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Zero-dimensional subschemes of projective spaces related to double points of linear subspaces and to fattening directions

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Abstract. Fix a linear subspace $V \subseteq \mathbb{P}^n$ and a linearly independent set $S \subset V$. Let $Z_{S,V} \subset V$ or $Z_{s,r}$ with $r := \dim(V)$ and $s = \sharp(S)$, be the zero-dimensional subscheme of V union of all double points $2p$, $p \in S$, of V (not of $\mathbb{P}^n$ if $n > r$). We study the Hilbert function of $Z_{S,V}$ and of general unions in $\mathbb{P}^n$ of these schemes. In characteristic 0 we determine the Hilbert function of general unions of $Z_{2,1}$ (easy), of $Z_{2,2}$ and, if $n = 3$, general unions of schemes $Z_{3,2}$ and $Z_{2,2}$.

Keywords: zero-dimensional scheme, Hilbert function, postulation

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Introduction

Fix $P \in \mathbb{P}^n$. Look at all possible zero-dimensional schemes Z with $Z_{\mathrm{red}} = \{P\}$ and invariant for the action of the group G_P of all $h \in \mathrm{Aut}(\mathbb{P}^n)$ such that $h(P) = P$. In characteristic zero we only get the infinitesimal neighborhoods mP of P in $\mathbb{P}^n$, $m > 0$, i.e the closed subschemes of $\mathbb{P}^n$ with $(\mathcal{I}_P)^m$ as its ideal sheaf. If we take two distinct points $P, Q \in \mathbb{P}^n$, $P \neq Q$, we also have a line (the line L spanned by the set $\{P, Q\}$) and it is natural to look at the zero-dimensional schemes $Z \subset \mathbb{P}^n$ such that $Z_{\mathrm{red}} = \{P, Q\}$ and $h^*(Z) \cong Z$ for all $h \in \mathrm{Aut}(\mathbb{P}^n)$ fixing P and Q (or, if we take non-ordered points fixing the set $\{P, Q\}$) and in particular fixing L. A big restriction (if $n > 1$) is to look only to the previous schemes Z which are contained in L, not just with $Z_{\mathrm{red}} = \{P, Q\} \subset L$. The easiest invariant zero-dimensional scheme (after the set $\{P, Q\}$) is the degree 4 zero-dimensional scheme $(2P + 2Q, L)$, i.e. the zero-dimensional subscheme $Z_{2,1}$ of L with 2 connected components, both of degree 2, and with $\{P, Q\}$ as its support. We call them $(2, 1)$-schemes. This is a kind of collinear zero-dimensional schemes and hence the Hilbert function of a general

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union of them is known ([1], [6]). We may generalize this construction in the following way.

For any linear space $V \subseteq \mathbb{P}^n$ and any $P \in V$ let $(2P, V)$ denote the closed subscheme of V with $(\mathcal{I}_{P,V})^2$ as its ideal sheaf. We have $(2P, V) = 2P \cap V$, $(2P, V)_{\mathrm{red}} = \{P\}$ and $\deg(2P, V) = \dim(V) + 1$. Fix integer $n \geq r \geq s - 1 \geq 0$. Fix an r-dimensional linear subspace $V \subseteq \mathbb{P}^n$ and a linearly independent set $S \subset V$ with $\sharp(S) = s$. Set $Z_{S,V} := \cup_{p \in S}(2p, V)$. $Z_{S,V}$ is a zero-dimensional scheme, $(Z_{S,V})_{\mathrm{red}} = S$, $Z_{S,V} \subset V$ and $\deg(Z_{S,V}) = s(r+1)$. Any two schemes $Z_{S',V'}$ and $Z_{S,V}$ with $\dim(V) = \dim(V')$ and $\sharp(S) = \sharp(S')$ are projectively equivalent. In the case $s = r + 1$ we may see $Z_{S,V}$ as the first order invariant of the linearly independent set S inside the projective space V (not the full projective space $\mathbb{P}^n$ if $r < n$) and V is exactly the linear span of S, so that it is uniquely determined by the set $S \subset \mathbb{P}^n$. In the case $s \leq r$, $Z_{S,V}$ is not uniquely determined by S. The scheme $Z_{S,V}$ prescribes some infinitesimal directions at each point of S, so that each connected component of $Z_{S,V}$ spans V. $Z_{S,V}$ is the minimal zero-dimensional subscheme $B \subset \mathbb{P}^n$ such that $B_{\mathrm{red}} \supseteq S$ and the linear span of each connected component A of B spans a linear space $V_A \supseteq V$. In all cases $Z_{S,V}$ depends only on S and V and not on the projective space containing V. An (s, r)-scheme of $\mathbb{P}^n$ or a scheme $Z_{s,r}$ of $\mathbb{P}^n$ is any scheme $Z_{S,V} \subset \mathbb{P}^n$ for some S, V with $\dim(V) = r$ and $\sharp(S) = s$. Set $Z_s := Z_{s,s-1}$. We have $\deg(Z_{s,r}) = s(r+1)$. If $\sharp(S) = 1$ we say that $Z_{S,V}$ is a 2-point of V. A scheme $Z_{1,2}$ is called a planar 2-point.

Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme. Consider the exact sequence

$$0 \to \mathcal{I}_Z(t) \to \mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{O}_Z(t) \to 0 \tag{0.1}$$

The exact sequence (0.1) induces the map $r_{Z,t} : H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_Z(t))$ (the restriction map). We say that Z has *maximal rank* if for each $t \in \mathbb{N}$ the restriction map $r_{Z,t}$ is either injective (i.e. $h^0(\mathcal{I}_Z(t)) = 0$) or surjective (i.e. $h^1(\mathcal{I}_Z(t)) = 0$). For each $t \in \mathbb{N}$ let $h_Z(t)$ be the rank of the restriction map $r_{Z,t}$. We have $h_\emptyset(t) = 0$ for all $t \in \mathbb{N}$. If $Z \neq \emptyset$, then $h_Z(0) = 1$ and the function $h_Z(t)$ is strictly increasing until it stabilizes to the integer $\deg(Z)$. The *regularity index* ρ of Z is the first $t \in \mathbb{N}$ such that $h_Z(t) = \deg(Z)$, i.e. such that $h^1(\mathcal{I}_Z(t)) = 0$. By the Castelnuovo - Mumford lemma the homogeneous ideal of Z is generated in degree $\leq \rho(Z) + 1$ and $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \geq \rho(Z)$. If Z is contained in a proper linear subspace $W \subset \mathbb{P}^n$, then each $h_Z(t)$ does not depend on whether one sees Z as a subscheme of $\mathbb{P}^n$ or of W and a minimal set of generators of the homogeneous ideal of Z in $\mathbb{P}^n$ is obtained lifting to $\mathbb{P}^n$ a minimal set of generators of the homogeneous ideal of Z in W and adding $n - \dim(W)$ linear equations. The integer $h_{Z_{S,V}}(t)$ only depends on r, s and t (see Remark 2). See Proposition 1 for the Hilbert function of each $Z_{S,V}$.

We study the Hilbert function of general unions of these schemes. There are obvious cases with non-maximal rank for $\mathcal{O}_{\mathbb{P}^n}(2)$, but we compute the exact values of $h_Z(2)$ (see Propositions 2 and 3). For $Z_{3,2}$ exceptional cases arise also with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$ if $d = 3$ and $n \leq 4$ (Proposition 4) and one case with $d = 4$ and $n = 3$ (as expected by the Alexander-Hirschowitz theorem [1], [4]) (see Theorem 3).

For the schemes $Z_{2,2}$ we prove the following results (only in characteristic zero).

Theorem 1. *Fix integers $n \geq 2$, $d \geq 3$ and $k \geq 2$. Let $Z \subset \mathbb{P}^n$ be a general union of k schemes $Z_{2,2}$. Then either $h^0(\mathcal{I}_Z(d)) = 0$ or $h^1(\mathcal{I}_Z(d)) = 0$.*

Theorem 2. *Fix integers $n \geq 2$, $d \geq 3$ and $k \geq 2$. Let $Z \subset \mathbb{P}^n$ be a general union of k schemes $Z_{2,2}$ and one planar 2-point. Then either $h^0(\mathcal{I}_Z(d)) = 0$ or $h^1(\mathcal{I}_Z(d)) = 0$, except in the case $(n, k, d) = (2, 2, 4)$ in which $h^0(\mathcal{I}_Z(4)) = h^1(\mathcal{I}_Z(4)) = 1$.*

For general unions of an arbitrary number of schemes $Z_{3,2}$ and $Z_{2,2}$ we prove the case $n = 3$ (see Theorem 3 for $\mathcal{O}_{\mathbb{P}^3}(d)$, $d \geq 3$).

We also explore the Hilbert function of general unions of zero-dimensional schemes and general lines (see Lemma 6 for a non-expected easy case with non maximal rank).

We work over an algebraically closed field $\mathbb{K}$ with characteristic 0. We heavily use this assumption to apply several times Remark 3.

1 Preliminaries

For any closed subscheme $Z \subset \mathbb{P}^n$ and every hyperplane $H \subset \mathbb{P}^n$ let $\mathrm{Res}_H(Z)$ be the closed subscheme of $\mathbb{P}^n$ with $\mathcal{I}_Z : \mathcal{I}_H$ as its ideal sheaf. For each $t \in \mathbb{Z}$ we have a residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z\cap H,H}(t) \to 0 \tag{1.1}$$

We say that (1.1) is the residual exact sequence of Z and H. We have $\mathrm{Res}_H(Z) \subseteq Z$. If Z is zero-dimensional, then $\deg(Z) = \deg(Z \cap H) + \deg(\mathrm{Res}_H(Z))$.

For any scheme $Z \subset \mathbb{P}^n$ let $h_Z : \mathbb{N} \to \mathbb{N}$ denote the Hilbert function of Z, i.e. for each $t \in \mathbb{N}$ let $h_Z(t)$ denote the rank of the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_Z(t))$.

Remark 1. Let $V \subseteq \mathbb{P}^n$ be an r-dimensional linear subspace. Assume $Z \subset V$. Since for each $t \in \mathbb{N}$ the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(V, \mathcal{O}_V(t))$ is surjective, the Hilbert function of Z is the same if we see Z as a subscheme of $\mathbb{P}^n$ or if we see it as a subscheme of the r-dimensional projective space V.

Remark 2. Fix integers $n \geq r > 0$ and s with $1 \leq s \leq r+1$. Fix linear spaces $V_i \subseteq \mathbb{P}^n$, $i = 1,2$, and sets $S_i \subset V_i$, $i = 1,2$, such that $\dim(V_i) = r$, $\sharp(S_i) = s$ and each S_i is linearly independent. Since there is $h \in \mathrm{Aut}(\mathbb{P}^n)$ with $h(V_1) = V_2$ and $h(S_1) = S_2$, Z_{S_1,V_1} and Z_{S_2,V_2} have the same Hilbert function.

Proposition 1. *Let $Z := Z_{S,V}$ be an (s,r)-scheme.*

(i) If $s = 1$, then $h_Z(t) = r+1$ for all $t \geq 1$ and the homogeneous ideal of $Z_{S,V}$ is generated by forms of degree ≤ 2.

(ii) If $s \geq 2$, then $h_Z(0) = 1$, $h_Z(1) = r+1$, $h_Z(2) = (r+1)s - s(s-1)/2$, $h_Z(t) = s(r+1)$ for all $t \geq 3$ and the homogeneous ideal of $Z_{S,V}$ is generated by forms of degree ≤ 4, but not of degree ≤ 3. Outside S the scheme-theoretic base locus of $|\mathcal{I}_Z(3)|$ is the union of all lines spanned by 2 of the points of S. Outside S the scheme-theoretic base locus of $|\mathcal{I}_Z(2)|$ is the linear span of S.

Proof. Since $Z \subset V$, we may assume $n = r$, i.e. $V = \mathbb{P}^r$ (Remark 1).

Part (i) is well-known. The case $r = 1$, $s \geq 2$ is also obvious, by the cohomology of line bundles on $\mathbb{P}^1$. Hence we may assume $r \geq 2$, $s \geq 2$ and use induction on the integer r. We assume that Proposition 1 is true for all pairs (s', r') with $1 \leq r' < r$ and $1 \leq s' \leq r'+1$. For any (s,r)-scheme Z we have $h_Z(0) = 1$ and $h_Z(1) = r+1$. Since S is linearly independent, we have $h^1(\mathcal{I}_S(t)) = 0$ for all $t > 0$.

(a) Assume $s \leq r$. Let $H \subset \mathbb{P}^r$ be a hyperplane containing S. We have $Z \cap H = Z_{S,H}$ and $\mathrm{Res}_H(Z) = S$. From the residual exact sequence (1.1) and the inductive assumption we get $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \geq 3$, $h^1(\mathcal{I}_Z(2)) = h^1(H, \mathcal{I}_{Z\cap H,H}(2))$ and $h^0(\mathcal{I}_Z(2)) = h^0(\mathcal{I}_{Z\cap H, V\cap H}(2)) + r + 1 - s$. The inductive assumption gives $h^1(H, \mathcal{I}_{Z\cap H,H}(2)) = s(s-1)/2$. Hence $h_Z(2) = \deg(Z) - s(s-1)/2 = s(r+1) - s(s-1)/2$. We also get that outside S the base locus of $|\mathcal{I}_Z(t)|$, $t = 2,3$, and of $|\mathcal{I}_{Z\cap H}(t)|$ are the same. By the Castelnuovo-Mumford's lemma the homogeneous ideal of Z is generated in degree ≤ 4. It is not generated in degree ≤ 3, because $|\mathcal{I}_Z(3)|$ has a one-dimensional base locus.

