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A LEMMA ON SCHAUDER BASES AND ITS APPLICATIONS (*)

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1. **INTRODUCTION.** In this note we prove a general result on Schauder bases in locally convex spaces and apply the same to conclude the nuclearity of spaces having different type of absolute bases and vice-versa.

Throughout this note, we write $\{x_n;f_n\}$ for an arbitrary *Schauder basis* (S.b.) in a Hausdorff locally convex space (X,T) (to be abbreviated hereafter as l.c.s.). The symbol D_T stands for the collection of all seminorms generating the topology T of an l.c.s. (X,T). For various unexplained terms from the theory of sequence spaces and Schauder bases, the reader is referred to [4] and [5]. Further, we also quote [1], [3] and [6] for several notions relating to different types of absolute bases. In particular, an S.b. $\{x_n;f_n\}$ for an l.c.s. (X,T) is called: (i) p-Köthe if $\Delta = \delta \equiv \{\{f_n(x)\} : x \in X\}$ and T is also generated by $D_p = \{\nu_p : \nu \in D_T\}$, where $0 < p < \infty$, with

$$\Delta = \bigcap_{\nu \in D_T} \{(\alpha_n) : [\alpha_n \nu(x_n)] \in \ell^p\},$$

$$\nu_p(x) = [\sum_{n \geq 1} \{|f_n(x)|\nu(x_n)\}^p]^{1/p}.,$$

(ii) ∞-Köthe if whenever $\{\alpha_n \nu(x_n)\} \in \ell^\infty$ for each ν in D_T

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and $\beta \in c_0$, the series $\sum_{n\geq 1} \alpha_n \beta_n x_n$ converges and T is also given by $D_\infty = \{\nu_\infty : \nu \in D_T\}$ with

$$\nu_\infty(x) = \sup_{n\geq 1} \{|f_n(x)|\nu(x_n)\} ,$$

and (iii) *uniformly equicontinuous* (u.e.) if for each ν in D_T there exists μ so that $\nu_\infty(x) \leq \mu(x)$.

We follow [7] for nuclear spaces. In particular, we recall (cf. [1] and [2])

PROPOSITION 1.1. *Let $\{x_n, f_n\}$ be an u.e. S.b. for an l.c.s. (X,T). Then (X,T) is nuclear if and only if for each ν in D_T, there exists μ in D_T so that*

$$\sum_{n\geq 1} \nu(x_n)/\mu(x_n) < \infty , \quad (0/0 = 0).$$

2. THE MAIN LEMMA. Consider any non-negative sequences $\{a_n\}, \{b_n\}$, integer $N \geq 1$ and positive reals α, β and γ with $1/\gamma = 1/\alpha + 1/\beta$. Then it is a simple consequence of Hölder's inequality to conclude

$$(2.1) \quad \left[\sum_{n=1}^{N} (a_n b_n)^{\gamma}\right]^{1/\gamma} \leq \left[\sum_{n=1}^{N} a_n^{\alpha}\right]^{1/\alpha} \left[\sum_{n=1}^{N} b_n^{\beta}\right]^{1/\beta}.$$

We now prove

LEMMA 2.2. *Let $\{x_n; f_n\}$ be an S.b. for an l.c.s.* (X,T). *Then the following two conditions are equivalent:*

(2.3) Let there be $r > 0$ so that for each ν in D_T there exists μ in D_T such that

$$\sum_{n\geq 1} [\nu(x_n)/\mu(x_n)]^r < \infty.$$

(2.4) *For each* $s > 0$ *and* ν *in* D_T, *there exists* μ *in* D_T *such that*

$$\sum_{n\geq 1}[\nu(x_n)/\mu(x_n)]^s < \infty .$$

Proof. Clearly, we need to prove only that (2.3) implies (2.4). Assume, therefore, the truth of the statement (2.3). Let s be any positive number. If $s \geq r$, there is nothing to prove. So, let $s < r$. Choose the least positive integer k with $ks \geq r$. Let $\nu_o = \nu$ be chosen arbitrary from D_T. Determine $\nu_1,\ldots,\nu_k$ in D_T so that

