Note di Matematica Vol. V, 37-49(1985)

# TENSORTOPOLOGIES AND EQUICONTINUITY (*)

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Dedicated to Gottfried Köthe on the occasion of his 80th birthday on December 25, 1985.

**Summary**. *The behavior of the various known tensortopologies with respect to equicontinuity will be studied. In particular, it will be shown that in the category of all locally convex spaces the tensortopology of hypocontinuity on bounded sets is the finest of all tensortopologies which respect equicontinuity of sets of linear mappings.*

1. Let LOC be the category of all locally convex spaces, the objects being (not necessarily Hausdorff) locally convex spaces and the morphisms linear continuous maps. A *tensortopology* $\mu$ assigns to each pair $(E,F)\in LOC\times LOC$ a locally convex topology $\mu(E,F)$ on the algebraic tensorproduct $E\otimes F$ of E and F (shorthand: $E\otimes_\mu F$) such that (see [3]):

(1) the bilinear map $E\times F \to E\otimes_\mu F$ is separately continuous;

(2) if $U^o \subset E'$ and $V^o \subset F'$ are equicontinuous sets of linear functionals on E resp. F, then

$$U^o \otimes V^o := \{\varphi \otimes \psi \mid \varphi\in U^o,\ \psi \in V^o$$

is equicontinuous on $E\otimes_\mu F$;

(3) if $S\in L(E_1,E_2)$ and $T\in L(F_1,F_2)$ are linear continuous operators then

$$S\otimes_\mu T : E_1\otimes_\mu F_1 \to E_2\otimes_\mu F_2$$

is continuous (the *mapping property*).

(*) This paper was written while I visited the Department of Mathematics of the University of Lecce. I thank the Department and in particular V.B.Moscatelli for the kind invitation.

In particular, $E \times F \to E \otimes_\mu F$ is a functor LOC x LOC → LOC which acts on the underlying vectorspaces as the algebraic tensorproduct. Obviously, this definition can as well be given for subclasses of LOC, e.g. for finite-dimensional spaces, normed or Banach-spaces, dual spaces (with the dual mappings as morphisms), etc.. Note that, if $E'$ or $F'$ is $\{0\}$, then $\Phi=0$ is the only separately continuous bilinear functional on $E \times F$ and $E \otimes_\mu F$ has the indiscrete topology for all tensortopologies $\mu$.

Tensortopologies respect complemented subspaces and complemented quotients, but in general do not respect dense subspaces nor the embeddings $E \hookrightarrow E''_e$ (even for normed spaces: take, as an example, the inductive topology defined below and normed spaces $E$ and $F$ such that $E \otimes_\iota F \neq E \otimes_\pi F$, where $\pi$ points at the projective topology).

In studying topological-geometric properties of locally convex tensorproducts, in particular if one wants to take advantage of the bounded approximation property (:= there is an equicontinuous net of finite-rank operators converging pointwise to the identity), it is sometimes useful (see e.g. Defant-Govaerts [1]) to consider *uniform tensortopologies* : these are tensortopologies $\mu$ which satisfy

(3') the *uniform mapping property*: If $C \subset L(E_1,E_2)$ and $D \subset L(F_1,F_2)$ are equicontinuous, then

$$C \otimes D := \{S \otimes T \mid S \in C,\ T \in D\}$$

is equicontinuous in $L(E_1 \otimes_\mu F_1, E_2 \otimes_\mu F_2)$.

Clearly, (3') implies (3) - and (2), the latter provided there is an $E_o \otimes_\mu F_o$ which does not have the indiscrete topology.

If $\mu$ is a uniform tensortopology, E and F have the bounded approximation property, then it is immediate that $E \otimes_\mu F$ and the completion $E \tilde{\otimes}_\mu F$ have the b.a.p. as well.

A tensortopology is uniform if it satisfies (3') for families C and D of projections and injections (i.e. continuous, injective mappings which are open onto their image); this can be easily deduced from the diagram (obvious definitions)

$$\begin{array}{ccc} \ell^\infty(E_1) \otimes_\mu \ell^\infty(F_1) & \xrightarrow{\Pi C \otimes \Pi D} & \ell^\infty(E_2) \otimes_\mu \ell^\infty(F_2) \\ \uparrow I_S \otimes I_T & & \downarrow P_S \otimes P_T \\ E_1 \otimes_\mu F_1 & \xrightarrow{S \otimes T} & E_2 \otimes_\mu F_2 \end{array}$$