(b) Assume $s = r+1 \geq 3$. Since $V = \mathbb{P}^r$, S spans $\mathbb{P}^r$ and every quadric hypersurface of $\mathbb{P}^r$ has as its singular locus a proper linear subspace of $\mathbb{P}^r$, we have $h^0(\mathcal{I}_Z(2)) = 0$ and hence $h_Z(2) = \binom{r+2}{2}$ and $h^1(\mathcal{I}_Z(2)) = (r+1)^2 - \binom{r+2}{2} = (r+1)r/2 = s(s-1)/2$. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Let H be the hyperplane spanned by S'. We have $Z \cap H = Z_{S',H}$ and $\mathrm{Res}_H(Z) = 2p \cup S'$. We have $h^0(\mathcal{I}_{2p\cup S'}(1)) = 0$ and so $h^1(\mathcal{I}_{2p\cup S'}(1)) = s-1$. We have $\mathrm{Res}_H(2p \cup S') = 2p$. Since $h^1(\mathcal{I}_{2p}(x)) = 0$ for all $x > 0$ and $S' \subset H$ is linearly independent, the residual exact sequence of $2p \cup S'$ with respect to H gives $h^1(\mathcal{I}_{2p\cup S'}(t)) = 0$ for all $t \geq 2$. Hence (1.1) and the inductive assumption gives $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \geq 3$. The scheme-theoretic base locus of $|\mathcal{I}_Z(2)|$ is $V = \mathbb{P}^r$. The scheme-theoretic base locus E of $|\mathcal{I}_Z(3)|$ contains the union T of all lines spanned by two of the points of S, which in turn contains Z. Since Z is zero-dimensional, we have $h^1(Z, \mathcal{I}_{Z\cap H,Z}(3)) = 0$ and so the restriction map $H^0(\mathcal{O}_Z(3)) \to H^0(\mathcal{O}_{Z\cap H}(3))$ is surjective. Since $h^1(\mathcal{I}_Z(3)) = 0$, the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(3)) \to H^0(\mathcal{O}_{Z\cap H}(3))$ is surjective. Therefore $E \cap H$ is the scheme-theoretic base locus of $|\mathcal{I}_{Z\cap H,H}(3)|$ in H. By the inductive assumption we get $E \cap H = T'$ outside S'.

Now we check that $E_{\text{red}} = T$. If $r = 2$, then this is true (even scheme-theoretically), because $h^0(\mathcal{I}_Z(3)) = 1$ and so $|\mathcal{I}_Z(3)| = \{T\}$. Now assume $r \geq 3$ and that this assertion is true for lower dimensional projective spaces. Fix $q \in \mathbb{P}^r \setminus T$. Let $S'' \subseteq S$ be a minimal subset of S whose linear span contains q. Since S is linearly independent, if $S_1 \subseteq S$ and the linear span of S_1 contains q, then $S_1 \supseteq S''$. Since $q \notin T$, we have $\sharp(S'') \geq 3$. Take 3 distinct points p_1, p_2, p_3 of S'' and let H_i, $i = 1, 2, 3$, be the linear span of $S \setminus \{p_i\}$. Since any two points of S are contained in at least one of the hyperplanes H_1, H_2 or H_3, we have $T \subset H_1 \cup H_2 \cup H_3$. Since each point of S is contained in at least two of the hyperplanes H_i, we have $Z \subset H_1 \cup H_2 \cup H_3$. Since $p_i \in S''$, $i = 1, 2, 3$, we have $q \notin H_i$ and so $q \notin (H_1 \cup H_2 \cup H_3)$. Thus $q \notin E_{\text{red}}$. Hence $E_{\text{red}} = T$.

To conclude the proof of (ii) it is sufficient to prove that E is reduced outside S. For any set $B \subset \mathbb{P}^r$ let $\langle B \rangle$ denote its linear span. Fix $p \in T \setminus S$ and call p_1 and p_2 the points of S such that p is contained in the line ℓ spanned by $\{p_1, p_2\}$. Since $\ell \subset T$, it is sufficient to prove that ℓ is the Zariski tangent space T_pE of E at p. Assume the existence of a line $R \subset T_pE$ such that $p \in R$ and $R \neq \ell$. Let $S_R \subseteq S$ be a minimal subset of S whose linear span contains R. Since S is linearly independent, any $B \subseteq S$ with $R \subset \langle B \rangle$ contains S_R. Set $\alpha := \sharp(S_R)$. Since $R \nsubseteq T$, we have $\alpha \geq 3$. Let G be the set of all $h \in \text{Aut}(\mathbb{P}^r)$ such that $h(u) = u$ for all $u \in S$. Note that $g(Z_{S,V}) = Z_{S,V}$ for all $g \in G$. Set $G_p := \{g \in G : g(p) = p\}$. G_p acts transitively on the set Δ_R of all lines $L \subset \langle S_R \rangle$ such that $p \in L$ and S_R is the minimal subset of S whose linear span contains L. Since G_p acts transitively on Δ_R, each $L \in \Delta_R$ is contained in T_pE. Since T_pE is closed, it contains all lines $L_1 \subset \langle S_R \rangle$ such that $p \in L_1$. Hence (changing if necessary R) we reduce to the case $\alpha = 3$. Assume for the moment $\{p_1, p_2\} \subset S_R$ and write $S_R = \{p_1, p_2, p_3\}$. Set $\Pi := \langle S_R \rangle$. $E \cap \Pi$ contains $T \cap \Pi$, i.e. the 3 distinct lines of the plane Π spanned by 2 of the points of S_R. $E \cap \Pi \supsetneq T \cap \Pi$, because T_pE contains the line $\langle \{p, p_3\} \rangle$ and so the scheme $E \cap \Pi$ contains the tangent vector of $\langle \{p, p_3\} \rangle$ at p. Since $T \cap \Pi$ is a cubic curve, we get $\Pi \subset E$. Hence $E_{\text{red}} \neq T$, a contradiction. Now assume $S_R \cap \{p_1, p_2\} = \emptyset$. Since $p \in \ell$, we have $\langle S_R \rangle \cap \ell \neq \emptyset$ and so $S_R \cup \{p_1, p_2\}$ is not linearly independent, a contradiction. Now assume $\sharp(S_R \cap \{p_1, p_2\}) = 1$. Since $p \in \ell$, we get $\ell \subset \langle S_R \rangle$ and hence $\{p_1, p_2\} \subset S_R$, a contradiction. QED

Remark 3. Let X be an integral projective variety with $\dim(X) > 0$, $\mathcal{L}$ a line bundle on X and $V \subseteq H^0(\mathcal{L})$ any linear subspace. Take a general $p \in X_{\mathrm{reg}}$ and a general tangent vector A of X at p. We have $\dim(H^0(\mathcal{I}_A \otimes \mathcal{L}) \cap V) = \max\{0, \dim(V) - 2\}$, because (in characteristic zero) any non-constant rational map $X \dashrightarrow \mathbb{P}^r$, $r \geq 1$, has non-zero differential at a general $p \in X_{\mathrm{reg}}$.

Lemma 1. *Let* $V \subseteq H^0(\mathcal{O}_{\mathbb{P}^n}(2))$, $n \geq 2$, *be any linear subspace such that* $\dim(V) \geq n+2$. *Let* $L \subset \mathbb{P}^n$ *be a general line. Then* $\dim(V \cap H^0(\mathcal{I}_L(2))) = \dim(V) - 3$.

Proof. Since two general points of $\mathbb{P}^n$ are contained in a line, we have $\dim(V \cap H^0(\mathcal{I}_L(2))) \leq \max\{0, \dim(V) - 2\}$, without any assumption on $\dim(V)$. Let $\mathcal{B}$ denote the scheme-theoretic base locus. Since $\dim(V) \geq n+2 > h^0(\mathcal{O}_{\mathbb{P}^n}(1))$, we have $\dim(\mathcal{B}) \leq n-2$. Hence $\mathcal{B} \cap L = \emptyset$ for a general line L. Let $f : \mathbb{P}^n \setminus \mathcal{B} \to \mathbb{P}^r$, $r = \dim(V) - 1$, be the morphism induced by V. We have $\dim(V \cap H^0(\mathcal{I}_L(2))) = \dim(V) - 2$ if and only if $f(L)$ is a line. Assume that this is the case for a general L. Since any two points of $\mathbb{P}^n$ are contained in a line, we get that the closure Γ of $f(\mathbb{P}^n \setminus \mathcal{B})$ in $\mathbb{P}^r$ is a linear space. Since Γ spans $\mathbb{P}^r$, we get $\Gamma = \mathbb{P}^r$. Hence $\dim(V) = r+1 \leq n+1$, a contradiction. QED

Remark 4. Let $Z \subset \mathbb{P}^n$, $n \geq 2$, be a general union of k schemes $Z_{2,1}$. We have k general lines L_i, $1 \leq i \leq k$, of $\mathbb{P}^n$ and on each L_i a general subscheme of L_i with 2 connected components, each of them with degree 2. Set $T := L_1 \cup \cdots \cup L_k$. We have $h^0(\mathcal{I}_Z(2)) = h^0(\mathcal{I}_T(2))$, where $T \subset \mathbb{P}^n$ is a general union of k lines. If $n = 2$, then $h^0(\mathcal{I}_T(2)) = 0$ if $k \geq 3$ and $h^0(\mathcal{I}_T(2)) = \binom{4-k}{2}$ if $k = 1, 2$. If $n \geq 3$, then $h^0(\mathcal{I}_T(2)) = \max\{0, \binom{n+2}{2} - 3k\}$ ([7]).

Fix an integer $d \geq 3$. If $n = 2$ assume $4s \leq ds + 1 - \binom{s-1}{2}$ for all s with $2 \leq s \leq \min\{k, d+1\}$. Note that the family of all schemes Z has a degeneration Z' in which Z' has k connected components W_i, $1 \leq i \leq k$, with $W_i \subset L_i$ and $(W_i)_{\mathrm{red}}$ a general point of L_i. In the terminology of [2] each W_i is a collinear jet. By semicontinuity we have $h^0(\mathcal{I}_Z(d)) \leq h^0(\mathcal{I}_{Z'}(d))$ and $h^1(\mathcal{I}_Z(d)) \leq h^1(\mathcal{I}_{Z'}(d))$. Hence either $h^0(\mathcal{I}_Z(d)) = 0$ or $h^1(\mathcal{I}_Z(d)) = 0$ ([2]).

Notation Let $A \subseteq \mathbb{P}^n$ be a plane. Fix 3 non-collinear points $p_1, p_2, p_3 \in A$. Let $L, R \subset A$ be lines with $L \cap \{p_1, p_2, p_3\} = \{p_3\}$ and $R \cap \{p_1, p_2, p_3\} = \{p_2\}$. Let $Z[8] \subset \mathbb{P}^n$ denote any scheme projectively equivalent to $(2p_1, A) \cup (2p_2, A) \cup (2p_3, L)$. Let $Z[7] \subset \mathbb{P}^n$ denote any scheme projectively equivalent to $(2p_1, A) \cup (2p_2, A) \cup \{p_3\}$. In both cases p_3 is called the vertex of $Z[8]$ or of $Z[7]$ and L is called the vertex line of $Z[8]$. Let $Z[5] \subset \mathbb{P}^n$ (resp. $Z[4] \subset \mathbb{P}^n$, resp. $Z'[5] \subset \mathbb{P}^n$) denote any scheme projectively equivalent to $(2p_1, A) \cup (2p_2, R)$ (resp. $(2p_1, A) \cup \{p_2\}$, resp. $(2p_1, A) \cup \{p_2, p_3\}$).

Lemma 2. *Fix integers* $n \geq 2$, $d \geq 3$, $x \geq 0$ *and* $c \geq 0$ *and a zero-dimensional scheme* $\Gamma \subset \mathbb{P}^n$ *such that either* $h^0(\mathcal{I}_{\Gamma \cup W}(d)) = 0$ *or* $h^1(\mathcal{I}_{\Gamma \cup W}(d)) =$

0, where $W \subset \mathbb{P}^n$ is a general union of $x+c$ schemes $Z_{3,2}$. Let $Z \subset \mathbb{P}^n$ be a general union of x schemes $Z_{3,2}$ and c schemes $Z[8]$. Then either $h^0(\mathcal{I}_{\Gamma\cup Z}(d)) = 0$ or $h^1(\mathcal{I}_{\Gamma\cup Z}(d)) = 0$.

Proof. We use induction on c, the case $c = 0$ being true for all x by assumption. Assume $c > 0$ and set $e := h^0(\mathcal{I}_\Gamma(d)) - 9x - 8c$. First assume $e > 0$. Let $Z' \subset \mathbb{P}^n$ be a general union of $x+1$ schemes $Z_{3,2}$ and $c-1$ schemes $Z[8]$. The inductive assumption gives $h^0(\mathcal{I}_{\Gamma\cup Z'}(d)) = e-1$ and $h^1(\mathcal{I}_{\Gamma\cup Z'}(d)) = 0$. Since Z is general, we may find Z' with $Z' \supset Z$ and $h^1(\mathcal{I}_{\Gamma\cup Z'}(d)) = 0$. Thus $h^1(\mathcal{I}_{\Gamma\cup Z}(d)) = 0$. Now assume $e \le 0$. We need to prove that $h^0(\mathcal{I}_{\Gamma\cup Z}(d)) = 0$. Decreasing if necessary c we may assume $e \ge -7$. Let $E \subset \mathbb{P}^n$ be a general union of x schemes $Z_{3,2}$ and $c-1$ schemes $Z[8]$. Let $A \subset \mathbb{P}^n$ be a general plane. Let $U \subset A$ be a general scheme $Z_{3,2}$. Note that $(\Gamma \cup E) \cap U = \emptyset$ even if $n = 2$. The inductive assumption gives $h^1(\mathcal{I}_{\Gamma\cup E}(d)) = 0$, $h^0(\mathcal{I}_{\Gamma\cup E}(d)) = 8+e$ and $h^0(\mathcal{I}_{\Gamma\cup E\cup U}(d)) = 0$, i.e. U imposes $8+e$ independent conditions to $H^0(\mathcal{I}_{\Gamma\cup E}(d))$. Let U' be a minimal subscheme of U with $h^0(\mathcal{I}_{\Gamma\cup E\cup U'}(d)) = 0$. If $U' \subsetneq U$, then we may find $Z[8] \supseteq U'$ and so $h^0(\mathcal{I}_{\Gamma\cup Z}(d)) = 0$. Now assume $U' = U$. We need to find a contradiction. Write $U = U_1 \cup U_2 \cup U_3$ with $U_i = (2p_i, U)$ and p_1, p_2, p_3 general in A. If $8+e \le 6$ we use Remark 3 and that U contains 3 general tangent vectors. Assume $8+e = 7$. We get $h^0(\mathcal{I}_{\Gamma\cup E\cup U_i}(d)) < h^0(\mathcal{I}_{\Gamma\cup E}(d)) - 2$ for at least one index i, say $h^0(\mathcal{I}_{\Gamma\cup E\cup U_1}(d)) = h^0(\mathcal{I}_{\Gamma\cup E}(d)) - 3$; then we use Remark 3 and that we may find $Z[8] \subset A$ containing U_1 and 2 general tangent vectors of A. Now assume $8+e = 8$. In this case we first get $h^0(\mathcal{I}_{\Gamma\cup E\cup U_i\cup U_j}(d)) = h^0(\mathcal{I}_{\Gamma\cup E}(d)) - 6$ for some $i \ne j$ and then apply once Remark 3. QED

2 Proof of Theorems 1 and 2

Unless otherwise stated from now on a 2-point means a planar 2-point.