$$\sum_{n\geq 1}[\nu_{i-1}(x_n)/\nu_i(x_n)]^r \leq M<\infty ,$$

for $i = 1,\ldots,k$. Fix $N \geq 1$. Then a repeated use of (2.1) for k-1 times yields

$$[\sum_{n=1}^{N}\{\nu(x_n)/\nu_k(x_n)\}^s]^{1/s} \leq [\sum_{n=1}^{N}\{\nu_o(x_n)/\nu_1(x_n)\}^r]^{1/r}\ldots$$

$$[\sum_{n=1}^{N}\{\nu_{k-2}(x_n)/\nu_{k-1}(x_n)\}^r]^{1/r}[\sum_{n=1}^{N}\{\nu_{k-1}(x_n)/\nu_k(x_n)\}^m]^{1/m} ,$$

where $1/s = (k-1)/r+1/m$. The inequality in (2.4) results from the foregoing inequality.

3. APPLICATIONS. As an application of Lemma 2.2 we may prove some results on the nuclearity of spaces. To begin with, we have

THEOREM 3.1. *Let* $\{x_n;f_n\}$ *be an u.e. S.b. for an l.c.s.* (X,T). *Then the following statements are equivalent:*

(i) (X,T) *is nuclear.*

(ii) *There exists* $r > 0$ *so that to each* ν *in* D_T *there corresponds a* μ *in* D_T *such that*

$$\sum_{n \geq 1} \{\nu(x_n)/\mu(x_n)\}^r < \infty .$$

(iii) *For every* $s > 0$ *and* ν *in* D_T *there exists a* μ *in* D_T *such that*

$$\sum_{n \geq 1} \{\nu(x_n)/\mu(x_n)\}^s < \infty .$$

Proof. (i) $\Rightarrow$ (ii): this follows from Proposition 1.1 with r=1.
(ii) $\Rightarrow$ (iii): cf. Lemma 2.2.
(iii) $\Rightarrow$ (i): let s=1 and apply Proposition 1.1.

As a consequence of Theorem 3.1, we derive the modified version of Grothendieck-Pietsch criterion contained in

PROPOSITION 3.2. *A sequence space* $(\lambda, \eta(\lambda,\mu))$, *where* $\eta(\lambda\ \mu)$ *is the normal topology on* λ *and* μ *is a normal subspace of the Köthe dual* λ^* *of* λ, *is nuclear if and only if, for each* $r>0$ *and positive sequence* $\{a_n\}$ *in* μ *there exists a positive sequence* $\{b_n\}$ *in* μ *such that* $\{a_n/b_n\} \in \ell^r$.

Proof. Observe that $\{e^n;e^n\}$ is an u.e. S.b for $(\lambda, \eta(\lambda,\mu))$, (where $e^n = \{0,0,\ldots,1,0,0,\ldots\}$, 1 being placed at the n-th co-ordinate) and $\eta(\lambda,\mu)$ is generated by the positive elements of μ.

Using Lemma 2.2, we give a simple proof of the following result which extends a similar result given in [8], p. 216.

THEOREM 3.3. *Every l.c.s.* (X,T) *having a* p*-Köthe basis* $(0 < p < 1)$ *is nuclear. Conversely, an u.e.S.b* $\{x_n;f_n\}$ *in a nuclear* *l.c.s.* (X,T) *is* p *-Köthe for each* p, $0 < p < 1$.

Proof. Choose ν in D_T arbitrarily. Then there exists μ in D_T so that

(*) $\left[\sum_{n\geq 1}\{|f_n(x)|\nu(x_n)\}^p\right]^{1/p} \leq \sum_{n\geq 1}|f_n(x)|\mu(x_n)$, $\forall x \in X$.

