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TOPOLOGICAL VECTOR SPACES OVER TOPOLOGICAL DIVISION RINGS: PROJECTIVE AND INDUCTIVE LIMITS (*)

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INTRODUCTION. The theory of topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$ without conditions of local convexity has been developed lastly by Adasch, Ernst and Keim [1], Iyahen [6] and Waelbroeck [12]. They introduced the notion of a "string" in a topological vector space which made the development of a theory "without duality" easier. The extension of this notion to topological vector spaces over valued division rings (fields) has been done by Prolla [8] and allowed a characterization of barrelled, bornological and quasi-barrelled spaces.

In the present paper we are concerned with topological vector spaces over Hausdorff non-discrete topological division rings which have been introduced by Nachbin in [7]. The main contents of our paper is the study of inductive limits of such topological vector spaces (section 3). Projective limits are treated only so far as results are needed for inductive limits (section 2). A basic result (theorem 3.6) is a characterization of fundamental systems of neighborhoods of zero of the inductive limit topology in terms of fundamental systems

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of neighborhoods of zero of those topological vector spaces which generate the inductive limit. For this characterization we have to assume that the underlying topological division ring is locally right-bounded, cf. proposition 3.8. Our characterization seems to be new even for topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$, cf. Iyahen [6] who stated such a characterization only for countable families of topological vector spaces.

§ 1 - NOTATIONS AND BASIC RESULTS

We say that a topology τ on a vector space E is τ_F- *compatible* if (E,τ) is a topological vector space over (F,τ_F).

The following result will be used many times in the text. We give its explicit statement for easy of reference.

THEOREM 1.1 - *Let* (E,τ) *be a* TVS. *If* $\mathcal{V}$ *is a fundamental system of* τ*-neighborhoods of* 0 *in* E, *then* $\mathcal{V}$ *is a filter basis on* E *satisfying the following conditions*

(V1) *for each* $W \in \mathcal{V}$ *there is* $U \in \mathcal{V}$ *such that* $U+U \subset W$;

(V2) *for each* $W \in \mathcal{V}$ *there is a* τ_F*-neighborhood* V *of* 0 *in* F *and there is* $U \in \mathcal{V}$ *such that* $VU \subset W$;

(V3) *for each* $W \in \mathcal{V}$ *and for each* $\lambda \in F, \lambda \neq 0$, *there is* $U \in \mathcal{V}$ *such that* $U \subset \lambda W$;

(V4) *for each* $x \in E$ *and for each* $W \in \mathcal{V}$, *there is a* τ_F*-neighborhood* V *of* 0 *in* F *such that* $Vx \subset W$.

Conversely, given a filter basis $\mathcal{V}$ *on* E *satisfying* (V1) - (V4), *there is a unique* τ_F*-compatible topology on* E *for which* $\mathcal{V}$ *is a fundamental system of neighborhoods of* 0.

PROOF: see [3] th. 3.14.

If (E,τ) and (G,η) are TVS and A is a linear map from E into G, by $A^{-1}(\eta)$ we denote the τ_F-compatible topology on E for which the set $\mathcal{B} = \{A^{-1}(U);\ U \in \mathcal{V}\}$ is a fundamental system of neighborhoods of 0 in E, where $\mathcal{V}$ is a fundamental system of η-neighborhoods of 0 in G. We have that $A^{-1}(\eta)$ is the coarsest τ_F-compatible topology on E for which A is continuous and it is called the *inverse-image topology of* η *by* A. In particular, when E is a subspace of G and A is the canonical embedding I_E from E into G, $I_E^{-1}(\eta)$ is called the *induced topology on* E *by* η and is denoted by η_E. If A is surjective, then $A(\tau)$ denotes the τ_F-compatible topology on G for which the set $\mathcal{W} = \{A(U); U\in\mathcal{U}\}$, where $\mathcal{U}$ is a fundamental system of τ-neighborhoods of 0 in E, is a fundamental system of neighborhoods of 0 in G.

Let E be a vector space and let $\{\tau_\alpha; \alpha\in\Lambda\}$ be a non-empty family of τ_F-compatible topologies on E. By

I) $\tau := \sup\{\tau_\alpha; \alpha\in\Lambda\}$ we denote the τ_F-compatible topology on E which satisfies the following conditions:

a) $\tau_\alpha \subset \tau$ for every $\alpha\in\Lambda$;

b) if η is a τ_F-compatible topology on E such that $\tau_\alpha \subset \eta$ for every $\alpha\in\Lambda$, then $\tau\subset\eta$.

τ is called the *least upper bound of the topologies* τ_α.

II) $\xi := \inf\{\tau_\alpha; \alpha\in\Lambda\}$ we denote the τ_F-compatible topology on E which satisfies the following conditions:

a) $\xi\subset\tau_\alpha$ for every $\alpha\in\Lambda$;

b) if μ is a τ_F-compatible topology on E such that $\mu\subset\tau_\alpha$ for every $\alpha\in\Lambda$, then $\mu\subset\xi$.

ξ is called the *greatest lower bound of the topologies* τ_α.

Let (E,τ) and (G,η) be TVS. By $\mathcal{L}(E;G)$ we denote the set of all continuous linear maps from E into G. For a subset H of $\mathcal{L}(E;G)$

we say that H is *equicontinuous* if one of the following equivalent conditions is fulfilled:

(a) For each η-neighborhood V of O in G $\bigcap_{T\in H} T^{-1}(V)$ is a τ-neighborhood of O in E;

(b) For each η-neighborhood V of O in G, there is a τ-neighborhood U of O in E such that $\bigcup_{T\in H} T(U) \subset V$.

Let (F,τ_F) be a topological division ring and let $\mathcal{V}$ be a fundamental system of τ_F-neighborhoods of O in F. We say that a subset M of F is *right-bounded* if for each $U\in\mathcal{V}$ there is $V\in\mathcal{V}$ such that $MV\subset U$.

LEMMA 1.2 (Kowalsky-Grünbaum) - *A subset* M *of a topological division ring* (F,τ_F) *is right-bounded if, and only if, for each basic* τ_F*-neighborhood* U *of* O *in* F *there is* $\lambda\in F\setminus\{O\}$ *such that* $M\lambda\subset U$.

We say that a topological division ring (F,τ_F) is *locally right-bounded* if there is a right-bounded τ_F-neighborhood of O in F.

§ 2 - PROJECTIVE LIMITS

THEOREM 2.1 - *Let* E *be a vector space,* $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ *be a family of topological vector spaces and* $(A_\alpha)_{\alpha\in\Lambda}$ *be a family of linear maps from* E *into* E_α. *For each* $\alpha\in\Lambda$, *let* $\mathcal{B}_\alpha$ *be a fundamental system of* τ_α*-neighborhoods of* 0 *in* E_α. *For each finite set* $\{\alpha_1,\dots,\alpha_n\}\subset\Lambda$ *and* $U_{\alpha_i}\in\mathcal{B}_{\alpha_i}$, $i=1,\dots,n$, *consider the subset of* E *defined by*

$$(1) \qquad U := \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(U_{\alpha_i}).$$

Let $\mathcal{F}$ *be the set of all subsets* U *of* E *defined by* (1). *Then there is a* τ_F*-compatible topology on* E *for which* $\mathcal{F}$ *is a fundamental system of neighborhoods of* 0 *in* E *and it is the coarsest* τ_F*-compatible topology on* E *for which all the maps* A_α, $\alpha\in\Lambda$, *are continuous.*

PROOF: It is obvious that $\mathcal{F}$ is a filter basis on E. Let $U\in\mathcal{F}$. Then $U=\bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(U_{\alpha_i})$ for some finite set $\{\alpha_1,\dots,\alpha_n\}\subset\Lambda$, where $U_{\alpha_i}\in\mathcal{B}_{\alpha_i}$, $i=1,\dots,n$. So by 1.1 we have that:

(a) for each $i=1,\dots,n$, there is $W_{\alpha_i}\in\mathcal{B}_{\alpha_i}$ with $W_{\alpha_i}+W_{\alpha_i}\subset U_{\alpha_i}$ and setting $W=\bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(W_{\alpha_i})$ we get $W\in\mathcal{F}$ and $W+W\subset U$;

(b) for each $i=1,\dots,n$, there are a τ_F-neighborhood V_{α_i} of 0 in F and $W_{\alpha_i}\in\mathcal{B}_{\alpha_i}$ with $V_{\alpha_i}W_{\alpha_i}\subset U_{\alpha_i}$ and putting $V:=\bigcap_{i=1}^{n} V_{\alpha_i}$ and $W:=\bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(W_{\alpha_i})$ we get that V is a τ_F-neighborhood of 0 in F, $W\in\mathcal{F}$ and $VW_{\alpha_i}\subset U_{\alpha_i}$, $i=1,\dots,n$, and so,

$$VW = V\bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(W_{\alpha_i}) \subset \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(U_{\alpha_i}) = U;$$

(c) for each $\lambda \in F, \lambda \neq 0$, and for each $i=1,\dots,n$, there is $W_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ such that $W_{\alpha_i} \subset \lambda U_{\alpha_i}$ and setting $W := \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(W_{\alpha_i})$ we get $W \in \mathcal{F}$ and $W \subset \lambda U$;

(d) for each $x \in E$ and for each $i=1,\dots,n$ there is a τ_F-neighborhood V_{α_i} of 0 in F such that $V_{\alpha_i} A_{\alpha_i}(x) \subset U_{\alpha_i}$ and setting $V := \bigcap_{i=1}^{n} V_{\alpha_i}$ we get that V is a τ_F-neighborhood of 0 in F with $Vx \subset U$.

