tilde{a}_{51} - \tilde{a}_{12}\tilde{a}_{24}\tilde{a}_{45}\tilde{a}_{51}\tilde{a}_{33} \\
&= a_{12}(-\mu + a_{23})(-\mu a_{24} + a_{34})a_{45}a_{51} - a_{12}a_{24}a_{45}a_{51}(\mu^2 - 2a_{23}\mu + 1) \\
&= a_{12}\Big(-\frac{a_{13}}{a_{12}} + a_{23}\Big)\Big(-\frac{a_{13}}{a_{12}}a_{24} + a_{34}\Big)a_{45}a_{51} \\
&\qquad - a_{12}a_{24}a_{45}a_{51}\Big(\frac{a_{13}^2}{a_{12}^2} - 2a_{23}\frac{a_{13}}{a_{12}} + 1\Big) \\
&= \frac{a_{13}^2}{a_{12}}a_{24}a_{45}a_{51} - a_{13}a_{34}a_{45}a_{51} - a_{23}a_{13}a_{24}a_{45}a_{51} + a_{12}a_{23}a_{34}a_{45}a_{51} \\
&\quad - a_{24}a_{45}a_{51}\frac{a_{13}^2}{a_{12}} + 2a_{24}a_{45}a_{51}a_{23}a_{13} - a_{12}a_{24}a_{45}a_{51}
\end{aligned}$$

$$= a_{13}a_{32}a_{24}a_{45}a_{51} + a_{12}a_{23}a_{34}a_{45}a_{51} - a_{12}a_{24}a_{45}a_{51} - a_{13}a_{34}a_{45}a_{51}$$
$$= W_A(5).$$

Note that $\tilde{a}_{33}$ cannot be omitted in the expression of $W_{\tilde{A}}(5)$ since it is not equal to 1, while $a_{33} = 1$ and $a_{22} = 1$ leads to their disappearance respectively from the last two terms in the expression of $W_A(5)$. We deduce that the hypothesis $\det A \geq 4W_A(5)$ is equivalent to $\det \tilde{A} \geq 4W_{\tilde{A}}(5)$, so $\tilde{A}$ is completely positive by CASE $G(A) = G_2$, and A is also obviously completely positive.

Assume now that $\mu = a_{34}/a_{24}$. The same matrix $\tilde{A}$ as above can be used, which in this case has $G(\tilde{A})$ exactly equal to G_2, since $\tilde{a}_{34} = 0$. Also in this case, with similar calculation, one can prove that $\tilde{A}$ is non-negative and that $W_{\tilde{A}}(5) = W_A(5)$, so the conclusion holds in a similar way as above. QED

It should be pointed out that, in case $G(A) \cong G_2$, condition $\det A[1,2|2,3] \leq 0$ in Theorem 1.4(2) is equivalent to requiring $W_A(5) \leq 0$, since $W_A(5) = a_{34}a_{45}a_{51} \det A[1,2|2,3]$; so in that case condition (2) in Theorem 3.1 is automatically verified by any matrix $A \in \mathcal{DNN}_5$.

We now provide two examples of completely positive matrices to show that the necessary conditions in Theorems 1.4 and 1.5 are wrong. Recall that, given a square matrix A, the principal submatrix A_k denotes the $k \times k$ upper-left corner of A.

Example 3.2. Let us consider the non-negative matrix

$$A = \begin{pmatrix} 1 & 1/2 & 1/5 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 1/5 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1 \end{pmatrix}.$$

Since all the principal submatrices A_k $(1 \leq k \leq 5)$ have positive determinant, A is positive definite, hence it is doubly non-negative. Note that $\det A[1,2|2,3] = 1/4 - 1/5 = 1/20 > 0$. Straightforward calculations give

$$4 \cdot W_A(5) = 1/40 < \det(A) = 3/25 < 4 \cdot A_{|5|} = 1/8$$

and, as shown in the proof of Theorem 3.1, $\det(A^-) = 19/200 = \det(A) - 4 \cdot W_A(5)$. The matrix A is completely positive, either by Barioli's Theorem 1.6, since A^- is positive definite, or by our Theorem 3.1.

Example 3.3. Let us consider the doubly non-negative matrix

$$A = \begin{pmatrix} 1 & 1/2 & 1/5 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 1/3 & 0 \\ 1/5 & 1/2 & 1 & 1/2 & 0 \\ 0 & 1/3 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1 \end{pmatrix}.$$

Note that $a_{13}/a_{12} = 2/5 < a_{23} = 1/2$. Straightforward calculations give

$$4 \cdot W_A(5) = -13/120 < \det(A) = 119/1800 < 4 \cdot A_{|5|} = 19/120.$$

The matrix A is completely positive, either by Barioli's Theorem 1.6, since A^- is positive definite, or by our Theorem 3.1.

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