For all positive integers n, d set $u_{d,n} := \lfloor \binom{n+d}{n}/6 \rfloor$ and $v_{d,n} = \binom{n+d}{n} - 6u_{d,n}$. We have

$$6u_{d,n} + v_{d,n} = \binom{n+d}{d}, \ 0 \le v_{d,n} \le 5 \tag{2.1}$$

Note that if $Z \subset \mathbb{P}^n$ is a disjoint union of x schemes $Z_{2,2}$ we have $h^0(\mathcal{O}_Z(d)) \le h^0(\mathcal{O}_{\mathbb{P}^n}(d))$ if and only if $x \le u_{d,n}$. If $d \ge 2$ and $n \ge 2$ from (2.1) for the integers d and $d-1$ we get

$$6(u_{d,n} - u_{d-1,n}) + v_{d,n} - v_{d-1,n} = \binom{n+d-1}{n-1} \tag{2.2}$$

From (2.2) we get that $u_{d,n-1} = u_{d,n} - u_{d-1,n}$ and $v_{d,n-1} = v_{d,n} - v_{d-1,n}$ if $v_{d,n} \ge v_{d-1,n}$, while $u_{d,n-1} = u_{d,n} - u_{d-1,n} - 1$ and $v_{d,n-1} = 6 + v_{d,n} - v_{d-1,n}$ if $v_{d,n} < v_{d-1,n}$.

Proposition 2. *Fix integers $n \ge 2$ and $k > 0$. Let $Z \subset \mathbb{P}^n$ be a general union of k schemes $Z_{2,2}$.*

(a) Assume $n = 2$. If $k = 1$, then $h^0(\mathcal{I}_Z(2)) = 1$. If $k \ge 2$, then $h^0(\mathcal{I}_Z(2)) = 0$.

(b) Assume $n = 3$. If $k = 1$, then $h^0(\mathcal{I}_Z(2)) = 5$. If $k = 2$, then $h^0(\mathcal{I}_Z(2)) = 1$. If $k \ge 3$, then $h^0(\mathcal{I}_Z(2)) = 0$.

(c) If $n \ge 4$, then $h^0(\mathcal{I}_Z(2)) = \max\{0, \binom{n+2}{2} - 5k\}$.

Proof. Let W be any scheme of type $Z_{2,2}$ and let A be the plane containing W. We have $h^0(A, \mathcal{I}_{W,A}(2)) = 1$ (the only conic of A containing W is a double line) and hence $h^1(A, \mathcal{I}_{W,A}(2)) = 1$. Hence $h^0(\mathcal{I}_Z(2)) \ge \max\{0, \binom{n+2}{2} - 5k\}$ and $h^1(\mathcal{I}_Z(2)) \ge k$. Let J be the line spanned by $\{p_1, p_2\} := W_{\text{red}}$. We have $h^0(\mathcal{I}_W(2)) = H^0(\mathcal{I}_B(2))$, where B is the union of $(2p_1, A)$ and any degree 2 scheme with p_2 as its reduction, contained in A and not contained in J. Therefore it is sufficient to prove that a general union Z' of k degree 5 schemes projectively equivalent to B satisfies $h^0(\mathcal{I}_{Z'}(2)) = \max\{0, \binom{n+2}{2} - 5k\}$, except in the case $(n, k) = (3, 2)$. B is a scheme $Z[5]$.

If $n = 2$, then the result is obvious. Now assume $n = 3$. If $k = 1$, then Z is contained in a plane and from the case $n = 2, k = 1$ we get $h^1(\mathcal{I}_Z(2)) = 1$. If $k = 2$, then $h^0(\mathcal{I}_Z(2)) \ge 1$, because Z is contained in a reducible quadric. Let A_1, A_2 be the two planes containing the two schemes $Z_{2,2}$ of Z and call $N_i \subset A_i$ the scheme $Z_{2,2}$ contained in A_i and let L_i be the line spanned by $(N_i)_{\text{red}}$. Fix $p \in A_1 \setminus L_1$. By the case $n = 2$ we have $h^0(A_1, \mathcal{I}_{N_1 \cup \{p\}, A_1}(2)) = 0$. Taking $p = A_1 \cap L_2$ we get that $f_{|A_1} \equiv 0$ for each $f \in H^0(\mathcal{I}_Z(2))$ vanishes on A_1. Similarly $f_{|A_2} \equiv 0$. Hence $|\mathcal{I}_Z(2)| = \{A_1 \cup A_2\}$. The case $k = 2$ obviously implies the case $k \ge 3$.

Now assume $n \ge 4$ and that Proposition 2 is true in $\mathbb{P}^{n-1}$. It is sufficient to do the cases $k = \lfloor \binom{n+2}{2}/5 \rfloor$ and $k = \lceil \binom{n+2}{2}/5 \rceil$ and in particular we may assume that $k \ge \lceil (n+1)/3 \rceil$. Fix a hyperplane $H \subset \mathbb{P}^n$.

(i) First assume $n+1 \equiv 0, 2 \pmod 3$. Write $n+1 = 3a+2b$ with $a \in \mathbb{N}$ and $0 \le b \le 1$. Let A_i, $1 \le i \le a+b$, be general planes. If $1 \le i \le a$, let $L_i \subset A_i$ be a general line and let p_{i1} be a general point of L_i; set $\{p_{i2}\} := L_i \cap H$ and let v_i be the connected zero-dimensional scheme with p_{i2} as its support and contained in the line $H \cap A_i$. For $i = 1, \dots, a$ set $B_i := (2p_{i1}, A_i) \cup v_i$. Each B_i is a degree 5 subscheme of A_i projectively equivalent to the scheme B described in the first paragraph of the proof. If $b = 1$ set $L_{a+1} := A_{a+1} \cap H$, fix two general points p_{a+11} and p_{a+12} of L_{a+1}, and set $B_{a+1} := (2p_{a+1}, A_{a+1}) \cup v_{a+12}$, where v_{a+12} is a degree 2 zero-dimensional scheme contained in A_{a+1}, not contained L_{a+1} and with p_{a+12} as its support. Let $E \subset H$ be a general union of $k-a-b$ schemes $Z[5]$. Set $F := E \cup \bigcup_{i=1}^{a+b} B_i$. We have $\text{Res}_H(F) = \cup_{i=1}^{a} (2p_{i1}, A_i) \cup G$ with $G = \emptyset$ if $b = 0$ and $G = \{p_{a+11} \cup p_{a+12}\}$ if $b = 1$. Thus $h^i(\mathcal{I}_{\text{Res}_H(F)}(1)) = 0$, $i = 0, 1$. Set

$G' := \emptyset$ if $b = 0$ and $G' = (2p_{a+11}, L_{a+1}) \cup \{p_{a+12}\}$ if $b = 1$. The scheme $F \cap H$ is the union of E, G' and a general tangent vectors. The inductive assumption gives that either $h^0(H, \mathcal{I}_{E,H}(2)) = 0$ or $h^1(\mathcal{I}_{E,H}(2)) = 0$. If $h^0(\mathcal{I}_{E,H}(2)) = 0$, then we get $h^0(\mathcal{I}_F(2)) = 0$, proving Proposition 2 in this case. Now assume $h^1(H, \mathcal{I}_{E,H}(2)) = 0$. If $b = 0$, then it is sufficient to use Remark 3. Now assume $b = 1$. By Remark 3 we have $h^0(H, \mathcal{I}_{E\cup G',H}(2)) \leq \max\{0, h^0(\mathcal{I}_{E,H}(2)) - 2\}$ and to prove the proposition in this case it is sufficient to exclude the case $h^0(H, \mathcal{I}_{E\cup G',H}(2)) = h^0(\mathcal{I}_{E,H}(2)) - 2 > 0$, i.e. the case in which a general line of H imposes only 2 conditions to $|\mathcal{I}_{E,H}(2)|$.

First assume $n = 4$. We have $a = b = 1$ and it is sufficient to prove that $h^0(\mathcal{I}_Z(2)) = 0$ when $k = 3$. Since $k - a - b = 1$, we have $h^0(H, \mathcal{I}_{E,H}(2)) = 5$ and so $h^0(H, \mathcal{I}_{E\cup L,H}(2)) = 2$ for a general line $L \subset H$ by the case $n = 3$ of Lemma 1.

Now assume $n \geq 5$ and $b = 1$. We have $h^1(H, \mathcal{I}_{E\cup U,H}(2)) = 0$ for a general degree 5 scheme $U \subset H$ projectively equivalent to B by the inductive assumption. Since the base locus of $|\mathcal{I}_U(2)|$ contains the line spanned by U_{red}, we get $h^0(H, \mathcal{I}_{E\cup G'}(2)) = h^0(H, \mathcal{I}_{E,H}(2)) - 3$. Apply a times Remark 3.

(ii) Now assume $n \equiv 0 \pmod 3$. Write $a := n/3 - 1$. Let A_i, $1 \leq i \leq a+2$, be general planes. If $1 \leq i \leq a$, let $L_i \subset A_i$ be a general line and p_{i1} a general point of L_i; set $\{p_{i2}\} := L_i \cap H$ and let v_i be the connected zero-dimensional scheme with p_{i2} as its support and contained in the line $H \cap A_i$. For $i = 1, \dots, a$ set $B_i := (2p_{i1}, A_i) \cup v_i$. For $i = a+1, a+2$ set $L_i := A_i \cap H$, fix two general points p_{i1} and p_{i2} of L_i, and set $B_i := (2p_{i1}, A_i) \cup v_{i2}$, where v_{i2} is a degree 2 zero-dimensional scheme contained in A_i, not contained L_i and with p_{i2} as its support. Let $E \subset H$ be a general union of $k - a - 2$ schemes $Z[5]$. Set $F := E \cup \bigcup_{i=1}^{a+2} B_i$. We conclude as above using Remark 3 and twice Lemma 1. QED

Proposition 3. *Fix integers $n \geq 2$ and $k > 0$. Let $Z \subset \mathbb{P}^n$ be a general union of k schemes $Z_{2,2}$ and one planar 2-point.*

(a) If $n = 2$, then $h^0(\mathcal{I}_Z(2)) = 0$.

(b) Assume $n = 3$. If $k = 1$, then $h^0(\mathcal{I}_Z(2)) = 2$. If $k \geq 2$, then $h^0(\mathcal{I}_Z(2)) = 0$.

(c) Assume $n \geq 4$. Then $h^0(\mathcal{I}_Z(2)) = \max\{0, \binom{n+2}{2} - 5k - 3\}$.

Proof. Part (a) and the second half of part (b) follow from Proposition 2. Assume $n = 3$ and $k = 1$. Write $Z = U \sqcup M$ with U a $(2,2)$-scheme and M a planar 2-point. Since $h^0(\mathcal{I}_U(2)) = 5$ (Proposition 2), we have $h^0(\mathcal{I}_Z(2)) \geq 2$. Let N be the plane spanned by M and let L be the line spanned by U_{red}. Fix a general $p \in N$. For a general Z we have $U \cap N = \emptyset$ and $L \cap N$ is a general point of N. Since L is in the base locus $\mathcal{B}$ of $|\mathcal{I}_{Z\cup\{p\}}(2)|$, we have $N \subset \mathcal{B}$.