Let $r=p/(1-p)$ and for any integer $N \geq 1$, let

$$Y_N = \sum_{i=1}^{N} \{\nu(x_i)^r / \mu(x_i)^{r+1}\}x_i .$$

Then from (*)

$$\sum_{n=1}^{N} \{\nu(x_n)/\mu(x_n)\}^r \leq 1.$$

Since N is arbitrary, the nuclearity of (X,T) follows from Lemma 2.2, (2.3) $\Rightarrow$ (2.4) and Proposition 1.1.

For converse, fix p in (0,1). Let $r=p/(1-p)$, $s=1/(1-p)$ and $t=1/p$ so that $s,t > 1$ and $\frac{1}{s} + \frac{1}{t} = 1$.

Let $\nu \in D_T$ Then, by Theorem 3.1 (iii), we can find μ in D_T such that

$$\sum_{n\geq 1}\{\nu(x_n)/\mu(x_n)\}^r < \infty .$$

Moreover, applying Proposition 1.1, one can easily verify the absolute convergence of the series $\sum_{n\geq 1}f_n(x)x_n$ in (X,T) for each x in X. Hence by Hölder's inequality we have

$$\left[\sum_{n\geq 1}\{|f_n(x)|\nu(x_n)\}^p\right]^{1/p} \leq \left[\sum_{n\geq 1}\{\nu(x_n)/\mu(x_n)\}^r\right]^{1/r} \sum_{n\geq 1}|f_n(x)|\mu(x_n)$$

Also for $\nu \in D_T$ and $0 < p < 1$, the inequality

$$\sum_{n\geq 1}|f_n(x)|\nu(x_n) \leq [\sum_{n\geq 1}\{|f_n(x)|\nu(x_n)\}^p]^{1/p}$$

is always true. As the base is 1-Köthe (cf. [1], p.218), it is p-Köthe for each p, 0<p<1. This completes the proof.

PROPOSITION 3.4. *Let* $\{x_n;f_n\}$ *be an S.b. which is both* p-*Köthe* $(1\leq p\leq\infty)$ *and* ∞-*Köthe for an l.c.s.* (X,T) . *Then* (2.3) *is satisfied with* r = p , *hence* (X,T) *is nuclear.*

Proof. Let $\nu\epsilon D_T$. Then there exists μ in D_T with $\nu_p(x)\leq\mu_\infty(x)$ for all x in X. For each integer N $\geq$ 1, let

$$Y_N = \sum_{i=1}^{N}\frac{x_i}{\mu(x_i)}$$

Then

$$[\sum_{i=1}^{N}\{\nu(x_i)/\mu(x_i)\}^p]^{1/p} \leq 1, \qquad \forall N \geq 1,$$

and the result is proved as before.

PROPOSITION 3.5. *Let* $\{x_n;f_n\}$ *be* 1-*Kothe and* 1-*Köthe base for an l.c.* TVS(X,T) . *Then* (2.4) *is satisfied and* $\{x_n;f_n\}$ *is* p-*Kothe for each* p *with* $1\leq p\leq\infty$.

Proof. It suffices to consider the case when $1<p<\infty$. Let $\nu\epsilon D_T$. Then there exists η in D_T so that $\nu_p(x) \leq \nu_1(x) \leq\eta(x)$. Following the proof of Proposition 3.4, we find that (2.3) is satisfied with r=1 and hence (2.4). Consequently there exists $\mu\epsilon D_T$ such that

$$\sum_{n\geq 1}|f_n(x)|\nu(x_n) \leq [\Sigma\{\nu(x_n)/\mu(x_n)\}^q|^{1/q}|\sum_{n\geq 1}\{|f_n(x)|\mu(x_n)\}^p]^{1/p}$$

where 1=1/p+1/q. Thus $\{x_n;f_n\}$ is an ℓ^p-base and $\nu(x)\leq k\mu_p(x)$,

$\nu_p(x) \leq \eta(x)$. Therefore $\{x_n;f_n\}$ is p-Köthe.

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