From (a) - (d) above and 1.1, there is a unique τ_F-compatible topology on E, which we will denote by τ, for which $\mathcal{F}$ is a fundamental system of neighborhoods of 0 in E. From the definition of τ, it is clear that for each $\alpha \in \Lambda$ the linear map A_α from (E,τ) into (E_α,τ_α) is continuous. Now let τ^1 be another τ_F-compatible topology on E such that for each $\alpha \in \Lambda$ $A_\alpha : (E,\tau^1) \to (E_\alpha,\tau_\alpha)$ is continuous and let U be a τ-neighborhood of 0 in E. Then there is $V \in \mathcal{F}$ such that $V \subset U$, where $V = \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(U_{\alpha_i})$, $U_{\alpha_i} \in \mathcal{B}_{\alpha_i}$, *for some finite set* $\{\alpha_1,\dots,\alpha_n\} \subset \Lambda$. From the continuity of $A_{\alpha_i} : (E,\tau^1) \to (E_{\alpha_i},\tau_{\alpha_i})$, $i=1,\dots,n$, it follows that V is a τ^1-neighborhood of 0 in E, which implies that U is also a τ^1-neighborhood of 0 in E.

DEFINITION 2.2 - The τ_F-compatible topology on E defined and described in 2.1 above is called the *projective topology* on E with respect to the family $((E_\alpha,\tau_\alpha),A_\alpha)_{\alpha\in\Lambda}$. A TVS (E,τ) generated as described in 2.1 is called the *projective limit of the topological vector spaces* (E_α,τ_α) *with respect to the linear maps* A_α *and denoted by*

$$(E,\tau) = \operatorname*{proj}_{\alpha\in\Lambda} ((E_\alpha,\tau_\alpha),A_\alpha).$$

EXAMPLE 2.3

2.3.1 - Let (E,τ) be a TVS and let H be a vector space. Let A be a linear map from H into E. It is clear that $(H,A^{-1}(\tau)) = \text{proj}((E,\tau),A)$. In particular, when H is a vector subspace of E, A is the canonical embedding I_H from H into E and τ_H is the induced topology on E by τ, we have $(H,\tau_H) = \text{proj}((E,\tau),I_H)$

2.3.2 - Let $(E,\tau) = \prod_{\alpha\in\Lambda} (E_\alpha,\tau_\alpha)$ be the topological product of the family of TVS $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$. If we denote the canonical projection from $\prod_{\alpha\in\Lambda} E_\alpha$ onto E_α by P_α, $\alpha\in\Lambda$, we have $(E,\tau) = \text{proj}_{\alpha\in\Lambda}((E_\alpha,\tau_\alpha),P_\alpha)$.

2.3.3 - Let E be a vector space and let $\{\tau_\alpha,\alpha\in\Lambda\}$ be a family of τ_F-compatible topologies on E. If $\tau = \sup\{\tau_\alpha, \alpha\in\Lambda\}$, then $(E,\tau) = \text{proj}_{\alpha\in\Lambda}((E,\tau_\alpha),i_\alpha)$, where for each $\alpha\in\Lambda$ i_α is the identity map on E.

PROPOSITION 2.4 - *Let* $(E,\tau) = \text{proj}_{\alpha\in\Lambda}((E_\alpha,\tau_\alpha),A_\alpha)$, *where for each* $\alpha\in\Lambda$ (E_α,τ_α) *is a Hausdorff* TVS. *Then* (E,τ) *is a Hausdorff* TVS *if, and only if,* $\bigcap_{\alpha\in\Lambda} \ker(A_\alpha) = \{0\}$.

PROOF: Since τ is a Hausdorff topology on E if, and only if, $\bigcap_{V\in\mathcal{B}} V = \{0\}$, where $\mathcal{B}$ is a fundamental system of τ-neighborhoods of 0 in E, it is enough to prove, under the assumptions, that $\bigcap_{V\in\mathcal{B}} V = \bigcap_{\alpha\in\Lambda} \ker(A_\alpha)$. Let $x\in \bigcap_{\alpha\in\Lambda}\ker(A_\alpha)$ and let $V\in\mathcal{B}$. Then $V = \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(U_{\alpha_i})$ where U_{α_i} is a basic τ_{α_i}-neighborhood of 0 in E_{α_i}, $i=1,\dots,n$. Since $A_{\alpha_i}(x) = 0 \in U_{\alpha_i}$, $i=1,\dots,n$, we have $x\in V$. Because we chose V arbitrarily in $\mathcal{B}$, it follows that

(a) $$\bigcap_{\alpha\in\Lambda}\ker(A_\alpha) \subset \bigcap_{V\in\mathcal{B}} V.$$

Conversely, let $x \in \bigcap_{V \in \mathcal{B}} V$ and $\alpha \in \Lambda$ be given. Let W_α be a τ_α-neighborhood of O in E_α. Since $\{\alpha\} \subset \Lambda$ is finite, we have $U := A_\alpha^{-1}(W_\alpha) \in \mathcal{B}$. Hence $x \in U$, which implies that $A_\alpha(x) \in W_\alpha$ and so, $A_\alpha(x) = 0$ because W_α is arbitrary and (E_α, τ_α) is a Hausdorff TVS. Since α was chosen arbitrarily, $x \in \bigcap_{\alpha \in \Lambda} \ker(A_\alpha)$. Thus

$$\text{(b)} \qquad \bigcap_{V \in \mathcal{B}} V \subset \bigcap_{\alpha \in \Lambda} \ker(A_\alpha)$$

REMARK: When we proved (a), we did not use the fact that (E_α, τ_α) is a Hausdorff TVS. Therefore, if (E, τ) is a Hausdorff TVS, then $\bigcap_{\alpha \in \Lambda} \ker(A_\alpha) = \{0\}$.

COROLLARY 2.5 - *Under the hypothesis in* 2.4, (E, τ) *is a Hausdorff* TVS *if, and only if, for each* $x \in E, x \neq 0$, *there are* $\alpha \in \Lambda$ *and a* τ_α*-neighborhood* W_α *of* O *in* E_α *such that* $A_\alpha(x) \notin W_\alpha$.

PROPOSITION 2.6: *If* $(E, \tau) = \operatorname{proj}_{\alpha \in \Lambda}((E_\alpha, \tau_\alpha), A_\alpha)$ *and if* (G, η) *is an arbitrary* TVS, *then a linear map* A *from* G *into* E *is continuous if, and only if, for each* $\alpha \in \Lambda$ *the linear map* $A_\alpha \circ A$ *from* G *into* E_α *is continuous.*

PROOF: Suppose that $A: (G, \eta) \to (E, \tau)$ is continuous. Then, from the definition of τ, it is immediate that for every $\alpha \in \Lambda$ $A_\alpha \; A : (G, \eta) \to (E_\alpha, \tau_\alpha)$ is continuous. Conversely, suppose that for each $\alpha \in \Lambda$ the map $A_\alpha \circ A : (G, \eta) \to (E_\alpha, \tau_\alpha)$ is continuous and let V be a τ-neighborhood of O in E. Then, from the définition of τ, there is a finite set $\{\alpha_1, \ldots, \alpha_n\} \subset \Lambda$ and, for each $i=1,\dots,n$, there is a basic τ_{α_i}-neighborhood U_{α_i} of 0 in E_{α_i} such that $U := \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(U_i) \subset V$. So $A^{-1}(U) \subset A^{-1}(V)$. Since by hypothesis $(A_{\alpha_i} \circ A)^{-1}(U_{\alpha_i})$ is a η-neighborhood of 0 in G for all $i=1,\dots,n$, it follows that $A^{-1}(V)$ is also a η-neighborhood of 0 in G.

PROPOSITION 2.7 - *If* $(E,\tau) = \underset{\alpha\in\Lambda}{\operatorname{proj}}((E_\alpha,\tau_\alpha),A_\alpha)$ *and if* (G,η) *is an arbitrary* TVS, *then a set* H *of linear maps from* G *into* E *is equicontinuous if, and only if, for each* $\alpha\in\Lambda$ $H_\alpha := \{A_\alpha \circ T; T\in H\}$ *is an equicontinuous subset of* $\mathcal{L}(G;E_\alpha)$.

PROOF: Suppose that H is an equicontinuous set of linear maps from G into E and let $\alpha\in\Lambda$ be given. Since for each $T\in H$, $T\in\mathcal{L}(G;E)$, we have $H_\alpha \subset \mathcal{L}(G;E_\alpha)$ by 2.6. Let V be a τ_α-neighborhood of 0 in in E_α. By hypothesis, there is a η-neighborhood U of 0 in G such that $T(U) \subset A_\alpha^{-1}(V)$ for every $T\in H$. Hence $(A_\alpha\circ T)(U) \subset V$ for every $T\in H$, which shows that $H_\alpha \subset \mathcal{L}(G;E_\alpha)$ is equicontinuous for each $\alpha\in\Lambda$.

Conversely, assume that H is a set of linear maps from G into E such that $H_\alpha = \{A_\alpha\circ T ; T\in H\}$ is an equicontinuous subset of $\mathcal{L}(G;E_\alpha)$ for each $\alpha\in\Lambda$. By 2.6 it follows that $H \subset \mathcal{L}(E;G)$. Let V be a τ-neighborhood of 0 in E. By definition of τ, there is a finite set $J = \{\alpha_1,\dots,\alpha_m\} \subset \Lambda$ and, for each $\alpha_i \in J$, there is a τ_{α_i}-neighborhood U_{α_i} of 0 in E_{α_i} such that $\bigcap_{i=1}^{m} A_{\alpha_i}^{-1}(U_{\alpha_i}) \subset V$. Let $1 \le i \le m$. Since U_{α_i} is a τ_{α_i}-neighborhood of 0 in E_{α_i}, there is a η-neighborhood V_i of 0 in G with $(A_\alpha \circ T)(V_i) \subset U_{\alpha_i}$, for every $T \in H$. Let $W = \bigcap_{i=1}^{m} V_i$ and $T \in H$. Then W is a η-neighborhood of O in G with $T(W) \subset \bigcap_{i=1}^{m} T(V_i) \subset \bigcap_{i=1}^{m} A_{\alpha_i}^{-1} (A_{\alpha_i} \circ T(V_i)) \subset \bigcap_{i=1}^{m} A_{\alpha_i}^{-1} (U_{\alpha_i}) \subset V$.