Since $h^0(\mathcal{I}_U(1)) = 1$, we get $h^0(\mathcal{I}_{Z\cup\{p\}}(2)) \le 1$ and so $h^0(\mathcal{I}_Z(2)) \le 2$. Now assume $n \ge 4$. Proposition 2 gives $h^0(\mathcal{I}_Z(2)) \ge \max\{0, \binom{n+2}{2} - 5k - 3\}$. By Proposition 2 and Remark 3 it is sufficient to do the case $k = \lfloor \binom{n+2}{2}/5 \rfloor$ and only for the integers $n \ge 4$ such that $\binom{n+2}{2} \equiv 3, 4 \pmod 5$. Fix a hyperplane $H \subset \mathbb{P}^n$. If $n+1 \equiv 0, 2 \pmod 3$ we use part (i) of the proof of Proposition 2 taking as $E \subset H$ a general union of one planar 2-point and $k-a-b$ scheme $Z[5]$; if $n \equiv 0 \pmod 3$ we use part (ii) of the proof of Proposition 2 with as $E \subset H$ a general union of a planar 2-point and $k-a-2$ schemes $Z[5]$. We explain now why this construction works. In the proof of Proposition 2 we constructed a zero-dimensional scheme $\mathcal{W} \subset \mathbb{P}^n$ for which we proved that either $h^0(H, \mathcal{I}_{\mathcal{W}\cap H,H}(2)) = 0$ or $h^1(\mathcal{I}_{\mathcal{W}\cap H,H}(2)) = 0$, $\deg(\mathrm{Res}_H(\mathcal{W})) = n+1$ and $\mathrm{Res}_H(\mathcal{W})$ is linearly independent. Thus $h^i(\mathcal{I}_{\mathrm{Res}_H(\mathcal{W})}(1)) = 0$, $i = 0, 1$. $\mathrm{Res}_H(\mathcal{W})$ does not depend on the scheme $E \subset H$ and so it is the same as in Proposition 2. Assume $n = 5c + 1$ with c a positive integer. We have $\binom{n+2}{2} = 5k + 3$ and so we need to prove that $h^i(\mathcal{I}_Z(2)) = 0$, $i = 0, 1$. So we need to prove that $h^i(H, \mathcal{I}_{\mathcal{W}\cap H,H}(2)) = 0$, $i = 0, 1$. We have the inductive assumption in H to handle E and then we continue as in steps (i) and (ii) of the proof of Proposition 2. QED

Lemma 3. *Let $G \subset \mathbb{P}^3$ be a general union of 3 planar 2-points. Then $h^1(\mathcal{I}_G(2)) = 0$ and $h^0(\mathcal{I}_G(2)) = 1$.*

Proof. Let A be the plane spanned by G_{red}. Keeping A fixed and moving G among the union of 3 planar 2-points with support on A we see that for a general G the scheme $G \cap A$ is a general union of 3 tangent vectors of A. Remark 3 gives $h^i(A, \mathcal{I}_{G\cap A}(2)) = 0$. Since $\mathrm{Res}_A(G) = G_{\mathrm{red}}$, we have $h^0(\mathcal{I}_{\mathrm{Res}_A(G)}(1)) = 1$ and $h^1(\mathcal{I}_{\mathrm{Res}_A(G)}(1)) = 0$. QED

Proofs of Theorems 1 and 2. First assume $n = 2$. Since any two points of $\mathbb{P}^n$ are collinear, in the case $n = 2$ of Theorem 1 (resp. Theorem 2) Z is a general union of $2k$ (resp. $2k+1$) general 2-points of $\mathbb{P}^2$. The Alexander-Hirschowitz list ([1], [4]) gives Theorems 1 and 2. We assume $n \ge 3$ and that Theorems 1 and 2 are true in $\mathbb{P}^{n-1}$. To prove Theorem 1 is sufficient to do the cases $k = \lfloor \binom{n+d}{n}/6 \rfloor = u_{d,n}$ and $k = \lceil \binom{n+d}{n}/6 \rceil$. Let $H \subset \mathbb{P}^n$ be a hyperplane.

(a) Assume $d = n = 3$. Since $u_{3,3} = 3$ and $v_{3,3} = 2$, to prove Theorem 1 it is sufficient to prove that $h^1(\mathcal{I}_Z(3)) = 0$ if $k = 3$ and $h^0(\mathcal{I}_Z(3)) = 0$ if $k = 4$. First assume $k = 3$. Let $Y \subset \mathbb{P}^3$ be a union of 3 schemes $Z_{2,2}$ such that one of them is contained in H and that each of the other ones have a point in its support contained in H and that Y is general with these restrictions. The scheme $Y \cap H$ is a general union of one scheme $Z_{2,2}$ (call it β) and 2 tangent vectors. We obviously have $h^1(H, \mathcal{I}_{\beta,H}(3)) = 0$ and hence $h^i(H, \mathcal{I}_{Y\cap H,H}(3)) = 0$, $i = 0, 1$, by Remark 3. Hence it is sufficient to prove that $h^1(\mathcal{I}_{\mathrm{Res}_H(Y)}(2)) = 0$. We have $\mathrm{Res}_H(Y) = U_1 \sqcup U_2$ with U_i spanning a general plane A_i and U_i union of a general 2-point $(2P_i, A_i)$ of A_i and a general point q_i of the line $A_i \cap H$. We have $h^1(\mathcal{I}_{(2P_1,A_1)\cup(2P_2,A_2)}(2)) = 0$ (Lemma 3) and so $h^0(\mathcal{I}_{(2P_1,A_1)\cup(2P_2,A_2)}(2)) = 4$. The scheme $(2P_1, A_1) \cup (2P_2, A_2)$ does not depend on H. Since any two points of $\mathbb{P}^3$ are collinear, moving H we may assume that (q_1, q_2) is a general element of $A_1 \times A_2$.

Since $h^0(\mathcal{I}_{A_1\cup(2P_2,A_2)}(2)) = h^0(\mathcal{I}_{(2P_2,A_2)}(1)) = 1$ and q_1 is general in A_1, we get $h^0(\mathcal{I}_{(2P_1,A_1)\cup(2P_2,A_2)\cup\{q_1\}}(2)) = 3$. Since $h^0(\mathcal{I}_{(2P_1,A_1)\cup(2P_2,A_2)\cup\{q_1\}\cup A_2}(2)) = h^0(\mathcal{I}_{(2P_2,A_2)\cup\{q_1\}}(1)) = 1$, we obtain that $h^0(\mathcal{I}_{(2P_1,A_1)\cup(2P_2,A_2)\cup\{q_1\}\cup\{q_2\}}(2)) = 2$, i.e. we have $h^1(\mathcal{I}_{\mathrm{Res}_H(Y)}(2)) = 0$.

Now assume $k = 4$. Since two general points of $\mathbb{P}^n$ are contained in a scheme $Z_{2,2}$ and $v_{3,3} = 2$, the case $k = 4$ of Theorem 1 follows from the case $k = 3$. For Theorem 2 it is sufficient to do the cases $k = 2$ (true because Z is contained in a general disjoint union of 3 schemes $Z_{2,2}$) and $k = 3$ (we use that a planar 2-point contains a tangent vector, Remark 3 and that $v_{3,3} = 2$).

(b) Assume $d = 3$ and $n \geq 4$. To prove Theorem 1 is sufficient to prove the cases $k = \lfloor \binom{n+3}{3}/6 \rfloor$ and $k = \lceil \binom{n+3}{3}/6 \rceil$. Fix any k disjoint schemes B_i projectively equivalent to $Z_{2,2}$. Let A_i be the plane containing B_i and let L_i be the line spanned by the reduction of B_i. We assume that $L_i \cap L_j = \emptyset$ for all i, j such that $i \neq j$. Since $\binom{6}{2} = 15$ and $\binom{n+2}{2} \geq 20$ for all $n \geq 5$, we may write $\binom{n+2}{2} = 5x + 4a$ with a, x non-negative integers and $0 \leq a \leq 4$. Now we check that $k \geq x + a$. Assume $k \leq x + a - 1$. Since $a \leq 4$, we get $5k \leq 5x + 5a - 5 \leq \binom{n+2}{2} - 1$, contradicting the inequality $6k \geq \binom{n+3}{3} - 5$. Let $G \subset \mathbb{P}^n$ be a general union of x schemes of type $Z_{2,2}$ and let $S \subset H$ be the intersection with H of the lines associated to each scheme $Z_{2,2}$. Since G is general, these x lines are x general lines of $\mathbb{P}^n$ and so S is a general subset of H. Fix a general planes A_i, $1 \leq i \leq a$, and let $B_i \subset A_i$ be a general scheme of type $Z_{2,2}$ with the restriction that one of the points of $(B_i)_{\mathrm{red}}$ is contained in $H \cap A_i$. Let $E \subset H$ be a general union of $k - a - x$ schemes $Z_{2,2}$ of H. Set $Y := G \cup E \cup \bigcup_{i=1}^a B_i$. For all $i = 1, \ldots, a$ the scheme $\mathrm{Res}_H(B_i)$ is a general union of a 2-point of A_i and a general point of the line $A_i \cap H$. Let M be the union of the reduced components of $\cup_{i=1}^a \mathrm{Res}_H(B_i)$. Since any $Z_{2,2}$ is a degeneration of a family of general planar 2 points, by Proposition 3 we have $h^1(\mathcal{I}_{\mathrm{Res}_H(Y)\setminus M}(2)) = 0$ and so $h^0(\mathcal{I}_{\mathrm{Res}_H(Y)\setminus M}(2)) = a$. $\mathrm{Res}_H(Y) \setminus M$ does not depend on H. Since $a \leq 4 \leq n$, any a points of $\mathbb{P}^n$ are contained in a hyperplane. Hence, writing $M = \{q_1, \ldots, q_a\}$ with $q_i \in A_i$, we may assume that $(q_1, \ldots, q_a)$ is a general element of $\times_{i=1}^a A_i$. As in step (a) we get $h^i(\mathcal{I}_{\mathrm{Res}_H(Y)}(2)) = 0$, $i = 0, 1$. Hence it is sufficient to prove that $h^0(H, \mathcal{I}_{S\cup(Y\cap H)}(3)) = \max\{0, \binom{n+2}{3} -$

$6(k-a-x)-x-2a\} = \max\{0, \binom{n+3}{3}-6k\}$. Since $S \subset H$ is general, it is sufficient to prove that $h^0(H, \mathcal{I}_{Y\cap H}(3)) = \max\{0, \binom{n+3}{3}-6k+x\}$. We have either $h^1(H, \mathcal{I}_{E,H}(3)) = 0$ or $h^0(H, \mathcal{I}_{E,H}(3)) = 0$ by step (a) and the inductive assumption. Therefore we may assume that $h^1(H, \mathcal{I}_{E,H}(3)) = 0$. Use Remark 3. To check Theorem 2 for $d = 3$ and $n \geq 4$ we leave to the reader at least 3 options. We may add in H one more planar 2-point or add it outside H using the integers x', a' with $\binom{n+2}{2} = 3+5x'+4a'$ or insert with its support on H, but not contained in H so that in the residual we have a general point (taking integers x'', a'' with $\binom{n+2}{2} = 1+5x''+4a''$ and whose intersection with H is a general tangent vector).

(c) Assume $d \geq 4$ and $n \geq 3$. We prove Theorem 1 for (n, d). By steps (a), (b) and induction on d we may assume that Theorems 1 and 2 are true in $\mathbb{P}^n$ for the integer $d-1$. In all cases for Theorem 1 it is sufficient to do the case $k = \lfloor \binom{n+d}{n}/6 \rfloor$ and $k = \lceil \binom{n+d}{n}/6 \rceil$. In particular we assume $6k \geq \binom{n+d}{n} - 5$.

(c1) First assume that $v_{d-1,n}$ is even. Write $\binom{n+d-1}{n} = 6x+4a$ with x, a non-negative integers and $0 \leq a \leq 2$. Since $6k \geq \binom{n+d}{n} - 5$ and $a \leq 2$, we have $k \geq a+x$. Let $G \subset \mathbb{P}^n$ be a general union of x schemes $Z_{2,2}$. Fix a general planes A_i, $1 \leq i \leq a$ and let $B_i \subset A_i$ be a general scheme of type $Z_{2,2}$ with the restriction that one of the points of $(B_i)_{\mathrm{red}}$ is contained in $H \cap A_i$. Let $E \subset H$ be a general union of $k-a-x$ schemes $Z_{2,2}$ of H. Set $Y := G \cup E \cup \bigcup_{i=1}^{a} B_i$. By Remark 3 and the inductive assumption on n, we may assume that either $h^0(H, \mathcal{I}_{Y\cap H,H}(d)) = 0$ or $h^1 H, (\mathcal{I}_{Y\cap H,H}(d)) = 0$.

Claim 1: We have $h^i(\mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) = 0$.

Proof of Claim 1: Since $\deg(\mathrm{Res}_H(Y)) = \binom{n+d-1}{n}$, it is sufficient to prove that $h^1(\mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) = 0$. For $1 \leq i \leq a$ write $B_i = (2p_{i1}, A_i) \cup (2p_{i2}, A_i)$ with $p_{i2} \in A_i \cap H$. We have $\mathrm{Res}_H(B_i) = (2p_{i1}, A_i) \cup \{p_{i2}\}$. The inductive assumption gives $h^1(\mathcal{I}_G(d-1)) = 0$. Hence Claim 1 is true if $a = 0$. Now assume $a = 1$. By the inductive assumption we have $h^0(\mathcal{I}_{G\cup B_1}(d-1)) = 0$. By the inductive assumption for Theorem 2 we have $h^1(\mathcal{I}_{G\cup(2p_{11},A_1)}(d-1)) = 0$. Since $h^0(\mathcal{I}_{G\cup(2p_{11},A_1)}(d-1)) > 0$ and $h^0(\mathcal{I}_{G\cup B_1}(d-1)) = 0$, A_1 is not contained in the base locus of $|\mathcal{I}_{G\cup(2p_{11},A_1)}(d-1)|$. For a general H we may assume that p_{12} is a general point of A_1. Hence $h^1(\mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) = 0$. Now assume $a = 2$. The inductive assumption gives $h^1(\mathcal{I}_{G\cup B_1}(d-1)) = 0$ and hence by semicontinuity $h^1(\mathcal{I}_{G\cup(2p_{11},A_1)\cup(2p_{21},A_2)}(d-1)) = 0$ and so $h^0(\mathcal{I}_{G\cup(2p_{11},A_1)\cup(2p_{21},A_2)}(d-1)) = 2$. The scheme $G \cup (2p_{11}, A_1) \cup (2p_{21}, A_2)$ does not depend from H. Since any two points of $\mathbb{P}^n$ are collinear, moving H we see that we may take as (p_{21}, p_{22}) a general element of $A_1 \times A_2$. Hence $h^0(\mathcal{I}_{G\cup(2p_{11},A_1)\cup(2p_{21},A_2)\cup\{p_{2i}\}}(d-1)) = 2$ if and only A_i is in the base locus of $|\mathcal{I}_{G\cup(2p_{11},A_1)\cup(2p_{21},A_2)}(d-1)|$. By monodromy for general A_1, A_2 if one of them is in the base locus, then so is the other one. But in this case we would have $h^0(\mathcal{I}_{G\cup B_1\cup B_2}(d-1)) = 2$, contradicting the inductive assumption. Now assume $h^0(\mathcal{I}_{G\cup(2p_{11},A_1)\cup(2p_{21},A_2)\cup\{p_{12}\}}(d-1)) = 1 = h^0(\mathcal{I}_{G\cup(2p_{11},A_1)\cup(2p_{21},A_2)\cup\{p_{21},p_{22}\}}(d-1))$. Since p_{22} is general in A_2, we get $1 = h^0(\mathcal{I}_{G\cup(2p_{11},A_1)\cup\{p_{11}\}\cup A_2}(d-1))$. The inductive assumption for Theorem 2 gives $h^0(\mathcal{I}_{G\cup(2p_{11},A_1)\cup B_2}(d-1)) = 0$, a contradiction. QED