For the tensortopologies NORM x NORM → NORM this can be improved: Let E be a normed space and $R(x_n):=(x_{n-1})$ the right-shift on $\ell^\infty(\mathbf{Z},E)$. Using $R^n I_o = I_n$ and $P_o R^{-n} = P_n$ for the canonial injections and projections, the same type of diagram yields the

**PROPOSITION:** *Let* $\mu$ : NORM x NORM → NORM *a tensortopology. If* $\|T_1 \otimes_\mu T_2\| = 1$ *for all surjective isometries* $T_1$ *and* $T_2$, *then* $\mu$ *is uniform.*

2. **Examples:** The following examples of tensortopologies (with the exception of (e)) were already studied by A.Grothendieck.

(a) The *inductive* topology $\iota$ which is characterized by the fact that a bilinear map $E \times F \to G$ is separately continuous if and only if its linearization $E \otimes_\iota F \to G$ is continuous. Property (1) (for a locally convex topology $\mu$ on $E \otimes F$) is equivalent to: $\iota$ is finer than $\mu$, notation: $\iota \supset \mu$. It is easy to see that $\iota$ is a tensortopology.

(b) The *injective* topology $\varepsilon$ which is the topology of uniform convergence on all $U^o \otimes V^o \subset E^* \otimes F^*$. Property (2) is equivalent to $\mu \supset \varepsilon$. The injective topology is even a uniform tensortopology; $\iota$ is the finest and $\varepsilon$ the coarsest tensortopology:

$$E \otimes_\iota F \to E \otimes_\mu F \to E \otimes_\varepsilon F.$$

In particular: $E \otimes_\mu F$ is Hausdorff if E and F are.

(c) The *projective* topology $\pi$ (a bilinear map $E \times F \to G$ is continuous if and only if its linearization $E \otimes_\pi F \to G$ is continuous) is a uniform tensortopology.

(d) If E and F are normed spaces, then $E \otimes_\varepsilon F$ and $E \otimes_\pi F$ are normed in a natural way and $\varepsilon(\cdot\,;E,F) \leq \pi(\cdot\,;E,F)$ for these norms. Grothendieck's metric theory of tensorproducts [5] deals with *tensornoms* $\alpha$ which, by definition, assign to each pair (E,F) of normed spaces a norm $\alpha(\cdot\,;E,F)$ on $E \otimes F$ such that

(1/2') $\varepsilon(\cdot\,;E,F) \leq \alpha(\cdot\,;E,F) \leq \pi(\cdot\,;E,F)$ on $E \otimes F$ (in this case $\alpha$ is called *reasonable*),

(3'') $\|S \otimes T : E_1 \otimes_\alpha F_1 \to E_2 \otimes_\alpha F_2\| \leq \|S\| \, \|T\|$ for all $S \in L(E_1,E_2)$ and $T \in L(F_1,F_2)$ (the *metric mapping property*).

The natural extension to locally convex spaces (tensorize the canonical normed quotient-spaces $E_p$ and $E_q$ of E and F) was introduced and studied by Harksen [6]; these so-called *tensornorm-topologies* $\alpha$ are uniform tensortopologies. Obvioulsy $\pi$ is the finest and $\varepsilon$ the coarsest tensornorm-topology.

Most of the usual tensornorms are *finitely generated*, i.e. for all $E,F \in$ NORM and $z \in E \otimes F$

$$\alpha(z;E,F) = \inf \alpha(z;M,N), \qquad (*)$$

where the infimum is taken over all finite-dimensional subspaces M of E and N of F such that $z \in M \otimes N$. *If* E *and* F *have the metric approximation property, then* (*) *holds for all tensornorms* $\alpha$*:* To see this, observe first that the right side of (*) defines a tensornorm $\vec{\alpha} \geq \alpha$. Let P and Q be finite-dimensional projections on E and F respectively of norm one coming from the m.a.p. and take $z \in E \otimes F$; then the metric mapping property gives

$$\begin{aligned}\vec{\alpha}(z;E,F) &\leq \vec{\alpha}(z-P\otimes Q(z);E,F) + \vec{\alpha}(P\otimes Q(z);E,F)\\ &\leq \vec{\alpha}(z-P\otimes Q(z);E,F) + \vec{\alpha}(P\otimes Q(z);PE,QF)\\ &= \vec{\alpha}(z-P\otimes Q(z);E,F) + \alpha(P\otimes Q(z);PE,QF)\\ &\leq \vec{\alpha}(z-P\otimes Q(z);E,F) + \alpha(z;E,F).\end{aligned}$$

Since the first term converges to zero (P and Q according to the m.a.p.), it follows that $\vec{\alpha} \leq \alpha$.