Since T was chosen arbitrarily, it follows that $H \subset \mathcal{L}(E;,G)$ is equicontinuous.

PROPOSITION 2.8 - *Let* $(E,\tau) = \operatorname{proj}_{\alpha \in \Lambda}((E_\alpha,\tau_\alpha),A_\alpha)$ *and assume that* $\bigcap_{\alpha \in \Lambda} \ker(A_\alpha) = \{O\}$. *The map* $J: E \to \prod_{\alpha \in \Lambda} (E_\alpha,\tau_\alpha)$ *defined by* $J(x) = (A_\alpha(x))_{\alpha \in \Lambda}$, $x \in E$, *is a topological isomorphism between* (E,τ) *and* $(J(E), \Pi_{J(E)})$, *where* Π *denotes the product topology on* $\prod_{\alpha \in \Lambda} (E_\alpha,\tau_\alpha)$.

PROOF: It is obvious that J is an injective linear map from E into $\prod_{\alpha \in \Lambda} E_\alpha$. In order to show that J is continuous, let $\alpha \in \Lambda$ be given and let P_α be the canonical projection from $\prod_{\alpha \in \Lambda} E_\alpha$ onto E_α. Since $P_\alpha \circ J = A_\alpha: E \to E_\alpha$ is continuous, the continuity of J follows from 2.6. Now consider $J^{-1}: J(E) \to E$. Since for each $\alpha \in \Lambda$ $A_\alpha \circ J^{-1} = P_\alpha|_{J(E)}$ and P_α is continuous by definition of π, it follows from 2.6 that J^{-1} is continuous.

§ 3 - INDUCTIVE LIMITS

DEFINITION 3.1 - Let $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ be a family of TVS, E be a vector space and $(A_\alpha)_{\alpha\in\Lambda}$ be a family of linear maps from E_α into E and assume that $E = \text{span} \bigcup_{\alpha\in\Lambda} A_\alpha(E_\alpha)$. Let $\mathcal{F}$ be the set of all τ_F-compatible topologies η on E such that all $A_\alpha: (E_\alpha,\tau_\alpha) \to (E,\eta), \alpha\in\Lambda$, are continuous. Let $\tau := \sup\{\eta; \eta\in\mathcal{F}\}$. Then τ is a τ_F-compatible topology τ on E and it is called the *inductive limit topology on* E *with respect to the family* $((E_\alpha,\tau_\alpha),A_\alpha)_{\alpha\in\Lambda}$. The topological vector space (E,τ) is called the *inductive limit of the topological vector spaces* (E_α,τ_α) *with respect to the linear maps* A_α and denoted by $(E,\tau) = \underset{\alpha\in\Lambda}{\text{ind}}((E_\alpha,\tau_\alpha),A_\alpha)$.

REMARK 3.2 - τ *is the finest* τ_F-*compatible topology on* E *for which all* A_α *are continuous*. In fact, by 2.3.3 $(E,\tau) = \underset{\eta\in\mathcal{F}}{\text{proj}}((E,\eta),i_\eta)$, where i_η is the identity map on E for each $\eta\in\mathcal{F}$, and $A_\alpha: (E_\alpha,\tau_\alpha) \to (E,\eta)$ is continuous for all $\alpha\in\Lambda$ and $\eta\in\mathcal{F}$, the continuity of all $A_\alpha: (E_\alpha,\tau_\alpha) \to (E,\tau)$ follows from 2.6. Now, by definition of τ, it is obviuos that $\tau\supset\xi$, for every τ_F-compatible topolgy ξ on E for which all $A_\alpha: (E_\alpha,\tau_\alpha) \to (E,\xi)$ are continuous.

EXAMPLES 3.3.

3.3.1 - Let (E,τ) be a TVS and M a subspace of E. Let Π be the canonical surjection from E onto the quotient space $E/_M$ and τ_q the quotient topology on $E/_M$. It is easy to verify that τ_q is a τ_F-compatible topology on $E/_M$ and because τ_q is the finest τ_F-compatible topology on $E/_M$ for which $\Pi: E \to E/_M$ is continuous, we have $(E/_M,\tau_q) = \text{ind}((E,\tau),\Pi)$.

3.3.2 - Let $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ be a family of TVS and define $E:=\bigoplus_{\alpha\in\Lambda} E_\alpha$. For each $\alpha\in\Lambda$, let I_α be the canonical embedding from E_α into E. The TVS $(E,\tau):=\operatorname{ind}_{\alpha\in\Lambda}((E_\alpha,\tau_\alpha),I_\alpha)$ is called the *direct sum of the family* $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ and it is denoted by $(E,\tau)=\bigoplus_{\alpha\in\Lambda}(E_\alpha,\tau_\alpha)$. The τ_F-compatible topology τ on E is called the *direct sum topology of the family* (E_α,τ_α).

3.3.3 - Let E be a vector space and let $\{\tau_\alpha,\alpha\in\Lambda\}$ be a non-empty family of τ_F-compatible topologies on E . If $\tau=\inf\{\tau_\alpha,\alpha\in\Lambda\}$, then $(E,\tau)=\operatorname{ind}_{\alpha\in\Lambda}((E,\tau_\alpha),i_\alpha)$ where $i_\alpha,\alpha\in\Lambda$, is the identity map on E.

PROPOSITION 3.4 - *Let* $(E,\tau)=\operatorname{ind}_{\alpha\in\Lambda}((E_\alpha,\tau_\alpha),A_\alpha)$ *and let* (G,η) *be an arbitrary* TVS. *A linear map* A *from* E *into* G *is continuous if, and only if, for each* $\alpha\in\Lambda$ *the linear map* $A\circ A_\alpha$ *from* E_α *into* G *is continuous.*

PROOF: Obviuosly the necessity of the condition holds true. Conversely, suppose that for each $\alpha\in\Lambda$ $A\circ A_\alpha$: $(E_\alpha,\tau_\alpha)\to(G,\eta)$ is continuous. Let V be a η-neighborhood of O in G. Then, by hypothesis, $(A\circ A_\alpha)^{-1}(V)=A_\alpha^{-1}(A^{-1}(V))$ is a τ_α-neighborhood of O in E_α for every $\alpha\in\Lambda$, which implies that A_α: $(E_\alpha,\tau_\alpha)\to(E,A^{-1}(\eta))$ is continuous. By definition of τ, $A^{-1}(\eta)\subset\tau$, which implies that $A:(E,\tau)\to(G,\eta)$ is continuous.

PROPOSITION 3.5 - *Let* $(E,\tau)=\operatorname{ind}_{\alpha\in\Lambda}((E_\alpha,\tau_\alpha),A_\alpha)$ *and* (G,η) *be an arbitrary* TVS. *A set* H *of linear maps from* E *into* G *is equicontinuous if, and only if, for each* $\alpha\in\Lambda$ $H_\alpha:=\{T\circ A_\alpha^{\bullet},T\in H\}$ *is an equicontinuous subset of* $\mathcal{L}(E_\alpha;G)$.

PROOF: Suppose that H is an equicontinous set of linear maps from E into G and let $\alpha \in \Lambda$ be given. Since, for each $T \in H$, $T \in \mathcal{L}(E;G)$, by 3.4 we have $H_\alpha \subset \mathcal{L}(E_\alpha;G)$. Let V be a η-neighborhood of O in G. By hypothesis, there is a τ-neighborhood U of O in E such that $T(U) \subset V$ for every $T \in H$. Thus H_α is equicontinuous because $A_\alpha^{-1}(U)$ is a τ_α-neighborhood of O in E_α and $(T \circ A_\alpha)(A_\alpha^{-1}(U)) \subset T(U) \subset V$ for for every $T \in H$.

Conversely, suppose that H is a set of linear maps from E into G such that H_α is an equicontinuous subset of $\mathcal{L}(E_\alpha;G)$ for each $\alpha \in \Lambda$. Let V be a η-neighborhood of O in G. By hypothesis, $\bigcap_{T\in H} (T \circ A_\alpha)^{-1}(V) = A_\alpha^{-1} \bigcap_{T\in H} T^{-1}(V)$ is a τ_α-neighborhood of O in E_α for each $\alpha \in \Lambda$. Let $\mathcal{U}$ be a fundamental system of neighborhoods of O in (G,η) and define $\mathcal{U}^H = \{\bigcap_{T\in H} T^{-1}(U),\ U \in \mathcal{U}\}$. It is obvious that $\mathcal{U}^H$ is a filter basis on E.