(c2) Now assume that $v_{d-1,n}$ is odd. Let $\Gamma \subset \mathbb{P}^n$ be a zero-dimensional scheme and let p be a general point of H. By the differential Horace lemma to prove that $h^1(\mathcal{I}_{\Gamma\cup K}(d)) = 0$ (resp. $h^0(\mathcal{I}_{\Gamma\cup K}(d)) = 0$ for a general planar 2-point $K \subset \mathbb{P}^n$ it is sufficient to prove that $h^1(H, \mathcal{I}_{(\Gamma\cap H)\cup\{p\},H}(d)) = h^1(\mathcal{I}_{\mathrm{Res}_H(\Gamma)\cup v}(d-1)) = 0$ (resp. $h^0(H, \mathcal{I}_{(\Gamma\cap H)\cup\{p\},H}(d)) = h^0(\mathcal{I}_{\mathrm{Res}_H(\Gamma)\cup v}(d-1)) = 0$), where $v \subset H$ is a general tangent vector of H with p as its support ([3]). Instead of K we may use a general $Z_{2,2}$. Instead of v we get a scheme $B \subset A$ with A general plane containing p, $\deg(B) = 5$ and B a disjoint union of the general planar 2-point of A and a general tangent vector of the line $A \cap H$. Let $B' \supset B$ denote the scheme $Z_{2,2}$ containing B. Since $v_{d-1,n}$ is odd, $\binom{n+d-1}{n}$ is odd and so we may write $\binom{n+d-1}{n} - 5 = 6x + 4a$ with x, a non-negative integers and $0 \le a \le 2$. Since $6k \ge \binom{n+d}{n} - 5$, and $a \le 2$, we have $k \ge x+a+1$. Let $G \subset \mathbb{P}^n$ be a general union of x schemes $Z_{2,2}$. Fix a general planes A_i, $1 \le i \le a$ and let $B_i \subset A_i$ be a general scheme of type $Z_{2,2}$ with the restriction that one of the points of $(B_i)_{\mathrm{red}}$ is contained in $H \cap A_i$. Let A_{a+1} be a general plane. Let B_{a+1} be a general scheme $Z_{2,2}$ whose reduction spans the line $A_{a+1} \cap H$. Let $E \subset H$ be a general union of $k-x-a-1$ schemes $Z_{2,2}$ of H. Set $Y := G \cup E \cup \bigcup_{i=1}^{a+1} B_i$. By the inductive assumption on n either $h^0(H, \mathcal{I}_{(Y\cap H)\cup\{p\}}(d)) = 0$ or $h^1(H, \mathcal{I}_{(Y\cap H)\cup\{p\}}(d)) = 0$. Therefore it is sufficient to prove that $h^i(\mathcal{I}_{\mathrm{Res}_H(Y)\cup B'}(d-1)) = 0$, $i = 0, 1$. Let γ (resp. β) be the union of the connected components of $\mathrm{Res}_H(Y) \cup B'$ contained in H (resp. not contained in H). Note that $\beta \cap H = \emptyset$. β is a general union of a 2-point of A and several $Z_{2,2}$. The inductive assumption for Theorem 2 gives $h^1(\mathcal{I}_\beta(d-1)) = 0$, i.e. $h^0(\mathcal{I}_\beta(d-1)) = \deg(\gamma)$. The scheme γ is a general union of a tangent vector of $H \cap A$ and a points of H. Since $\beta \cap H = \emptyset$, by Remark 3 we have $h^0(\mathcal{I}_{\beta\cup\gamma}(d-1)) = \max\{0, h^0(\mathcal{I}_\beta(d-2))\}$. Even when $d = 4, 5$ we have $h^0(\mathcal{I}_\beta(d-2)) = 0$ by the inductive assumption, because $\deg(\beta) = \binom{n+d-1}{n} - a - 2$ and $0 \le a \le 2$.

(d) To conclude we need to prove Theorem 2 for (n, d). By Remark 3 and Theorem 1 for (n, d) it is sufficient to do the case $k = \lfloor \binom{n+d}{n}/6 \rfloor$ and $\binom{n+d}{n} \equiv 3, 4, 5 \pmod 6$. If $v_{d-1,n}$ is even we make the same construction as in steps (c1) taking instead of E a general union $E' \subset H$ of $k-x-a$ schemes $Z_{2,2}$ and a planar 2-point. The case considered in (c2) is easier: take the planar 2-scheme outside H and define x, a by the relations $\binom{n+d-1}{n} - 3 = 6x + 4a$, $0 \le a \le 2$. QED

3 General unions of schemes $Z_{3,2}$

Consider the following assertion $H_{d,n}$:

Assertion $H_{d,n}$: For all $x, y \in \mathbb{N}$ either $h^0(\mathcal{I}_Z(d)) = 0$ or $h^1(\mathcal{I}_Z(d)) = 0$ for a general union $Z \subset \mathbb{P}^n$ of x schemes $Z_{3,2}$ and y schemes $Z_{2,2}$.

For all positive integers n, d set $a_{d,n} := \lfloor \binom{n+d}{n}/9 \rfloor$ and $b_{d,n} = \binom{n+d}{n} - 9a_{d,n}$. We have

$$9a_{d,n} + b_{d,n} = \binom{n+d}{d},\ 0 \le b_{d,n} \le 8 \tag{3.1}$$

If $d \ge 2$ and $n \ge 2$ from (3.1) for the integers d and $d-1$ we get

$$9(a_{d,n} - a_{d-1,n}) + b_{d,n} - b_{d-1,n} = \binom{n+d-1}{n-1} \tag{3.2}$$

From (3.2) we get that $a_{d,n-1} = a_{d,n} - a_{d-1,n}$ and $b_{d,n-1} = b_{d,n} - b_{d-1,n}$ if $b_{d,n} \ge b_{d-1,n}$, while $a_{d,n-1} = a_{d,n} - a_{d-1,n} - 1$ and $b_{d,n-1} = 9 + b_{d,n} - b_{d-1,n}$ if $b_{d,n} < b_{d-1,n}$.

Our original aim was the construction of exceptional cases for general unions of these zero-dimensional schemes and a prescribed number of lines. See Lemma 6 for one such case.

In the next section we prove $H_{d,3}$ for all $d \ge 5$ and give the list of all exceptional cases in $\mathbb{P}^3$ for $d = 3, 4$ (Theorem 3). We list the possible values $h^0(\mathcal{I}_Z(d))$ if $d = 3$, $n = 4$ and Z is unions of $Z_{3,2}$ (Proposition 4).

Lemma 4. *Fix integers $n \ge 2$ and $k > 0$. Let $Z \subset \mathbb{P}^n$ be a general union of k schemes $Z_{3,2}$.*

(a) If $n = 2$, then $h^0(\mathcal{I}_Z(2)) = 0$.

(b) Assume $n = 3$. If $k = 1$, then $h^0(\mathcal{I}_Z(2)) = 4$. If $k = 2$, then $h^0(\mathcal{I}_Z(2)) = 1$. If $k \ge 3$, then $h^0(\mathcal{I}_Z(2)) = 0$.

(c) Assume $n = 4$. If $k = 1$ (resp. $k = 2$, resp. $k \ge 3$), then $h^0(\mathcal{I}_Z(2)) = 9$ (resp. $h^0(\mathcal{I}_Z(2)) = 4$, resp. $h^0(\mathcal{I}_Z(2)) = 0$).

(d) If $n \ge 5$, then $h^0(\mathcal{I}_Z(2)) = \max\{0, \binom{n+2}{2} - 6k\}$.

Proof. Let W be a $Z_{3,2}$-configuration of Z and A the plane containing W. Since $H^0(\mathcal{I}_W(2)) = H^0(\mathcal{I}_A(2))$, we have $h^0(\mathcal{I}_Z(2)) = h^0(\mathcal{I}_T(2))$, where T is the union of the planes containing the $Z_{3,2}$-configurations of Z. T is a general union of k planes of $\mathbb{P}^n$. Thus parts (a) and (b) and the case $k = 1$ of parts (c) and (d) are obvious.

Now assume $n = 4$ and $k > 1$. If T_1 and T_2 are general planes, then every quadric hypersurface containing $T_1 \cup T_2$ is a cone with vertex containing the point $T_1 \cap T_2$. Taking the linear projection from the linear span of $\mathrm{Sing}(T)$ we get part (c), (in the case $k = 3$, because $h^0(\mathbb{P}^1, \mathcal{I}_\beta(2)) = 0$ for any degree 3 scheme $\beta \subset \mathbb{P}^1$).

Part (d) follows from [5, Theorem 3.2] applied to T. QED

Lemma 5. *Let $T \subset \mathbb{P}^3$ be a general union of one $Z_{2,2}$ and 3 lines. Then $h^1(\mathcal{I}_T(3)) = 0$. If Z is a general union of one $Z_{2,2}$ and 3 general collinear degree 4 schemes, then $h^1(\mathcal{I}_Z(3)) = 0$.*

Proof. Write $T = B \sqcup R$ with B zero-dimensional and R a union of 3 general lines. Note that $|\mathcal{I}_R(2)|$ is formed by a unique quadric, which is smooth and in particular that $h^1(\mathcal{I}_R(2)) = 0$. Let $H \subset \mathbb{P}^3$ be the plane spanned by B. Since $R \cap H$ is a general union of 3 points, we have $h^1(H, \mathcal{I}_{T \cap H, H}(3)) = 0$. Use that $\mathrm{Res}_H(T) = R$, $h^1(\mathcal{I}_R(2)) = 0$ and the residual exact sequence of T and H. The statement for Z follows from the one for T, because $h^0(\mathcal{O}_Z(3)) = h^0(\mathcal{O}_T(3))$ and $H^0(\mathcal{I}_W(3)) = H^0(\mathcal{I}_L(3))$ for any line $L \subset \mathbb{P}^3$ and any zero-dimensional scheme $W \subset L$ with $\deg(W) = 4$. QED

We found the following counterexample if we also add lines.

Lemma 6. *Let $T \subset \mathbb{P}^3$ be a general union of one $Z_{3,2}$ and 2 lines and let $Y \subset \mathbb{P}^3$ be a general union of one $Z_{3,2}$ and one line. Then $h^1(\mathcal{I}_T(3)) = 1$, $h^0(\mathcal{I}_T(3)) = 4$ and $h^1(\mathcal{I}_Y(3)) = 0$.*

Proof. Write $T = B \sqcup R \sqcup L$ with B a $Z_{3,2}$-scheme and R, L lines. Let H be the plane spanned by B. We have $\mathrm{Res}_H(T) = R \cup L$ and hence $h^0(\mathcal{I}_{\mathrm{Res}_H(T)}(2)) = 4$. Since $h^0(H, \mathcal{I}_{Z_{3,2}}(3)) = 1$ and $R \cap H$ is general in H, we have $h^0(H, \mathcal{I}_{H \cap T, H}(3)) = 0$. The residual exact sequence of T and H gives $h^0(\mathcal{I}_T(3)) = h^0(\mathcal{I}_{\mathrm{Res}_H(T)}(2)) = 4$. Since $h^0(\mathcal{I}_T(3)) = 4$, we have $h^1(\mathcal{I}_T(3)) = 1$.

Take $Y = B \cup R$. We have $h^i(\mathcal{I}_{Y \cap H, H}(3)) = 0$, $i = 0, 1$ and hence $h^1(\mathcal{I}_Y(3)) = h^1(\mathcal{I}_R(2)) = 0$. QED

Proposition 4. *Let $Z \subset \mathbb{P}^n$, $n \geq 2$, be a general union of k schemes $Z_{3,2}$.*

(i) Assume $n = 2$. We have $h^0(\mathcal{I}_Z(3)) = 1$ and $h^1(\mathcal{I}_Z(3)) = 0$ if $k = 1$, and $h^0(\mathcal{I}_Z(3)) = 0$ if $k \geq 2$.

(ii) Assume $n = 3$. We have $h^0(\mathcal{I}_Z(3)) = 11$ and $h^1(\mathcal{I}_Z(3)) = 0$ (resp. $h^0(\mathcal{I}_Z(3)) = 4$ and $h^1(\mathcal{I}_Z(3)) = 2$, resp. $h^0(\mathcal{I}_Z(3)) = 1$ and $h^1(\mathcal{I}_Z(3)) = 8$, resp. $h^0(\mathcal{I}_Z(3)) = 0$) if $k = 1$ and $h^1(\mathcal{I}_Z(3)) = 8$ (resp. $k = 2$, resp. $k = 3$, resp. $k \geq 4$).

(iii) Assume $n = 4$. We have $h^1(\mathcal{I}_Z(3)) = 0$ if $k \leq 3$, $h^0(\mathcal{I}_Z(k)) = 0$ if $k \geq 5$ and $h^0(\mathcal{I}_Z(3)) = 1$, $h^1(\mathcal{I}_Z(3)) = 2$ if $k = 4$.

Proof. Proposition 1 gives $|\mathcal{I}_Z(3)| = |\mathcal{I}_T(3)|$, where T is the union of the 3 lines spanned by each pair of points of the reduction of any $Z_{3,2} \subseteq Z$. Part (i) and all cases with $k = 1$ follow. Now assume $n > 2$ and $k \geq 2$. The values of $h^1(\mathcal{I}_Z(3))$ in parts (ii) and (iii) are uniquely determined by n, k and the values of $h^0(\mathcal{I}_Z(2))$.