Let $\bigcap_{T\in H} T^{-1}(U) \in \mathcal{U}^H$, $U \in \mathcal{U}$. Because $\mathcal{U}$ is a fundamental system of neighborhoods of O in (G,η) , it is easy to verify that $\mathcal{U}^H$ fulfils (V1) - (V3) in 1.1. In order to show that $\mathcal{U}^H$ fulfills also the condition (V4) of 1.1, let $x \in E$ be given. Then there are $\alpha_1,\dots,\alpha_k \in \Lambda$ and $x_{\alpha_i} \in E_{\alpha_i}$, $i=1,\dots,k$, such that $x = \sum_{i=1}^{k} A_{\alpha_i}(x_{\alpha_i})$. Choose $U_i \in U$ such that $\sum_{i=1}^{k} U_i \subset U$. Since $A_{\alpha_i}^{-1}(\bigcap_{T\in H} T^{-1}(U_i)) = \bigcap_{T\in H} (T\, A_{\alpha_i})^{-1}(U_i)$ is a τ_{α_i}-neighborhood of O in E_{α_i}, there is a τ_F-neighborhood V of O in F such that $Vx_{\alpha_i} \subset A_{\alpha_i}^{-1}(\bigcap_{T\in H} T^{-1}(U_i))$ for all $i = 1,\dots,k$. Then $V(A_{\alpha_i}(x_{\alpha_i})) \subset \bigcap_{T\in H} T^{-1}(U_i)$ and we infer

$$Vx \subset \sum_{i=1}^{k} V A_{\alpha_i}(x_{\alpha_i}) \subset \sum_{i=1}^{k} \bigcap_{T\in H} T^{-1}(U_i) \subset \bigcap_{T\in H} T^{-1}(\sum_{i=1}^{k} U_i) \subset \bigcap_{T\in H} T^{-1}(U).$$

So, according to 1.1, there is a unique τ_F-compatible topology $\tau(H)$ on E for which $\mathcal{U}^H$ is a fundamental system of neighborhoods of O in E. Since for each $\alpha \in \Lambda$ $A_\alpha^{-1}(\bigcap_{T\in H} T^{-1}(U)) = \bigcap_{T\in H} (T\circ A_\alpha)^{-1}(U)$, $U \in \mathcal{U}$, is a τ_α-neighborhood of O in E_α, all linear maps $A_\alpha:(E_\alpha,\tau_\alpha)\to(E,\tau(H))$ are continuous. Thus $\tau \supset \tau(H)$, which proves that each element of $\mathcal{U}^H$ is a τ-neighborhood of O in E.

THEOREM 3.6 - *Let* (F,τ_F) *be a locally right-bounded topological division ring and* $(E,\tau) = \operatorname{ind}_{\alpha\in\Lambda} ((E_\alpha,\tau_\alpha),A_\alpha)$. *For each* $\alpha \in \Lambda$, *let* $\mathcal{U}^\alpha$ *be a fundamental system of* τ_α*-neighborhoods of* O *in* E_α. *Let* Δ *be the set of all finite subsets of* $\mathbb{N}$ *and define*

$$\mathcal{U} := \{ \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{\alpha\in\Lambda} A_\alpha(U_k^\alpha) ; (U_k^\alpha)_{k=1}^{\infty} \subset \mathcal{U}^\alpha, U_{k+1}^\alpha + U_{k+1}^\alpha \subset U_k^\alpha, \ \alpha\in\Lambda\}$$

Then $\mathcal{U}$ *is a fundamental system of* τ*-neighborhoods of* O *in* E.

PROOF: It is easy to verify that $\mathcal{U}$ is a filter basis on E. Next we want to prove that $\mathcal{U}$ fulfils (V1)-(V4) in 1.1.. For this, let $U \in \mathcal{U}$,

$U = \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{\alpha\in\Lambda} A_\alpha(U_k^\alpha)$, for some $(U_k^\alpha)_{k=1}^{\infty} \subset \mathcal{U}^\alpha$ with $U_{k+1}^\alpha + U_{k+1}^\alpha \subset U_k^\alpha$, $\alpha\in\Lambda$.

a) Let $W = \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{\alpha\in\Lambda} A_\alpha(U_{2k}^\alpha)$. Since $(U_{2k}^\alpha)_{k=1}^{\infty} \subset \mathcal{U}^\alpha$ and $U_{2(k+1)}^\alpha + U_{2(k+1)}^\alpha \subset U_{2k+1}^\alpha + U_{2k+1}^\alpha \subset U_{2k}^\alpha$ for all $\alpha \in \Lambda$, we have $W \in \mathcal{U}$. Let $x \in W+W$. Then there are $J\in\Delta$, $x_k^{(i)} \in \bigcup_{\alpha\in\Lambda} A_\alpha(U_{2k}^\alpha)$,

$i=1,2;\ k \in J$, such that $x = \sum_{k\in J} x_k^{(1)} + \sum_{k\in J} x_k^{(2)}$. Let $y_{2k-1} = x_k^{(1)}$ and $y_{2k} = x_k^{(2)}$, $k \in J$. Then $y_{2k-1} \in \bigcup_{\alpha\in\Lambda} A_\alpha(U_{2k}^\alpha) \subset \bigcup_{\alpha\in\Lambda} A_\alpha(U_{2k-1}^\alpha)$ and $y_{2k} \in \bigcup_{\alpha\in\Lambda} A_\alpha(U_{2k}^\alpha)$. Thus $x = \sum_{j\in J'} y_j \in \sum_{j\in J'} \bigcup_{\alpha\in\Lambda} A_\alpha(U_j^\alpha) \subset U$, where $J' = (2J-1) \cup 2J$. So $W+W \subset U$.

Let V be the right-bounded τ_F-neighborhood of 0 in F.

(V2) - Inductively, we can choose $(W_k^\alpha)_{k=1}^\infty \subset \mathcal{U}^\alpha$ such that $W_{k+1}^\alpha + W_{k+1}^\alpha \subset W_k^\alpha$ and $VW_k^\alpha \subset U_k^\alpha$. Let $W := \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{\alpha\in\Lambda} A_\alpha(W_k^\alpha) \in \mathcal{U}$. Then $VW \subset \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{\alpha\in\Lambda} A_\alpha(VW_k^\alpha) \subset \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{\alpha\in\Lambda} A_\alpha(U_k^\alpha) = U$.

(V3) - Let $\lambda \in F$, $\lambda \neq 0$. For each $\alpha \in \Lambda$, define $W_1^\alpha = \lambda U_1^\alpha$. By induction, for each $\alpha \in \Lambda$ we can find a sequence $(W_k^\alpha)_{k=1}^\infty \subset \mathcal{U}^\alpha$ with $W_{k+1}^\alpha + W_{k+1}^\alpha \subset W_k^\alpha \subset \lambda U_k^\alpha$ for each $k \in \mathbb{N}$. Set $W := \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{\alpha\in\Lambda} A_\alpha(W_k^\alpha)$. Then $W \in \mathcal{U}$ and $W \subset \lambda U$.

(V4) - Let $x \in E$. Then there are $k_0 \in \mathbb{N}$ and $x_{\alpha_k} \in E_{\alpha_k}$, $k \in \{1,\dots,k_0\}$, such that $x = \sum_{k=1}^{k_0} A_{\alpha_k}(x_{\alpha_k})$. Let W be a τ_F-neighborhood of 0 in F such that for each k, $1 \le k \le k_0$, $Wx_{\alpha_k} \subset U_k^{\alpha_k}$. Then $Wx = \sum_{k=1}^{k_0} A_{\alpha_k}(Wx_{\alpha_k}) \subset \sum_{k=1}^{k_0} A_{\alpha_k}(U_k^{\alpha_k}) \subset U$.

Let $\tau_{\mathcal{U}}$ be the unique τ_F-compatible topology on E for which $\mathcal{U}$ is a fundamental system of neighborhoods of 0 in E. Obviously for each $\alpha \in \Lambda$ $A_\alpha: (E_\alpha,\tau_\alpha) \to (E,\tau_{\mathcal{U}})$ is continuous. From the definition of τ we have $\tau_{\mathcal{U}} \subset \tau$. Conversely, let U_1 be a τ-neighborhood of 0 in E and let $(U_n)_{n=1}^\infty$ be a sequence of τ-neighborhoods of 0 in E such that $U_{n+1} + U_{n+1} \subset U_n$ for all $n=1,2,\dots$. Since for each $\alpha \in \Lambda$ $A_\alpha^{-1}(U_n)$ is a

τ_α-neighborhood of O in E_α, we can choose, by induction on n, $U_n^\alpha \in \mathcal{U}^\alpha$ such that $U_n^\alpha \subset A_\alpha^{-1}(U_{n+1})$, $U_{n+1}^\alpha + U_{n+1}^\alpha \subset U_n^\alpha$. Then for every $n \in \mathbb{N}$ $\sum_{k=1}^{n} \bigcup_{\alpha\in\Lambda} A_\alpha(U_k^\alpha) \subset \sum_{k=1}^{n} U_{k+1} \subset U_1$, which proves that $\tau_{\mathcal{U}} \supset \tau$. Thus $\tau_{\mathcal{U}} = \tau$.

REMARK: The assumption $U_{n+1}^\alpha + U_{n+1}^\alpha \subset U_n^\alpha$ in the definition of U can be omitted , because for every sequence $(U_n^\alpha)_{n=1}^\infty$ of τ_α-neighborhoods of O in E_α we can always choose $W_n^\alpha \in \mathcal{U}^\alpha$ such that $W_n^\alpha \subset U_n^\alpha$ and $W_{n+1}^\alpha + W_{n+1}^\alpha \subset W_n^\alpha$.

PROPOSITION 3.7 - *Let* (F,τ_F) *be a right -bounded topological division ring,* $(E,\tau) = \operatorname{ind}_{n\in\mathbb{N}} ((E_n,\tau_n),A_n)$ *and* $\mathcal{U}^n$ *be a fundamental system of* τ_n*-neighborhoods of* O *in* $E_n, n \in \mathbb{N}$. *Let* Δ *be the set of all finite subsets of* $\mathbb{N}$. *Define*

$$\mathcal{U}' = \{ \bigcup_{J\in\Delta} \sum_{k\in J} A_k(U_k);\ U_k \in \mathcal{U}^k\}.$$

Then $\mathcal{U}'$ *is a fundamental system of* τ*-neighborhoods of* O *in* E.