(a) Assume $n = 3$. We have $\binom{6}{3} = 2 \cdot 9 + 2$. Let A_1, A_2, A_3, A_4 be general planes and B_i a general union of 3 lines of A_i. We have $h^0(\mathcal{I}_{A_1 \cup A_2}(3)) = 4$. We fix A_1 and B_1. For a general A_2, the line $A_1 \cap A_2$ is a general line of A_1. For a general B_2 the set $B_2 \cap (A_1 \cap A_2)$ contains a general point of $A_1 \cap A_2$. Hence $h^0(A_1, \mathcal{I}_{B_1 \cup (B_2 \cap A_2)}(3)) = 0$. In the same way we get $h^0(A_i, \mathcal{I}_{B_i \cap (B_j \cap A_i)}(3)) = 0$ for all $i \neq j$. Hence $h^0(\mathcal{I}_{B_1 \cup B_2}(3)) = h^0(\mathcal{I}_{A_1 \cup A_2}(3)) = 4$, $h^0(\mathcal{I}_{B_1 \cup B_2 \cup B_3}(3)) = h^0(\mathcal{I}_{A_1 \cup A_2 \cup A_3}(3)) = 1$. Therefore $h^0(\mathcal{I}_Z(3)) = 0$ if $k \geq 4$.

(b) Assume $n = 4$. We have $\binom{7}{3} = 3 \cdot 9 + 8$. Let $H \subset \mathbb{P}^4$ be a hyperplane. Let A_i, $1 \leq i \leq k$, be general planes. Let $E_i \subset A_i$ be a general reducible conic. Set $L_i := A_i \cap H$ and $T_i := E_i \cup L_i$.

Claim 1: We have $h^i(\mathcal{I}_{E_1 \cup E_2 \cup E_3}(2)) = 0$, $i = 0, 1$.

Proof of Claim 1: Since $h^0(\mathcal{O}_{E_1 \cup E_2 \cup E_3}(2)) = 15$, the claim is equivalent to prove that a general union of 3 reducible conics is contained in no quadric hypersurface. We degenerate $E_2 \cup E_3$ to $F := F_1 \cup F_2 \cup v_1 \cup v_2$ with $F_1 \cup F_2 \subset H$, $F_1 \cup F_2$ a connected nodal union of 4 lines with arithmetic genus 1, F_1 and F_2 reducible conics, v_1 and v_2 tangent vectors not contained in H and supported at the two points of $F_1 \cap F_2$. Set $Y := F \cup E_1$. We have $H \cap F = F_1 \cup F_2$ and $Y \cap H = F_1 \cup F_2 \cup (E_1 \cap H)$ with $H \cap E_1$ two general points of H. Hence $h^0(H, \mathcal{I}_{Y \cap H, H}(2)) = 0$. We have $\mathrm{Res}_H(Y) = E_1 \cup (F_1 \cap F_2)$. Since $h^0(\mathcal{I}_{E_1 \cup (F_1 \cap F_2)}(1)) = 0$ for general E_1 and $F_1 \cup F_2$, a residual exact sequence gives Claim 1. QED

Since $h^1(H, \mathcal{I}_{L_1 \cup L_2 \cup L_3}(3)) = 0$, Claim 1 proves the case $k = 3$. Now we check the case $k = 4$ and hence all cases with $k \geq 4$. We will also prove that $h^0(\mathcal{I}_U(3)) = 1$ for a general union of 4 planes and so $h^0(\mathcal{I}_Z(3)) > 0$ if $k = 4$. Fix a general $p \in L_1$. We take as A_4 a general plane containing p and let $T_4 \subset A_4$ be the union of 3 general lines. Set $Y := T_1 \cup T_2 \cup T_3 \cup T_4$. Take $G \in |\mathcal{I}_Y(3)|$. Since $p \notin T_4$, $G \supset A_4$. $A_2 \cap A_4$ and $A_3 \cap A_4$ are general points of A_2 and A_3, respectively. Hence $A_2 \cup A_3 \subset G$. Since $A_2 \cap A_1 \notin T_1$, we also get $A_1 \subset G$. Although $T_3 \cup A_4$ is not general, $W := A_1 \cup A_2 \cup A_3 \cup A_4$ is a general union of 4 planes. Hence to prove the case $k = 4$ it is sufficient to prove that $h^0(\mathcal{I}_W(3)) = 1$. Taking into account the 6 points $A_i \cap A_j$, $1 \leq i < j \leq 4$, show that $h^0(\mathcal{O}_W(3)) = 34$ (use 4 Mayer-Vietoris exact sequences to check that $h^1(\mathcal{O}_W(3)) = 0$). Hence $h^0(\mathcal{I}_W(3)) \geq 1$. Let $M \subset \mathbb{P}^4$ be a general hyperplane containing A_4. Since $\mathrm{Res}_M(W) = A_1 \cup A_2 \cup A_3$, Claim 1 gives $h^0(\mathcal{I}_{\mathrm{Res}_M(W)}(2)) = 0$. $W \cap M$ is the union of A_4 and 3 general lines. Hence $h^0(M, \mathcal{I}_{W \cap M}(3)) = 1$. The residual sequence of M gives $h^0(\mathcal{I}_W(3)) \leq 1$. Since $h^0(\mathcal{I}_W(3)) > 0$, the proof of the case $k = 4$ is finished. QED

4 General unions of schemes $Z_{3,2}$ and $Z_{2,2}$ in $\mathbb{P}^3$

The aim of this section is to prove the following result.

Theorem 3. *Take $x, y, d \in \mathbb{N}$ with $d \geq 3$. Let $Z \subset \mathbb{P}^3$ be a general union of x $Z_{3,2}$ and y $Z_{2,2}$. We have $h^0(\mathcal{I}_Z(d)) \cdot h^1(\mathcal{I}_Z(d)) = 0$, except in the following cases:*

(1) $(d, x, y) = (3, 2, 0)$, $h^0(\mathcal{I}_Z(3)) = 4$, $h^1(\mathcal{I}_Z(3)) = 2$:

(2) $(d, x, y) = (3, 3, 0)$, $h^0(\mathcal{I}_Z(3)) = 1$, $h^1(\mathcal{I}_Z(3)) = 8$;

(3) $(d, x, y) = (3, 1, 2)$, $h^0(\mathcal{I}_Z(3)) = 1$, $h^1(\mathcal{I}_Z(3)) = 2$;

(4) $(d, x, y) = (3, 2, 1)$, $h^0(\mathcal{I}_Z(3)) = 1$, $h^1(\mathcal{I}_Z(3)) = 5$;

(5) $(d, x, y) = (4, 4, 0)$, $h^0(\mathcal{I}_Z(4)) = 1$, $h^1(\mathcal{I}_Z(4)) = 2$.

In this section we take $n = 3$. Let $H \subset \mathbb{P}^3$ be a plane. Let $U(x, y) \subset \mathbb{P}^3$ be a general union of x schemes $Z_{3,2}$ and y schemes $Z_{2,2}$. Let $V(x, y) \subset \mathbb{P}^3$ be a general union of x schemes $Z_{3,2}$ and y schemes $Z[7]$ with vertex contained in H; note that the latter condition is not restrictive if $y \leq 3$, because any 3 points of $\mathbb{P}^3$ are contained in a plane. Set $V(x) = U(x) = U(x, 0) = V(x, 0)$.

Lemma 7. *Let $Z \subset \mathbb{P}^3$, be a general union of x schemes $Z_{3,2}$ and y schemes $Z_{2,2}$. Then either $h^0(\mathcal{I}_Z(3)) = 0$ or $h^1(\mathcal{I}_Z(3)) = 0$, except the cases with $y = 0$ listed in Proposition 4 and the following cases with $y > 0$:*

(1) $(x, y) = (1, 2)$ with $h^0(\mathcal{I}_Z(3)) = 1$ and $h^1(\mathcal{I}_Z(3)) = 2$;

(2) $(x, y) = (2, 1)$ with $h^0(\mathcal{I}_Z(3)) = 1$ and $h^1(\mathcal{I}_Z(3)) = 5$.

Proof. All cases with $y = 0$ are covered by Proposition 4. All cases with $x = 0$ are true by Theorem 1. The case $(x, y) = (0, 4)$ covered by Theorem 1 shows that $h^0(\mathcal{I}_Z(3)) = 0$ if $x + y \geq 4$.

Assume $x = y = 1$. Let A be the plane containing $Z_{3,2}$. Fix a general $p \in A$. We have $h^i(A, \mathcal{I}_{Z_{3,2} \cup \{p\}}(3)) = 0$, $i = 0, 1$. Let $B \subset \mathbb{P}^3$ be a general plane containing p. We take a general $Z_{2,2} \subset B$ such that $(Z_{2,2})_{\mathrm{red}} \cap A = \{p\}$. Let $L \subset B$ be a general line with $p \in L$. By the residual sequence of A and the differential Horace lemma applied to the connected component of $Z_{2,2}$ with p as its reduction it is sufficient to prove that $h^1(\mathcal{I}_{(2q,B) \cup (2p,L)}(2)) = 0$, where q is a general point of B. This is true for every $q \in B \setminus L$, because $h^1(B, \mathcal{I}_{(2q,B) \cup (2p,L),B}(2)) = 0$. In all cases with $x + y = 3$ we have $h^0(\mathcal{I}_Z(3)) > 0$, because Z is contained in $x + y$ planes.

Now assume $x = 1$ and $y = 2$; we want to prove that $h^0(\mathcal{I}_Z(3)) = 1$ and hence $h^1(\mathcal{I}_Z(3)) = 2$. Let $Z' \subset Z$ be any disjoint union of 3 schemes $Z_{2,2}$ contained in Z (with $Z \setminus Z'$ a 2-point of a plane A). Theorem 1 gives $h^1(\mathcal{I}_{Z'}(3)) = 0$ and so $h^0(\mathcal{I}_{Z'}(3)) = 2$. $\mathrm{Res}_A(Z')$ is a general union of two schemes $Z_{2,2}$ and so $h^0(\mathcal{I}_{\mathrm{Res}_A(Z')}(2)) = 1$ (Proposition 2). Since $Z \setminus Z'$ contains a general point of A and $h^0(\mathcal{I}_{\mathrm{Res}_A(Z')}(2)) < h^0(\mathcal{I}_{Z'}(3))$, we get $h^0(\mathcal{I}_Z(3)) < h^0(\mathcal{I}_{Z'}(3))$ and so $h^0(\mathcal{I}_Z(3)) = 1$.

Finally, we consider the case $x = 2$, $y = 1$. Take $Z'' \subset Z$ with Z'' union of one $Z_{3,2}$ and two $Z_{2,2}$. Since $h^0(\mathcal{I}_Z(3)) \geq 1$ and $h^0(\mathcal{I}_{Z''}(3)) = 1$ by the case $(x, y) = (1, 2)$ just done, we get $h^0(\mathcal{I}_Z(3)) = 1$ and hence $h^1(\mathcal{I}_Z(3)) = 5$. QED

Proposition 5. *Fix $x, y \in \mathbb{N}$. We have $h^0(\mathcal{I}_{U(x,y)}(4)) \cdot h^1(\mathcal{I}_{U(x,y)}(4)) = 0$ if and only if $(x, y) \neq (4, 0)$.*

We have $h^0(\mathcal{I}_{U(4)}(4)) = 1$, $h^1(\mathcal{I}_{U(4)}(4)) = 2$ and $h^1(\mathcal{I}_{V(3,1)}(4)) = 0$.

Proof. By Theorem 1 we may assume $x > 0$. Since $V(x, y)$ is contained in $x + y$ planes, we have $h^0(\mathcal{I}_{U(4)}(4)) > 0$. We first check that $h^0(\mathcal{I}_{U(4)}(4)) = 1$ and hence $h^1(\mathcal{I}_{U(4)}(4)) = 2$. We degenerate $U(4)$ to a general union Z' of one $Z_{3,2} \subset H$ and 3 general schemes $Z_{3,2}$ such that exactly one of the points of their support is contained in H. $Z' \cap H$ is a general union of 3 planar 2-points and 3 tangent vectors and so $h^i(H, \mathcal{I}_{Z' \cap H,H}(4)) = 0$, $i = 0, 1$. Since any 3 points of $\mathbb{P}^3$ are contained in a plane, $\mathrm{Res}_H(Z')$ may be considered as a general $V(0, 3)$. $V(0, 3)$ is contained in a union of 3 planes and so to prove that $h^0(\mathcal{I}_{Z'}(4)) \leq 1$ (and hence $h^0(\mathcal{I}_{U(4)}(4)) = 1$) it is sufficient to prove that $h^0(\mathcal{I}_{V(0,3)}(3)) \leq 1$. Fix $U(0, 3)$ and call A_1, A_2, A_3 the 3 planes spanned by the connected components of $U(0, 3)$. Since any 3 points of $\mathbb{P}^3$ are contained in a plane, $V(0, 3)$ has the Hilbert function of $U(0, 3) \cup \{P_1, P_2, P_3\}$, where each P_i is a general point of A_i. By Theorem 1 we have $h^1(\mathcal{I}_{U(0,3)}(3)) = 0$ and so $h^0(\mathcal{I}_{U(0,3)}(3)) = 2$. Hence there is $q \in A_1 \cup A_2 \cup A_3$ such that $h^0(\mathcal{I}_{U(0,3)\cup\{q\}}(3)) = 1$. Thus $|\mathcal{I}_{V(0,3)}(3)| = \{A_1 \cup A_2 \cup A_3\}$, i.e. $h^0(\mathcal{I}_{V(0,3)}(3)) = 1$. Thus $h^0(\mathcal{I}_{U(4)}(4)) = 1$. Since $h^0(\mathcal{I}_{U(4)}(4)) = 1$ we have $h^0(\mathcal{I}_{U(x,y)}(4)) = 0$ if $x \geq 4$ and $x + y > 4$. Now assume $x \leq 3$. To prove that $h^1(\mathcal{I}_{U(x,y)}(4)) \cdot h^0(\mathcal{I}_{U(x,y)}(4)) = 0$ it is sufficient to check the following pairs (x, y): $(1, 4)$, $(1, 5)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$.

(a) Take $(x, y) = (3, 2)$. We degenerate $U(x, y)$ to a general union Z' of one $Z_{3,2}$ contained in H, two $Z_{3,2}$ with a point of their reduction contained in H, one $Z_{2,2}$ with a point of its support in H and a general $Z_{2,2}$. Since $Z' \cap H$ is a general union of 3 planar 2-points and 3 tangent vectors, we have $h^i(H, \mathcal{I}_{Z' \cap H,H}(4)) = 0$, $i = 0, 1$ and so it is sufficient to prove that $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 0$. $\mathrm{Res}_H(Z')$ is a general union of one $Z_{2,2}$, two $Z[7]$ with vertex in H and one $Z[4]$ with vertex in H. Since $h^0(\mathcal{I}_{U(0,2)}(2)) = 1$ (Proposition 2), we have $h^0(\mathcal{I}_W(2)) = 0$ for a general union of one $Z_{2,2}$, two $Z[7]$ with vertex contained in H and one $Z[4]$ with vertex contained in H. Since any 3 points of $\mathbb{P}^3$ are contained in a plane, adding one vertex at each step we see that $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 0$ if $h^0(\mathcal{I}_E(3)) \leq 3$, where E is a general union of 3 $Z_{2,2}$ and one planar 2-point. Theorem 2 gives $h^0(\mathcal{I}_E(3)) = 0$.

(b) Take $(x, y) = (3, 1)$. We make the same construction taking now as Z' a general union of one $Z_{3,2}$ contained in H, two $Z_{3,2}$ with a point of their reduction contained in H, one $Z_{2,2}$ with a point of its support in H.

(c) Take either $(x, y) = (1, 5)$ or $(x, y) = (1, 4)$. We degenerate $U(x, y)$ to a general union Z' of $y-3$ schemes $Z_{2,2}$, one scheme $Z_{3,2}$ contained in H and 3 general $Z_{2,2}$ with one point of their support contained in H. Since $Z' \cap H$ is a general union of 3 planar 2-points and 3 tangent vectors, we have $h^i(H, \mathcal{I}_{Z' \cap H, H}(4)) = 0$, $i = 0, 1$, and so it is sufficient to prove that $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 0$ if $y = 5$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 0$ if $y = 4$. $\mathrm{Res}_H(Z')$ is a general union of $y - 3$ $Z_{2,2}$ and 3 $Z[4]$ with vertex contained in H. Let E be the union of the unreduced connected components of $\mathrm{Res}_H(Z')$. E is a general union of $y - 3$ $Z_{2,2}$ and 3 planar 2-points. We degenerate two general planar 2-points to two disjoint, but coplanar 2-points, i.e., to a scheme $Z_{2,2}$. In this way we degenerate E to a scheme E_1 to which we apply Theorem 2. Theorem 2 gives $h^1(\mathcal{I}_{E_1}(3)) = 0$ if $y = 4$ and $h^0(\mathcal{I}_{E_1}(3)) = 0$ if $y = 5$ and so $h^1(\mathcal{I}_E(3)) = 0$ if $y = 4$ and $h^0(\mathcal{I}_E(3)) = 0$ if $y = 5$. Hence $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 0$ if $y = 5$. Now assume $y = 4$ and hence $h^0(\mathcal{I}_E(3)) = 5$. Recall that to prove this case it is sufficient to prove that $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 2$, i.e. $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = h^0(\mathcal{I}_E(3)) - 3$. $\mathrm{Res}_H(Z') \backslash E$ is a general union of 3 points of H. Hence $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = \max\{2, h^0(\mathcal{I}_{E \cup H}(3))\}$. We have $h^0(\mathcal{I}_{E \cup H}(3)) = h^0(\mathcal{I}_E(2)) \leq 1$, because E contains 4 general tangent vectors of $\mathbb{P}^3$ and another general point of $\mathbb{P}^3$.

(d) Take either $(x, y) = (2, 3)$ or $(x, y) = (2, 2)$. We degenerate $U(x, y)$ to a general union Z' of one $Z_{3,2}$ contained in H, one $Z_{3,2}$ with a point of its reduction contained in H, two $Z_{2,2}$ with a point of its support in H and $y - 2$ general $Z_{2,2}$. Since $Z' \cap H$ is a general union of 3 planar 2-points and 3 tangent vectors, we have $h^i(H, \mathcal{I}_{Z \cap H, H}(4)) = 0$, $i = 0, 1$, and so it is sufficient to prove that $h^0(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 0$ if $y = 3$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(Z')}(3)) = 0$ if $y = 2$. $\mathrm{Res}_H(Z')$ is a general union of $y - 2$ schemes $Z_{2,2}$, one $Z[7]$ with vertex contained in H and two $Z[4]$ with vertex contained in H. Let E be the union of the unreduced connected components of $\mathrm{Res}_H(Z')$. As in the previous step we reduce to prove that $h^1(\mathcal{I}_E(3)) = 0$. To prove this h^1-vanishing it is sufficient to do the case $y = 3$. In this case E is a general union of 2 $Z_{2,2}$ and two planar 2-points and hence $U(0, 3)$ is a specialization of it. Use the semicontinuity theorem and Theorem 1.

(e) Now we check that $h^1(\mathcal{I}_{V(3,1)}(4)) = 0$, i.e. $h^0(\mathcal{I}_{V(3,1)}(4)) = 1$. We proved that $h^1(\mathcal{I}_{U(3,1)}(4)) = 0$, i.e. $h^0(\mathcal{I}_{U(3,1)}(4)) = 2$. Since $y \leq 3$ $V(3, 1)$ has the Hilbert function of a general union β of 3 $Z_{3,2}$ and one $Z[7]$. Assume $h^1(\mathcal{I}_\beta(4)) > 0$, i.e. (since $\beta \supset U(3, 1)$ and $\deg(\beta) = \deg(U(3, 1)) + 1$) assume

$H^0(\mathcal{I}_\beta(4)) = H^0(\mathcal{I}_{U(3,1)}(4))$. Write $U(3,1) = U \sqcup B$ with B the $Z_{2,2}$. Let A be the plane spanned by B. A general β is obtained from $U(3,1)$ adding a general point of A. Hence $H^0(\mathcal{I}_{A \cup U(3,1)}(4)) = H^0(\mathcal{I}_{U(3,1)}(4))$. Since $\mathrm{Res}_A(U(3,1)) = U$, we get $h^0(\mathcal{I}_U(3)) = 2$. Since $U = U(3,0)$, the case $(n,k) = (3,3)$ of Proposition 4 gives a contradiction. QED

Lemma 8. *Let $Z \subset \mathbb{P}^3$ be a general union of 2 $Z_{2,2}$ and 3 planar 2-points. Then $h^0(\mathcal{I}_Z(3)) = 0$.*

Proof. Let $A', A'' \subset \mathbb{P}^3$ be general planes. Fix one $Z_{2,2}$, $B \subset H$, and call B' a general $Z_{2,2}$ of A' with the only restriction that one of the points of B'_{red} is general in the line $A' \cap H$. Let D be a general planar 2-point of A'' with D_{red} contained in the line $A'' \cap H$. Let $G \subset \mathbb{P}^3$ be a general union of 2 planar 2-points. Set $Z' := G \cup B \cup B' \cup D$. By semicontinuity it is sufficient to prove that $h^0(\mathcal{I}_{Z'}(3)) = 0$. Since $Z' \cap H$ is a general union of B and 2 general tangent vectors of H, Theorem 1 and Remark 3 give $h^i(H, \mathcal{I}_{Z' \cap H}(3)) = 0$, $i = 0, 1$. $\mathrm{Res}_H(Z')$ is a general union of 2 planar 2 points (i.e. G), a scheme $Z[4]$ whose vertex is a general point of H (i.e. $\mathrm{Res}_H(B')$) and a general point of H (i.e. $\mathrm{Res}_H(D)$). Since $\mathrm{Res}_H(\mathrm{Res}_H(Z'))$ contains G, we have $h^0(\mathcal{I}_{\mathrm{Res}_H(\mathrm{Res}_H(Z'))}(1)) = 0$. Hence $h^0(\mathcal{I}_{\mathrm{Res}_H(Z') \cup H}(2)) = 0$. Since $\mathrm{Res}_H(D)$ is general in H, to prove the lemma it is sufficient to prove that $h^0(\mathcal{I}_{G \cup \mathrm{Res}_H(B')}(2)) \le 1$.

Since $G \cup \mathrm{Res}_H(B')$ contains 3 general planar 2-points, it is sufficient to quote Lemma 3. QED

Lemma 9. *$H_{5,3}$ is true.*

Proof. By Theorem 1 we may assume $x > 0$. We have $a_{5,3} = 6$ and $b_{5,3} = 2$. We first check that $h^1(\mathcal{I}_{U(6,0)}(5)) = 0$. We specialize $U(6,0)$ to a scheme Z' which is a general union of 3 general schemes $Z_{3,2}$, 2 schemes $Z_{3,2}$ contained in H and a scheme $Z_{3,2}$ whose reduction contains a point of H. By the case $n = 2$ of the Alexander-Hirschowitz theorem and Remark 3 we have $h^1(\mathcal{I}_{H \cap Z'}(3)) = 0$. Since $\mathrm{Res}_H(Z')$ is a scheme $V(3,1)$, the last assertion of Proposition 5 gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z')}(4)) = 0$ and hence $h^1(\mathcal{I}_{U(6,0)}(5)) = 0$ and $h^0(\mathcal{I}_{U(6,0)}(5)) = 2$. Hence $h^0(\mathcal{I}_{U(6,1)}(5)) = 0$ and $h^1(\mathcal{I}_{U(x,y)}(5)) = 0$ if $x + y \le 6$. We just solved all cases with $y \le 1$ and hence now we may assume that $y \ge 2$.

First assume $x \ne 5$. Let $F \subset H$ be a general union of one $Z_{3,2}$ and two $Z_{2,2}$, i.e. a general union of 7 2-points of H. Since $h^i(H, \mathcal{I}_{F,H}(5)) = 0$, $i = 0, 1$, it is sufficient to use that either $h^0(\mathcal{I}_{U(x-1,y-2)}(4)) = 0$ or $h^1(\mathcal{I}_{U(x-1,y-2)}(4)) = 0$ by Proposition 5.

Now we prove that $h^0(\mathcal{I}_{U(5,2)}(5)) = 0$. Let $E \subset H$ be a general union of two $Z_{3,2}$, i.e. a general union of 6 2-points of H. Fix general $p_1, p_2 \in H$. Let $G \subset \mathbb{P}^3$ be a general union of 3 $Z_{3,2}$. Let $v_i \subset H$ be a general tangent vector of H with

$(v_i)_{\mathrm{red}} = \{p_i\}$. An elementary case of the Alexander-Hirschowitz theorem and Remark 3 gives $h^0(H, \mathcal{I}_{E\cup v_1\cup v_2,H}(5)) = 0$, $i = 0,1$. Let $B_i \subset \mathbb{P}^3$ be a general $Z_{2,2}$ containing v_i and let A_i be the plane spanned by B_i. Note that $B_i \cap H = v_i$ and $\mathrm{Res}_H(B_i) = \{p_i\}\cup\{2o, A\}$ with o general in A_2. Fix a general plane $A_2 \subset \mathbb{P}^3$ with $p_2 \in A$. It is sufficient to prove that $h^0(\mathcal{I}_{G\cup(2o_1,A_1)\cup\{q_1\}\cup(2o_2,A_2\}\cup\{q_2\}}(4)) = 0$. By Proposition 5 we have $h^0(\mathcal{I}_{G\cup(2o_1,A_1)\cup(2o_2,A_2)}(4)) = 2$ (degenerate $(2o_1, A_1) \cup (2o_2, A_2)$ to a general $Z_{2,2}$). The scheme $G \cup (2o_1, A_1) \cup (2o_2, A_2)$ does not depend on H. Moving H we may assume that (q_1, q_2) is a general element of $A_1 \times A_2$. Hence it is sufficient to use that $h^0(\mathcal{I}_{G\cup A_1\cup A_2}(4)) = h^0(\mathcal{I}_G(2)) = 0$ by Lemma 4. QED

Lemma 10. *Fix an integer $d \geq 6$. Assume that $H_{d-1,3}$ is true. If $d \geq 7$ also assume that $H_{d-2,3}$ is true. Then $H_{d,3}$ is true.*

Proof. Increasing or decreasing if necessary x or y it is sufficient to do all cases with $|\binom{d+3}{3} - 9x - 6y| \leq 5$ and all cases with $y = 0$. To cover all pairs $(x, 0)$ it is sufficient to do the cases $x = \lfloor\binom{d+3}{3}/9\rfloor$ and $x = \lceil\binom{d+3}{3}/9\rceil$. Moreover, if we take $x = \lfloor\binom{d+3}{3}/9\rfloor$ (and hence we need to prove an h^1-vanishing) we may assume $b_{d,3} \leq 5$ (otherwise we check $h^1 = 0$ for the pair $(x, 1)$). If $b_{d-1,3} \leq b_{d,3}$, then $b_{d,3} - b_{d-1,3} = b_{d,2}$. If $b_{d-1,3} > b_{d,3}$, then $b_{d,2} = 9 + b_{d,3} - b_{d-1,3}$. We have $b_{d,2} = 1$ if $d \equiv 0,3,6 \pmod 9$, $b_{d,2} = 3$ if $d \equiv 1,5 \pmod 9$, $b_{d,2} = 6$ if $d \equiv 2,4 \pmod 9$ and $b_{d,2} = 0$ if $d \equiv 7,8 \pmod 9$. Let $E \subset H$ be a general union of $a_{d,2}$ schemes $Z_{3,2}$. We have $h^1(H, \mathcal{I}_{E,H}(d)) = 0$, $h^0(H, \mathcal{I}_{E,H}(d)) = b_{d,2}$ and $h^0(H, \mathcal{I}_{E,H}(d-1)) = 0$. We have $b_{6,2} = 1$. By Theorem 1 we may assume $x > 0$.