PROOF: From 3.6, it is enough to prove that $\mathcal{U}'$ is a filter basis on E generating the same filter as the set $\mathcal{U}$ defined there, of course when $\Lambda = \mathbb{N}$. It is clear that $\mathcal{U}'$ is a filter basis on E. Let $U \in \mathcal{U}$, $U = \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{n\in\mathbb{N}} A_n(U_k^n)$. Then $U \supset \bigcup_{J\in\Delta} \sum_{k\in J} A_k(U_k^k) \in \mathcal{U}'$, which proves that the filter generated by $\mathcal{U}'$ is finer than the filter generated by $\mathcal{U}$. Conversely, let $U \in \mathcal{U}'$, $U = \bigcup_{J\in\Delta} \sum_{k\in J} A_k(U_k)$. For each $n \in \mathbb{N}$, choose $(U_k^n)_{k=1}^\infty \subset \mathcal{U}^n$ with $U_1^n + U_1^n \subset U_n$ and $U_{k+1}^n + U_{k+1}^n \subset U_k^n$ for each $k \in \mathbb{N}$. Then $\sum_{k\in J} U_k^n \subset U_n$ for every $J \in \Delta$. Let

$W = \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{n\in\mathbb{N}} A_n(U_k^n) \in \mathcal{U}$. We claim that $W \subset U$. In fact, let $x \in W$. Then there is a $J \in \Delta$ such that for every $k \in J$ there is an $n_k \in \mathbb{N}$ such that $x \in \sum_{k\in J} A_{n_k}(U_k^{n_k})$. Let $J' = \{n_k : k \in J\}$. Then

$$x \in \sum_{k\in J} \sum_{n\in J'} A_n(U_k^n) = \sum_{n\in J'} A_n(\sum_{k\in J} U_k^n)$$

$$\subset \sum_{n\in J'} A_n(U_n) \subset \sum_{J\in\Delta} \sum_{k\in J} A_k(U_k) = U.$$

The assumption that (F,τ_F) is a locally right-bounded topological division ring is necessary in the following sense:

PROPOSITION 3.8 - *Let* (F,τ_F) *be a topological division ring and* $(E,\tau) = \oplus_{\mathcal{V}}(F,\tau_F)$, *where* $\mathcal{V}$ *is fundamental system of* τ_F-*neighborhoods of* 0 *in* F. *Then a set* $\mathcal{U}$ *of subsets of* F, *as defined in* 3.6, *is a fundamental system of neighborhoods of* 0 *in* (E,τ) *if, and only if,* F *is locally right-bounded.*

PROOF: It is clear that the condition is sufficient. Conversely, for each $V \in \mathcal{V}$, we can inductively construct a sequence $(U_k^V)_{k=1}^{\infty} \subset \mathcal{V}$, such that $U_{k+1}^V + U_{k+1}^V \subset U_k^V$, $k \geq 1$, and $U_1^V + U_1^V \subset V$. Let $U \in \mathcal{U}$. Then $U = \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{V\in\mathcal{V}} i_V(U_k^V)$, where $i_V : F \to E$ is the V-th coordinate map. By assumption, there are $(W_k^V)_{k=1}^{\infty} \subset \mathcal{V}$ with $W_{k+1}^V + W_{k+1}^V \subset W_k^V$, $k \geq 1$, and $W_1^V + W_1^V \subset V$ and $V' \in \mathcal{V}$ such that $V'W \subset U$, where $W = \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{V\in\mathcal{V}} i_V(W_k^V)$. Let $\tilde{V} \in \mathcal{V}$ and let $i^{\tilde{V}} : E \to F$ be the projection of the $\tilde{V}$-th component of E onto F. Then since $i^{\tilde{V}} i_V = \delta_{\tilde{V},V}\mathrm{id}_F$, $V'W_1^{\tilde{V}} \subset i^{\tilde{V}}(V' \bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{V\in\mathcal{V}} i_V(W_k^V))$

$$\subset i^{\tilde{V}}(U) = i^{\tilde{V}}(\bigcup_{J\in\Delta} \sum_{k\in J} \bigcup_{V\in\mathcal{V}} i_V(U_k^V)) = \bigcup_{J\in\Delta} \sum_{k\in J} U_k^{\tilde{V}} \subset \bigcup_{J\in\Delta} \tilde{V} = \tilde{V},$$

what means, since $\tilde{V} \in \mathcal{V}$ was arbitrary, that V' is a right-bounded neighborhood of O in F.

PROPOSITION 3.9 - *Let* Λ *be a finite subset of* $\mathbb{N}$ *and let* $(E,\tau) = \operatorname{ind}_{\alpha\in\Lambda}((E_\alpha,\tau_\alpha),A_\alpha)$. *For each* $\alpha \in \Lambda$, *let* $\mathcal{U}^\alpha$ *be a fundamental system of* τ_α*-neighborhoods of* O *in* E_α. *Then the set*

$$\mathcal{U} = \{ \sum_{\alpha\in\Lambda} A_\alpha(U_\alpha),\ U_\alpha \in \mathcal{U}^\alpha \}$$

is a fundamental system of τ*-neighborhoods of* O *in* E.

PROOF: It is similar to the proof of theorem 3.6, except for the proof of the condition (V2) of th. 1.1. . In order to prove (V2), let $U = \sum_{\alpha\in\Lambda} A_\alpha(U_\alpha) \in \mathcal{U}$. Then for each $\alpha \in \Lambda$ there is a τ_F-neighborhood V^α of O in F and $W_\alpha \in \mathcal{U}^\alpha$ such that $V^\alpha W_\alpha \subset U_\alpha$ and taking $V := \bigcap_{\alpha\in\Lambda} V^\alpha$, we have that V is a τ_F-neighborhood of O in F, $W = \sum_{\alpha\in\Lambda} A_\alpha(W_\alpha) \in \mathcal{U}$ and
$VW = V \sum_{\alpha\in\Lambda} A_\alpha(W_\alpha) = \sum_{\alpha\in\Lambda} A_\alpha(VW_\alpha) \subset \sum_{\alpha\in\Lambda} A_\alpha(U_\alpha) = U$.

PROPOSITION 3.10 - *Let* $(E,\tau) = \bigoplus_{\alpha\in\Lambda}(E_\alpha,\tau_\alpha)$. *Then for every non-empty finite subset* Φ *of* Λ *the direct sum topology of the family* $(E_\alpha,\tau_\alpha)_{\alpha\in\Phi}$ *and the induced topology by* τ *on* $\bigoplus_{\alpha\in\Phi} E_\alpha$ *coincide.*

PROOF: Let Φ be a finite subset of Λ and set $(H,\eta) := \bigoplus_{\alpha\in\Phi}(E_\alpha,\tau_\alpha)$. Let τ_H be the induced topology by τ on H. For each $\alpha_o \in \Phi$, let $\tilde{I}_{\alpha_o}$ be the canonical embedding

from $(E_{\alpha_o},\tau_{\alpha_o})$ into (H,η) and $\tilde{I}_{\alpha_o} = 0$ if $\alpha_o \notin \Phi$.
For each $\alpha_o \in \Lambda$, let I_{α_o} be the canonical embedding from $(E_{\alpha_o},\tau_{\alpha_o})$ into (E,τ). Let I be the canonical embedding from (H,η) into (E,τ). Then, from the definition of η and τ, we have that $\tilde{I}_{\alpha_o}$ and I_{α_o} are continuous and, since $I \circ \tilde{I}_{\alpha_o} = I_{\alpha_o}$, it follows that I is continuous by 3.4 which implies that $\eta \supset \tau_H$. Now, consider $i:(H,\tau_H) \to (H,\eta)$ the identity map on H. Then $i = P \circ I_H$ where P is the projection from (E,τ) onto (H,η) and I_H is the embedding from (H,τ_H) into (E,τ). Since I_H is continuous and P is continuous (the continuity of P follows from 3.4 because for each $\alpha \in \Lambda$ $P\circ I_\alpha=\tilde{I}_\alpha$), it follows that i is continuous, and so, $\eta \subset \tau_H$. Thus $\eta = \tau_H$.

COROLLARY 3.11 - *Let* $(E,\tau) = \bigoplus_{\alpha\in\Lambda} (E_\alpha,\tau_\alpha)$. *Then for each* $\alpha \in \Lambda$ *the induced topology by* τ *on* E_α *and* τ_α *coincide.*

PROPOSITION 3.12 - *Let* (E_α,τ_α) *be a family of topological vector spaces. If* $(E,\tau) = \bigoplus_{\alpha\in\Lambda} (E_\alpha,\tau_\alpha)$ *and* $(G,\eta) = \prod_{\alpha\in\Lambda} (E_\alpha,\tau_\alpha)$, *then:*

i) τ *is finer than the induced topology by* η *on* E;

ii) *for every finite subset* Φ *of* Λ, τ *and* η *coincide on* $\bigoplus_{\alpha\in\Phi} E_\alpha$.

PROOF: For each $\alpha \in \Lambda$, let I_α be the canonical embedding from E_α into E and P_α the projection from G onto E_α.
Proof of i)-Let I be the canonical embedding from (E,τ) into (G,η) and let $\alpha \in \Lambda$ be given. Since $I \circ I_\alpha$ is the canonical embedding from E_α into (G,η), which is continuous by (2.6),

by 3.4 we have that I is continuous. Thus τ is finer than the induced topology by η on E.