(a) Assume $b_{d,2} = 6$.

(a1) First assume $x \geq a_{d,2}$ and $y > 0$. Let $E_1 \subset H$ be a general union of E and one scheme $Z_{2,2}$. Since $h^i(H, \mathcal{I}_{E_1,H}(d)) = 0$, $i = 0,1$, it is sufficient to apply $H_{d-1,3}$ to $U(x - a_{d,2}, y - 1)$.

(a2) Now assume $x < a_{d,2}$ and hence $6y \geq \binom{d+2}{2}$. Set $w := 2\lfloor x/2\rfloor$. So $w = x$ if x is even and $w = x - 1$ if x is odd. Let $F \subset H$ be a general union of w schemes $Z_{3,2}$ and $3w/2 + 1$ schemes $Z_{2,2}$. Since $h^i(H, \mathcal{I}_{F,H}(d)) = 0$, $i = 0,1$, it is sufficient to apply $H_{d-1,3}$ to the scheme $U(x - w, y - 1 - 3w/2)$.

(a3) Now assume $y = 0$. Fix a general line $L \subset H$. Since $h^0(H, \mathcal{I}_{E,H}(d-1)) = 0$, the image of the restriction map $H^0(H, \mathcal{I}_{E,H}(d)) \to H^0(L, \mathcal{O}_L(d))$ has dimension 6. By Remark 3 for general tangent vectors $v, v' \subset L$ we have $h^0(H, \mathcal{I}_{E\cup v'\cup v,H}(d)) = 2$. Hence $h^i(H, \mathcal{I}_{E\cup v'\cup v\cup w}(d)) = 0$, $i = 0,1$, for a general tangent vector $w \subset H$ (Remark 3). Call S the reduction of $v \cup v'$ and $\{q\}$ the reduction of w. Let $M \subset \mathbb{P}^3$ be a general plane containing L. Let $N \subset \mathbb{P}^3$ be a general plane containing the line spanned by w. Let $G \subset \mathbb{P}^3$ be a general union of $x - a_{d,2} - 2$ schemes $Z_{3,2}$. Let $U \subset M$ be a general scheme $Z_{3,2}$ containing

S. Let $V \subset N$ be a general $Z_{3,2}$ containing q. Note that $U \cap H = v \cup v'$ and $V \cap N = w$ and that $\mathrm{Res}_H(U \cup V) = S \cup W'$ with W' a scheme $Z[7]$. By semicontinuity and the residual exact sequence of H it is sufficient to prove that either $h^0(\mathcal{I}_{G\cup W'\cup S}(d-1)) = 0$ or $h^1(\mathcal{I}_{G\cup W'\cup S}(d-1)) = 0$. Let $W'' \subset W'$ be the scheme $Z_{2,2}$ contained in W'. We have $W' = W'' \cup \{q\}$. Since G is general and each point of $\mathbb{P}^3$ is contained in a hyperplane, $G \cup W''$ has the Hilbert function of $U(x - a_{d,2} - 2, 1)$. By $H_{d-1,3}$ either $h^0(\mathcal{I}_{G\cup W''}(d-1)) = 0$ or $h^1(\mathcal{I}_{G\cup W''}(d-1)) = 0$. Therefore we may assume $h^1(\mathcal{I}_{G\cup W''}(d-1)) = 0$. The only restriction on $W' \cup S$ is that both S and the vertex q of W' are contained in H. Since any 3 points of $\mathbb{P}^3$ are contained in a plane, for a general S we have $h^0(\mathcal{I}_{G\cup W''\cup\{q\}\cup S}(d-1)) = \max\{h^0(\mathcal{I}_{G\cup W''}(d-1)) - 3, 0\}$ and so either $h^0(\mathcal{I}_{G\cup W'\cup S}(d-1)) = 0$ or $h^1(\mathcal{I}_{G\cup W'\cup S}(d-1)) = 0$.

(b) Assume $b_{d,2} = 3$ and so $d \geq 10$ and $d \equiv 1, 5 \pmod 9$.

(b1) Assume $y \geq 2$. Since $\binom{d+2}{2} \equiv 0 \pmod 3$, there are unique integers a, b such that $9a + 6b = \binom{d+2}{2}$ and $\min\{x, a_{d,2} - 1\} - 1 \leq a \leq \min\{x, a_{d,2} - 1\}$. Since $x > 0$, we have $a \geq 0$. Since $9x + 6y \geq \binom{d+3}{3} - 5$ and $y \geq 2$, we have $y \geq b$. Let $F \subset H$ be a general union of a schemes $Z_{3,2}$ and b schemes $Z_{2,2}$. Since $h^i(H, \mathcal{I}_{F,H}(d)) = 0$, $i = 0, 1$, it is sufficient to apply $H_{d-1,3}$ to the scheme $U(x - a, y - b)$.

(b2) Assume $y \leq 1$. Hence $x \geq a_{d,2} + 2$. Since $h^0(H, \mathcal{I}_{E,H}(d)) = 3$, we have $h^i(H, \mathcal{I}_{E\cup v\cup\{q\}}(d)) = 0$, $i = 0, 1$, for a general $q \in H$ and a general tangent vector $v \subset H$. Let $G \subset \mathbb{P}^3$ be a general union of $x - a_{d,2} - 2$ schemes $Z_{3,2}$ and y schemes $Z_{2,2}$. Let $L \subset H$ be the line spanned by v. Let $M \subset \mathbb{P}^3$ be a general plane containing L and let $N \subset \mathbb{P}^3$ be a general plane containing tL. Set $\{p\} := v_{\mathrm{red}}$. Let $W \subset M$ be a general scheme $Z[8]$ with p as its vertex and L as its vertex line. Let $U \subset N$ be a general scheme $Z[7]$ with q as its vertex. Let W' (resp. U') be the $Z_{3,2}$ scheme containing W (resp. U) and let W_1 (resp. U_1) be the $Z_{2,2}$ scheme contained in W (resp. U). Since $\mathrm{Res}_H(U') = U$, the differential Horace lemma applied to H, q and W' shows that it is sufficient to prove that either $h^1(\mathcal{I}_{G\cup W\cup U}(d-1)) = 0$ or $h^0(\mathcal{I}_{G\cup W\cup U}(d-1)) = 0$. Note that $\deg(U_1) + \deg(W') = \deg(U) + \deg(W)$. The inductive assumption gives that either $h^1(\mathcal{I}_{G\cup U_1\cup W'}(d-1)) = 0$ (case $9x+6y \leq \binom{d+3}{3}$) or $h^0(\mathcal{I}_{G\cup U_1\cup W'}(d-1)) = 0$ (case $9x + 6y \geq \binom{d+3}{3}$).

Assume for the moment $9x+6y < \binom{d+3}{3}$ and hence $h^0(\mathcal{I}_{G\cup W\cup U_1}(d-1)) > 0$. Since $h^1(\mathcal{I}_{G\cup U_1\cup W'}(d-1)) = 0$, we have $h^1(\mathcal{I}_{G\cup U_1\cup W}(d-1)) = 0$ and so it is sufficient to prove that $h^0(\mathcal{I}_{G\cup W\cup U}(d-1)) < h^0(\mathcal{I}_{G\cup W\cup U_1}(d-1))$. Since q is general in H, $G \cup U_1$ is a general union of $x - a_{d-2} - 2$ $Z_{3,2}$ and $y + 1$ $Z_{2,2}$. U is obtained from U_1 adding a general point of $N \cap H$; since for a fixed W we may take as H a general plane containing L, moving H we may assume that U is obtained from U_1 adding a general point of N. Hence $h^0(\mathcal{I}_{G\cup W\cup U}(d-1)) <$

$h^0(\mathcal{I}_{G\cup W\cup U_1}(d-1))$, unless $h^0(\mathcal{I}_{G\cup W\cup N}(d-1)) = h^0(\mathcal{I}_{G\cup W\cup U_1}(d-1))$, i.e. unless $h^0(\mathcal{I}_{G\cup W}(d-2)) = h^0(\mathcal{I}_{G\cup W\cup U_1}(d-1))$. The inductive assumption gives $h^0(\mathcal{I}_{G\cup W_1}(d-2)) = 0$ and so $h^0(\mathcal{I}_{G\cup W}(d-2)) = 0$, a contradiction.

Now assume $9x+6y \ge \binom{d+3}{3}$. Therefore we have $h^0(\mathcal{I}_{G\cup U_1\cup W'}(d-1)) = 0$. If $h^0(\mathcal{I}_{G\cup U_1\cup W}(d-1)) = 0$, then we are done, because $U_1 \subset U$. Now assume $h^0(\mathcal{I}_{G\cup U_1\cup W}(d-1)) > 0$ and hence $h^0(\mathcal{I}_{G\cup U_1\cup W}(d-1)) = 1$. Hence $h^0(\mathcal{I}_{G\cup U_1\cup W_1}(d-1)) > 0$. Since v and q are general in H, $G\cup U_1\cup W_1$ is a general union of $x-a_{d,2}-2$ $Z_{3,2}$ and $y+2$ $Z_{2,2}$. The inductive assumption gives $h^1(\mathcal{I}_{G\cup U_1\cup W_1}(d-1)) = 0$ and so (since $b_{d,2}=3$) $h^0(\mathcal{I}_{G\cup U_1\cup W_1}(d-1)) = \binom{d+3}{3} - 9x-6y+3$. Of course, we get $9x+6y \le \binom{d+3}{3}+2$ and $h^0(\mathcal{I}_{G\cup U_1\cup W_1}(d-1)) \le 2$. To get $h^0(\mathcal{I}_{G\cup U_1\cup W}(d-1)) = 0$ (i.e. a contradiction) it is sufficient to prove that $h^0(\mathcal{I}_{G\cup U_1\cup W_1\cup\{o,o'\}}(d-1)) = 0$ for a general $o \in M$ and a general $o' \in N$ (because $G\cup U_1\cup W_1$ does not depend on H, p (resp. q) is a general point of $N\cap H$ (resp. $M\cap H$) and hence varying H we may assume that (p,q) is a general element of $N\times M$ and for a general line L' through q we may find a hyperplane H' containing $L'\cup\{p\}$ (the linear span of $L'\cup\{p\}$)). This is very easy and we write down only the less trivial case $h^0(\mathcal{I}_{G\cup U_1\cup W_1}(d-1)) = 2$ and $h^0(\mathcal{I}_{G\cup U_1\cup W_1\cup\{o\}}(d-1)) = 1$. We have $h^0(\mathcal{I}_{G\cup U_1\cup W_1\cup\{o,o'\}}(d-1)) = 1$ if and only if $H^0(\mathcal{I}_{G\cup U_1\cup W_1\cup\{o\}}(d-1)) = H^0(\mathcal{I}_{G\cup U_1\cup\{o\}\cup N}(d-1))$. The latter vector space is 0, because $h^0(\mathcal{I}_{G\cup U_1\cup\{o\}}(d-2)) \le h^0(\mathcal{I}_{G\cup U_1}(d-2)) = 0$.

(c) Assume $b_{d,2} = 1$.

(c1) Assume $x \ge a_{d,2}+1$. Look at step (b2). We use $Z[8]$, but not $Z[7]$.

(c2) Assume $x \le a_{d,2}$. Write $w := x$ if x is even and $w := x-1$ if x is odd. We have $\binom{d+2}{2} - 9w \equiv 1 \pmod 6$ and $6y \ge \binom{d+2}{2} - 9u$. Let $F \subset H$ be a general union of w schemes $Z_{3,2}$, $(\binom{d+2}{2} - 9w)/6$ schemes $Z_{2,2}$ and one point. We have $h^i(H, \mathcal{I}_{F,H}(d)) = 0$, $i=0,1$. If x is odd, we use the differential Horace and reduce to prove that either $h^0(\mathcal{I}_G(d-1)) = 0$ or $h^1(\mathcal{I}_G(d-1)) = 0$, where G is a general union of one scheme $Z[8]$ and $y - (\binom{d+2}{2} - 9w - 1)/6$ schemes $Z_{2,2}$. Use $H_{d-1,3}$ and Lemma 2. If $x = w$ as G we take a general union of one scheme $Z[5]$ and $y - 1 - (\binom{d+2}{2} - 9w - 1)/6$ schemes $Z_{2,2}$.

(d) Assume $b_{d,2} = 0$. First assume $x \ge a_{d,2}$. We use that $h^i(H, \mathcal{I}_{E,H}(d)) = 0$, $i = 0,1$, and apply $H_{d-1,3}$ to $U(x-a_{d,2}, y)$. Now assume $x < a_{d,2}$. Set $u = x$ if $a_{d,2}-x$ is even and $u = x-1$ if $a_{d,2}-x$ is odd. Since $x > 0$, in all cases we have $u \ge 0$ and $\binom{d+2}{2} - 9u \equiv 0 \pmod 6$ and $6y \ge \binom{d+2}{2} - 9u$. Let $F \subset H$ be a general union of u schemes $Z_{3,2}$ and $(\binom{d+2}{3} - 9u)/6$ schemes $Z_{2,2}$. Since $h^i(H, \mathcal{I}_{F,H}(d)) = 0$, $i=0,1$, it is sufficient to apply $H_{d-1,3}$ to the scheme $U(x-u, y - (\binom{d+2}{3} - 9u)/6)$. QED

Proof of Theorem 3: Use Proposition 4 and Lemma 7 for the case $d=3$, Proposition 5 for the case $d=4$ and Lemmas 9 and 10 for the cases $d \ge 5$. QED

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