Proof of ii) - Let $\Phi \subset \Lambda$ finite and let $H := \bigoplus_{\alpha\in\Phi} E_\alpha$. By (i), it is enough to prove that η is finer than τ on H. For this, let U be a τ_H-neighborhood of 0 in H. By 3.10 and 3.9, there are τ_α-neighborhoods U_α of 0 in E_α, $\alpha \in \Phi$, such that

$U \supset \sum_{\alpha\in\Phi} I_\alpha(U_\alpha)$.

Then $M = \bigcap_{\alpha\in\Phi} P_\alpha^{-1}(U_\alpha)$ is a η-neighborhood of 0 in G by 2.3.2 and 2.1. So $V := M \cap \bigoplus_{\alpha\in\Phi} E_\alpha$ is a neighborhood of 0 for the induced topology by η on $\bigoplus_{\alpha\in\Phi} E_\alpha$ with

$$V = (\bigcap_{\alpha\in\Phi} P_\alpha^{-1}(U_\alpha)) \cap \bigoplus_{\alpha\in\Phi} E_\alpha = \bigcap_{\alpha\in\phi} P_\alpha^{-1}(I_\alpha^{-1}(I_\alpha(U_\alpha))) \cap \bigoplus_{\alpha\in\Phi} E_\alpha =$$

$$= \bigcap_{\alpha\in\phi} [(I_\alpha U_\alpha + \bigoplus_{\substack{\beta\in\Lambda \\ \beta\neq\alpha}} E_\beta) \cap \bigoplus_{\alpha\in\phi} E] = \bigcap_{\alpha\in\phi} [I_\alpha(U_\alpha) + \bigoplus_{\substack{\beta\in\phi \\ \beta\neq\alpha}} E_\beta] =$$

$= \bigoplus_{\alpha\in\phi} U_\alpha \subset \sum_{\alpha\in\phi} I_\alpha(U_\alpha) \subset U$, which proves that U is a neighborhood of 0 for the induced topology by η on $H = \bigoplus_{\alpha\in\Phi} E_\alpha$.

COROLLARY 3.13 - *The direct sum topology and product topology coincide when we consider a finite family of* TVS.

COROLLARY 3.14 - *Let* $(E,\tau) = \bigoplus_{\alpha\in\Lambda} (E_\alpha, \tau_\alpha)$. *Then* (E,τ) *is a Hausdorff* TVS *if, and only if, for each* $\alpha \in \Lambda$ (E_α,τ_α) *is a Hausdorff* TVS.

PROPOSITION 3.15 - *If* $(E,\tau) = \operatorname{ind}_{\alpha\in\Lambda}((E_\alpha,\tau_\alpha), A_\alpha)$, *then* (E,τ) *is linearly and topologically isomorphic to the quotient space* $\bigoplus_{\alpha\in\Lambda} (E_\alpha,\tau_\alpha)/_N$, *where* N *is the kernel of the map*

$J: \bigoplus_{\alpha\in\Lambda} E_\alpha \to E$ *with* $J(x) := \sum_{\alpha\in\Lambda} A_\alpha(x_\alpha)$, *for each* $x = (x_\alpha)_{\alpha\in\Lambda} \in \bigoplus_{\alpha\in\Lambda} E_\alpha$.

PROOF: Let $(G,\eta) = \bigoplus_{\alpha\in\Lambda}(E_\alpha,\tau_\alpha)$ and it is clear that $J: G \to E$ is a linear map. Thus we can consider the quotient space $G/_N$ equipped with the quotient topology $\dot{\eta}$. Let π_J be the canonical surjection from (G,η) onto $(G/_N,\dot{\eta})$ and let $\hat{J}$ be the linear map from $G/_N$ into E induced by J, i.e., $\hat{J}\circ\pi_J = J$. Since J is a continuous surjection (from the hypothesis made on E we have that J is surjective and its continuity follows from 3.4 because for each $\alpha \in \Lambda$ $J\circ I_\alpha = A_\alpha$, where I_α is the canonical embedding from E_α into G), we have that $\hat{J}$ is a continuous bijection. Let $\hat{J}^{-1}: E \to G/_N$ be the inverse map of $\hat{J}$. Then $\hat{J}^{-1}$ is a linear map and its continuity comes from the continuity of $\hat{J}^{-1}\circ A_\alpha = \pi_J\circ I_\alpha$, $\alpha \in \Lambda$ and 3.4.

REMARK 3.16 - Let $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ be a family of TVS and let Φ and ψ be disjoined subsets of Λ with $\Lambda = \Phi \cup \psi$. If we define $(E,\tau) := \bigoplus_{\alpha\in\Lambda}(E_\alpha,\tau_\alpha)$, $(G,\eta) := \bigoplus_{\sigma\in\Phi}(E_\alpha,\tau_\alpha)$ and $(H,\xi) := \bigoplus_{\alpha\in\psi}(E_\alpha,\tau_\alpha)$, then, since (E,τ) is the topological direct sum of (G,η) and (H,ξ), we have that (E,τ) is the topological product of (G,η) and (H,ξ) by corollary 3.13.

COROLLARY 3.17 - *Let* $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ *be a family of Hausdorff* TVS. *Then, for each subset* Φ *of* Λ, $\bigoplus_{\alpha\in\Phi} E_\alpha$ *is a closed subspace of* $(E,\tau) = \bigoplus_{\alpha\in\Lambda}(E_\alpha,\tau_\alpha)$.

PROOF: Let $\Phi \subset \Lambda$ be given. Let ψ be a subset of Λ with

$\Phi \cap \psi = \phi$ and $\Lambda = \phi \cup \psi$. By 3.16 we have

$(E,\tau) = (\underset{\alpha\in\Phi}{\oplus}(E_\alpha,\tau_\alpha)) \times (\underset{\alpha\in\psi}{\oplus}(E_\alpha,\tau_\alpha))$, which implies that

$\underset{\alpha\in\psi}{\oplus}(E_\alpha,\tau_\alpha)$ is linearly and topologically ismorphic to the quotient space of (E,τ) by $\underset{\alpha\in\Phi}{\oplus} E_\alpha$. Since for each $\alpha\in\psi$ (E_α,τ_α) is a Hausdorff TVS, it follows from 3.14 that $\underset{\alpha\in\psi}{\oplus}(E_\alpha,\tau_\alpha)$ is a Hausdorff TVS. So the quotient space is a Hausdorff TVS, which implies that $\underset{\alpha\in\Phi}{\oplus} E_\alpha$ is τ-closed in E.

COROLLARY 3.18 - *Under the hypothesis made in* 3.17, *for each* $\alpha \in \Lambda$ E_α *is a closed subspace of* (E,τ).

DEFINITION 3.19 - If $(E,\tau) = \underset{n\in\mathbb{N}}{\text{ind}}((E_n,\tau_n),I_n)$ where

(i) $(E_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of subspaces of E with $E = \bigcup_{n=1}^{\infty} E_n$,

(ii) I_n is the canonical embedding from E_n into E for each $n\in\mathbb{N}$,

(iii) the induced topology on E_n by τ_{n+1} coincides with τ_n for each $n\in\mathbb{N}$,

then we say that (E,τ) is the *strict inductive limit* of the sequence $(E_n)_{n\in\mathbb{N}}$ and we denote it by $(E,\tau) = \underset{\overrightarrow{n\in\mathbb{N}}}{\text{ind}}(E_n,\tau_n)$.

REMARK 3.20 - If $(E,\tau) = \underset{\overrightarrow{n\in\mathbb{N}}}{\text{ind}}(E_n,\tau_n)$, then $(E,\tau) = \underset{\overrightarrow{k\in\mathbb{N}}}{\text{ind}}(E_{n_k},\tau_{n_k})$ for every subsequence $(n_k')_{k\in\mathbb{N}}$.

EXAMPLE 3.21 - If $(E,\tau) = \bigoplus_{n \in N} (E_n, \tau_n)$, then $(E,\tau) = \underrightarrow{\text{ind}}_{j \in \mathbb{N}} (G_j, \beta_j)$, where $(G_j,\beta_j) = \bigoplus_{1 \le n \le j} (E_n,\tau_n)$ for each $j \in \mathbb{N}$. In fact, it is obvious that the family $\{G_j, j \in \mathbb{N}\} = \{\bigoplus_{1 \le n \le j} E_n, j \in \mathbb{N}\}$ is a strictly increasing sequence of subspaces of E with $E = \bigcup_{j=1}^{\infty} G_j = \bigcup_{j=1}^{\infty} E_1 \oplus E_2 \oplus \dots \oplus E_j$. Then, from 3.10, it follows that the induced topolgy on G_j by β_{j+1} coincides with β_j for each $j \in \mathbb{N}$. Let $(E,\tau') := \underrightarrow{\text{ind}}_{j \in \mathbb{N}} ((G_j,\beta_j),A_j)$, where for each $j \in \mathbb{N}$ $A_j : G_j \to E$ is the canonical embedding. We claim that $\tau = \tau'$. The mappings $A_j : (G_j,\beta_j) \to (E,\tau')$ and $I_{nj} : (E_n,\tau_n) \longrightarrow (G_j,\beta_j)$, $n \le j$, are continuous. Thus $A_j \circ I_{nj} : (E_n,\tau_n) \to (E,\tau')$ is continuous for each $n \in \mathbb{N}$. Since the embedding $I_n : (E_n,\tau_n) \to E$ is equal to $A_j \circ I_{nj}$ and τ is the finest topology on E for which all τ_n are continuous we conclude that $\tau' \subset \tau$. Now, if i is the identity map from (E,τ') into $E,\tau)$, then its continuity follows from 3.4 because for each $j \in \mathbb{N}$, $i \circ A_j : (G_j,\beta_j) \to (E,\tau)$ is continuous. This implies that $\tau \subset \tau'$. So $\tau = \tau'$.

PROPOSITION 3.22 - *Suppose that* (F,τ_F) *is a locally right-bounded topological division ring and let* $(E,\tau) = \underrightarrow{\text{ind}}_{n \in \mathbb{N}} (E_n,\tau_n)$. *Then for every* $n \in \mathbb{N}$ *the induced topology on* E_n *by* τ *coincides with* τ_n.

PROOF: Let $n \in \mathbb{N}$ be given. Let τ_n' be the induced topology by τ on E_n and let I_n be the canonical embedding from E_n into E. Since τ_n' is the coarsest τ_F-compatible topology on E_n for which I_n is continuous and by hypothesis

$I_n: (E_n, \tau_n) \to (E, \tau)$ is continuous, we have $\tau'_n \subset \tau_n$. Now we want to show that $\tau_n \subset \tau'_n$. For this, let W_n be a τ_n-neighborhood of 0 in E_n and let U_n be a basic τ_n-neighborhood of 0 in E with

$$(1) \qquad \underbrace{U_n + U_n + \dots + U_n}_{(n+1)\text{-terms}} \subset W_n.$$

Since the induced topology on E_n by τ_{n+1} coincides with τ_n, there is U'_{n+1}, a basic τ_{n+1}-neighborhood of 0 in E_{n+1}, with $U'_{n+1} \cap E_n \subset U_n$. Therefore, there is U_{n+1}, a basic τ_{n+1}-neighborhood of 0 in E_{n+1}, such that

$$(2) \qquad (U_{n+1} + U_{n+1}) \cap E_n \subset U'_{n+1} \cap E_n \subset U_n.$$

In an analogous fashion, there is U_{n+1}, a basic τ_{n+1}-neighborhood of 0 in E_{n+2}, such that $(U_{n+2} + U_{n+2}) \cap E_{n+1} \subset U_{n+1}$. From this and (2) it follows that $(U_{n+2}+U_{n+2}+U_{n+1}) \cap E_n \subset U_n$. If we continue in this way, we can find U_{n+j}, a basic τ_{n+j}-neighborhood of 0 in E_{n+j}, $j \geq 1$, such that for every $r \in \mathbb{N}$,

$$(U_{n+1} + U_{n+2} + \dots + U_{n+r} + U_{n+r}) \cap E_n \subset U_n.$$

So, $\left(\bigcup_{r\geq 1} \sum_{1\leq j\leq r} U_{n+j} \right) \cap E_n \subset U_n$ and using (1) we have

$$\Big(\bigcup_{r\geq 1} \sum_{1\leq j\leq r} U_{n+j} + \underbrace{U_n + U_n + \dots + U_n}_{n\text{-terms}} \Big) \cap E_n \subset$$

$$\subset \Big(\bigcup_{r\geq 1} \sum_{1\leq j\leq r} U_{n+j} \Big) \cap E_n + \underbrace{U_n + \dots + U_n}_{n\text{-terms}} \subset \underbrace{U_n + \dots + U_n}_{(n+1)\text{-terms}} \subset W_n.$$

By 3.7, $W := \bigcup_{r\geq 1} (\underbrace{U_n + \dots + U_n}_{n\text{-terms}} + \sum_{1\leq j\leq r} U_{n+j})$ is a τ-neighborhood of 0 in E. Therefore $W \cap E_n$ is a τ'_n-neighborhood of 0 in E_n cointained in W_n, which implies that $\tau_n \subset \tau'_n$.

COROLLARY 3.23 - *The strict inductive limit of a sequence of* TVS *over a locally right-bounded topological division ring is a Hausdorff* TVS *if, and only if, each element of the sequence is a Hausdorff* TVS.

COROLLARY 3.24 - *Suppose that* (F,τ_F) *is a locally right-bounded topological division ring and let* $(E,\tau) = \underset{\overrightarrow{n\in\mathbb{N}}}{\text{ind}}(E_n,\tau_n)$. *If for each* $n \in \mathbb{N}$ E_n *is closed in* (E_{n+1},τ_{n+1}), *then* E_n *is closed in* (E,τ).

PROOF: Let $n \in \mathbb{N}$ be given and suppose that E_n is closed in (E_{n+1},τ_{n+1}). It is clear, by induction, that E_n is closed in (E_{n+p},τ_{n+p}), $p\geq 1$. Let $x \in E$ be such that $x \notin E_n$. Then there is $p\geq 1$ such that $x \in E_{n+p}$. Since E_n is closed in (E_{n+p},τ_{n+p}), there is V_{n+p}, a τ_{n+p}-neighborhood of 0 in E_{n+p}, such that $(x+V_{n+p}) \cap E_n = \emptyset$. By 3.22, there is a τ-neighborhood V of 0 in E such that $V \cap E_{n+p} = V_{n+p}$. Then $(x+V) \cap E_n = \emptyset$, i.e., $\complement E_n$ is τ-open in E.

LEMMA 3.25 - *Let* $(x_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* (E,τ) *and let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a convergent sequence to* 0 *in* (F,τ_F). *Then* $(\lambda_n x_n)_{n\in\mathbb{N}}$ *is a convergent sequence to* 0 *in* (E,τ).

PROOF: Let $B = \{x_n;\ n \in \mathbb{N}\}$ and let W be a τ-neighborhood of O in E. Since B is τ-bounded, there is a τ_F-neighborhood V of O in F such that $VB \subset W$. Because $\lambda_n \to O$ in F when $n \to \infty$, there is $n_o \in \mathbb{N}$ such that $\lambda_n \in V$ for every $n \geq n_o$. Let $n \geq n_o$ be given. Then $\lambda_n B \subset W$, which implies that $\lambda_n x_n \in W$.

PROPOSITION 3.26 - *Suppose that* (F,τ_F) *is a metrizable locally right-bounded topological division ring and let* $(E,\tau) = \underset{n\in\mathbb{N}}{\underrightarrow{\text{ind}}}\ (E_n,\tau_n)$, *where for each* $n \in \mathbb{N}$ E_n *is closed in* (E_{n+1},τ_{n+1}). *Let* B *be a non-empty subset of* E. *Then* B *is bounded in* (E,τ) *if, and only of, there is* $n \in \mathbb{N}$ *such that* B *is bounded in* (E_n,τ_n).

PROOF: The sufficiency of the condition is immediate because the canonical embedding from (E_n,τ_n) into (E,τ) is continuous for every $n \in \mathbb{N}$ and in this case it is not necessary to suppose that (F,τ_F) is metrizable. Conversely, suppose that B is bounded in (E,τ) and $B \not\subset E_n$ for every $n \in \mathbb{N}$. By 3.20, without lost of generality, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in B with $x_n \in E_{n+1}$ and $x_n \notin E_n$, $n=1,2,\ldots$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $F\setminus\{O\}$ with $\lambda_n \to O$ when $n \to \infty$. Since by 3.24 for every $n \in \mathbb{N}$ E_n is closed in (E,τ) we can find a strictly decreasing sequence $(U_{n+1})_{n\in\mathbb{N}}$ of basic τ-neighborhoods of O in E with $U_{n+1} + U_{n+1} \subset U_n$ and such that $\lambda_n x_n \notin U_{n+1} + E_n$. Let $W_n := U_{n+1} \cap E_n$, $n=1,2,\ldots$. Then, by 3.22, W_n is a τ_n-neighborhood of O in E_n. Let $U = \bigcup_{J\in\Delta} \sum_{k\in J} W_k$, where Δ is the set of all finite subsets of $\mathbb{N}$. So, by 3.7, U is a τ-neighborhood of O in E and since $U \subset \bigcup_{J\in\Delta} (E_n + \sum_{\substack{k\in J\\ k>n+1}} W_k) \subset \bigcup_{J\in\Delta} (E_n+U_{n+1}) = E_n+U_{n+1}$, we have $\lambda_n x_n \notin U$ for all $n \in \mathbb{N}$. But this is a contradiction because, by 3.25, $\lambda_n x_n \to 0$ in (E,τ) when $n\to\infty$. So $B \subset E_{n_o}$ for some $n_o \in \mathbb{N}$. We have also that B is a τ_{n_o}-bounded in E_{n_o} because, by 3.22, τ coincides with τ_{n_o} in E_{n_o} and, by hypothesis, B is τ-bounded in E.

PROPOSITION 3.27 - *Suppose that* (F,τ_F) *is a metrizable locally right-bounded topological division ring and let* $(E,\tau) = \bigoplus_{\alpha\in\Lambda} (E_\alpha,\tau_\alpha)$ *be a Hausdorff* TVS. *Let* B *be a non-empty subset of* E. *Then* B *is bounded in* (E,τ) *if, and only if, there are a finite subset* $\{\alpha_1,\dots,\alpha_n\} \subset \Lambda$ *and bounded subsets* M_{α_i} *of* $(E_{\alpha_i},\tau_{\alpha_i})$, $1\le i\le n$, *such that* $B \subset \sum_{i=1}^{n} I_{\alpha_i}(M_{\alpha_i})$, *where* I_α *is the canonical embedding from* E_α *into* E *for each* $\alpha \in \Lambda$.

PROOF: It is immediate that the condition is sufficient and in this case it is not necessary to suppose that (F,τ_F) is metrizable. Conversely, suppose that B is bounded in (E,τ). We claim that there is a finite subset $\Delta \subset \Lambda$ such that $P_\alpha(B) = \{0\}$ for all $\alpha\in\Lambda\setminus\Delta$, where P_α is the projection from (E,τ) onto (E_α,τ_α) for each $\alpha \in \Lambda$, and in this case $B \subset \sum_{\alpha\in\Delta} I_\alpha(P_\alpha(B))$. For this, suppose that there is a countable subset $\Lambda_o := \{\alpha_i, i \in \mathbb{N}\} \subset \Lambda$ such that $P_{\alpha_i}(B)\neq\{0\}$, $\alpha_i \in \Lambda_o$. Let $P_{\Lambda_o} := (P_{\alpha_i})_{i\in\mathbb{N}}$ be the projection from E onto $\bigoplus_{\alpha_i\in\Lambda_o} E_{\alpha_i}$. Since B is bounded and P_{Λ_o} is continuous we have that $P_{\Lambda_o}(B)$

is bounded in $\bigoplus_{\alpha \in \Lambda_o} (E_\alpha, \tau_\alpha) = \underset{\overrightarrow{k \in \mathbb{N}}}{\text{ind}} \bigoplus_{j=1}^{k} (E_{\alpha_j}, \tau_{\alpha_j})$ (see example 3.21). But for every $k \in \mathbb{N}$ $P_{\Lambda_o}(B) \not\subset \bigoplus_{j=1}^{k} (E_{\alpha_j}, \tau_{\alpha_j})$, which is a contradiction by 3.26.

COROLLARY 3.28 - *Suppose that* (F, τ_F) *is a metrizable locally right-bounded topological division ring and let* $(E_\alpha, \tau_\alpha)_{\alpha \in \Lambda}$ *be an infinite family of Hausdorff* TVS *over it. Then* $(E, \tau) = \bigoplus_{\alpha \in \Lambda} (E_\alpha, \tau_\alpha)$ *is not metrizable.*

PROOF: Let Φ and Ψ be disjoint subsets of Λ such that $\Lambda = \Phi \cup \Psi$. Then, since $(E,\tau) = (\bigoplus_{a \in \Phi} (E_\alpha, \tau_\alpha)) \times (\bigoplus_{\alpha \in \Psi} (E_\alpha, \tau_\alpha))$ (see 3.16), it is enough to prove the assertion when we consider a countable subset Ψ of Λ. Then, suppose that Ψ is the set $\mathbb{N}$ of all natural numbers and let $(G, \eta) := \bigoplus_{n \in \mathbb{N}} (E_n, \tau_n)$. Suppose also that (G, η) is metrizable and let $(U_j)_{j \in \mathbb{N}}$ be a monotonically decreasing fundamental sequence of η-neighborhoods of 0 in G. For each $j \in \mathbb{N}$, let $(G_j, \beta_j) = \bigoplus_{1 \leq n \leq j} (E_n, \tau_n)$. Then, $(G, \eta) = \underset{\overrightarrow{j \in \mathbb{N}}}{\text{ind}} (G_j, \beta_j)$ and for each $j \in \mathbb{N}$ G_j is a proper subspace of G closed in (G_{j+1}, τ_{j+1}). Let $(x_j)_{j \in \mathbb{N}}$ be a sequence in G such that $x_j \notin G_j$ and $x_j \in U_j$, $j \in \mathbb{N}$. Since $x_j \notin G_j$ for each $j \in \mathbb{N}$, it follows from 3.26, that $(x_j)_{j \in \mathbb{N}}$ is a non-bounded sequence in (G, η), which contradicts the fact that $(U_j)_{j \in \mathbb{N}}$ is a monotonically decreasing fundamental sequence of η-neighborhoods of 0 in G and $x_j \in U_j$ for every $j \in \mathbb{N}$.

COROLLARY 3.29 - *Let* $(E_\alpha,\tau_\alpha)_{\alpha\in\Lambda}$ *be a family of sequentially complete (resp. quasi-complete) Hausdorff* TVS *over the same metrizable locally right-bounded toplogical division ring* (F,τ_F). *Then* $(E,\tau) = \bigoplus_{\alpha\in\Lambda} (E_\alpha,\tau_\alpha)$ *is a sequentially complete (resp. quasi-complete)* TVS.

PROOF: Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (E,τ). Then $B = \{x_n : n\in\mathbb{N}\}$ is a bounded subset of (E,τ). Let P_α be the projection from E to E_α. By 3.27 there is a finite subset $\Lambda_o := \{\alpha_1,\dots,\alpha_k\}$ such that $P_\alpha(x) = 0$ for $x\in B$ if $\alpha\notin\Lambda_o$. Since the P_α's are (uniformly) continuous, $\mathfrak{z}_n^{\alpha_i} := P_{\alpha_i}(x_n)$, $n\in\mathbb{N}$, defines a Cauchy sequence in $(E_{\alpha_i}, \tau_{\alpha_i})$. By hypothesis there is $\mathfrak{z}^{\alpha_i}\in E_{\alpha_i}$ such that $\mathfrak{z}_n^{\alpha_i}\to\mathfrak{z}^{\alpha_i}$, $1\le i\le k$. Set $\mathfrak{z}^\alpha = 0$ if $\alpha\in\Lambda\setminus\Lambda_o$ and $x := (\mathfrak{z}^\alpha)_{\alpha\in\Lambda}$. Then $x_n\to x$ in (E,τ) when $n\to\infty$.

PROPOSITION 3.30 - *Let* (F,τ_F) *be locally right-bounded topological division ring and let* $(E,\tau) = \underrightarrow{\mathrm{ind}}_{n\in\mathbb{N}} (E_n,\tau_n)$ *and* (G,η) *be* TVS *over it. Suppose that* (E,τ) *is a topological subspace of* (G,η). *If for each* $n\in\mathbb{N}$ E_n *is closed in* (G,η), *then* E *is closed in* (G,η).

PROOF: Suppose that $E \neq \overline{E}^\eta$. Then there is $x\in\overline{E}^\eta$ such that $x\notin E$. Hence, from the hypothesis, it follows that $x\notin E_n$

for every $n \in \mathbb{N}$. So since for each $n \in \mathbb{N}$ E_n is closed in (G,η), we can find a sequence $(W'_n)_{n \in \mathbb{N}}$ of η-neighborhoods of 0 in G such that $(x+W'_n) \cap E_n = \emptyset$ and $W'_{n+1} + W'_{n+1} + W'_{n+1} \subset W'_n$, $n \geq 1$. Let $W_n := W'_n \cap E$, $n \geq 1$. Then $(W_n)_{n \in \mathbb{N}}$ is a sequence of τ-neighborhoods of 0 in E with $(x+W_n) \cap E_n = \emptyset$ and $W_{n+1} + W_{n+1} + W_{n+1} \subset W_n$ for every $n \in \mathbb{N}$. Let $U_n = W_n \cap E_n$, $n \geq 1$. Then for every $n \in \mathbb{N}$ U_n is a τ_n-neighborhood of 0 in E_n and setting $U := \bigcup_{J \in \Delta} \sum_{i \in J} I_i(U_i)$ where Δ is the set of all finite subsets of $\mathbb{N}$ and for each $i \in \mathbb{N}$ I_i is the canonical embedding from E_i into E, we have that U is a τ-neighborhood of 0 in E by 3.7. It is easy to prove that $\overline{U}^\eta$ is a neighborhood of 0 in $\overline{E}^\eta$ with respect to the induced topology $\eta_{\overline{E}^\eta}$. Since $x \in \overline{E}^\eta$ we infer that $(x+\overline{U}^\eta) \cap E \neq \emptyset$. Hence there is some $n_o \in \mathbb{N}$ such that $(x+\overline{U}^\eta) \cap E_{n_o} \neq \emptyset$. We claim that $U \subset E_{n_o} + W_{n_o+1} + W_{n_o+1}$. In fact, if $z \in U$, then for some $k \in \mathbb{N}$, which can be chosen greater than n_o, $z \in \sum_{1 \leq i \leq k} I_i(U_i)$.

Therefore $z \in \sum_{1 \leq i \leq n_o} I_i(U_i) + \sum_{n_o+1 \leq i \leq k} I_i(U_i) \subset$

$\subset E_{n_o} + \sum_{n_o+1 \leq i \leq k} W_i \subset E_{n_o} + W_{n_o+1} + W_{n_o+1}$. So

$\overline{U}^\eta \subset U + W'_{n_o+1} \subset E_{n_o} + W'_{n_o}$. Since $(x+\overline{U}^\eta) \cap E_{n_o} \neq \emptyset$,

it follows that $(x+W'_{n_o}) \cap E_{n_o} \neq \emptyset$, which is impossible.

So $E = \overline{E}^\eta$.

COROLLARY 3.31 - *Let* (F,τ_F) *be a complete locally right-bounded topological division ring and let* $(E,\tau) = \underset{n \in \mathbb{N}}{\underrightarrow{\operatorname{ind}}} (E_n,\tau_n)$, *where for each* $n \in \mathbb{N}$ (E_n,τ_n) *is a complete Hausdorff* TVS *over* (F,τ_F). *Then* (E,τ) *is complete.*

PROOF: By 3.23 (E,τ) is a Hausdorff TVS. Let $(\hat{E},\hat{\tau})$ be a completion of (E,τ). Since (E,τ) is a topological subspace of $(\hat{E},\hat{\tau})$ and for each $n \in \mathbb{N}$ E_n is closed in $(\hat{E},\hat{\tau})$, from 3.30 it follows that E is closed in $(\hat{E},\hat{\tau})$. Thus (E,τ) is complete.

COROLLARY 3.32 - *Let* (F,τ_F) *be a complete Hausdorff locally right-bounded topological division ring and let* $(E_n,\tau_n)_{n \in \mathbb{N}}$ *be a sequence of complete Hausdorff* TVS *over* (F,τ_F)*.Then* $(E,\tau) = \bigoplus_{n \in \mathbb{N}} (E_n,\tau_n)$ *is a complete* TVS.

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