Since there are Banach-spaces without the metric approximation property, there are relevant tensornorms which are not finitely generated: Take for an example the tensornorm $\alpha$ which is induced by the embedding

$$E \otimes F \subset (E' \otimes_{\varepsilon} F')'$$

i.e., the norm on $E \otimes F$ considered as a subspace of the integral operators $E' \to F$. Assume $\alpha$ were finitely generated; since it coincides on finite-dimensional spaces with $\pi$ (see e.g. [8], p.296(9)) and $\pi$ is finitely generated, this would imply $\alpha = \pi$ and, by [8], p.312(2), all Banach-spaces would have the metric approximation property.

(e) The topologies of hypocontinuity due to L. Schwartz [9]: Let $E, F \in LOC$ and $a_1(E)$, resp. $a_2(F)$, be covers of E, resp. F, by absolutely convex subsets such that $a_1(E)$ and $a_2(F)$ are filtrating with respect to inclusion. For every $G \in LOC$, a bilinear map $E \times F \to G$ is called $(a_1(E), a_2(F))$-hypocontinuous if its restrictions to all $A_1 \times F$ and $E \times A_2$ (for $A_1 \in a_1(E)$ and $A_2 \in a_2(F)$) are continuous (induced topology). It is not difficult to see that the locally convex topology $\eta$ on $E \otimes F$ of uniform convergence on all equi-$(a_1(E), a_2(F))$-hypocontinuous sets of bilinear forms $E \times F \to \mathbb{K}$ has the following properties:

(1) *$\eta$ is the finest locally convex topology $\nu$ on $E \otimes F$ such that $E \times F \to (E \otimes F, \nu)$ is $(a_1(E), a_2(F))$-hypocontinuous.*

(2) *A bilinear map $E \times F \to G$ is $((a_1(E), a_2(F))$-hypocontinuous if and only if its linearization $(E \otimes F, \eta) \to G$ is continuous.*

If $a_1(E)$ and $a_2(F)$ consist of bounded sets only, a bilinear map $\Phi : E \times F \to G$ is $(a_1(E), a_2(F))$-hypocontinuous if and only if for every zero-neighbourhood $W \in \mathcal{U}_G(0)$, every $A_1 \in a_1(E)$ and $A_2 \in a_2(F)$, there are $U \in \mathcal{U}_E(0)$ and $V \in \mathcal{U}_F(0)$ such that

$$\Phi(A_1, V) \subset W \text{ and } \Phi(U, A_2) \subset W.$$

Denoting by $[\![A]\!]$ the normed space span A (with the Minkowski-gauge functional $m_A$) this is equivalent to: All restrictions of $\Phi$

$$[\![A_1]\!] \times F \to G$$

$$E \times [\![A_2]\!] \to G$$

are continuous.

According to [3], a *cover-prescription* a (on LOC) assigns to each $E \in LOC$ a cover a(E) of E as before such that

$$T(a(E_1)) \subset a(E_2)$$

whenever $T \in L(E_1, E_2)$. If $a_1$ and $a_2$ are two cover-prescriptions, $E \otimes_{a_1 a_2} F$ denotes $E \otimes F$ equipped with the unique locally convex topology coming from the covers $a_1(E)$ and $a_2(F)$ of E and F respectively. It is easily seen that the assignment $(E,F) \longrightarrow E \otimes_{a_1 a_2} F$ is a tensortopology: the $(a_1, a_2)$-*hypocontinuous tensortopology*. If $a_1 = a_2 = \{$ finite-dimensional subspaces $\}$, one obtains the inductive topology $\iota$ and, if $a_1 = a_2 = \{$ all subspaces $\}$, the projective topology $\pi$ : obviously all hypocontinuous topologies are between $\iota$ and $\pi$.

3. For $b := \{$ bounded, absolutely convex subsets $\}$ the (b,b)-hypocontinuous tensortopology is denoted by $\beta$. Since equicontinuous sets map bounded sets into bounded sets, it is easy to see that $\beta$ is a uniform tensortopology.

**PROPOSITION:** $\beta$ *is the finest uniform tensortopology on* LOC x LOC. Since $\iota \neq \beta$ (e.g. for some normed spaces), the inductive tensor-topology $\iota$ is *not* uniform.

*Proof.* For a uniform tensortopology $\mu$ and $E,F\in LOC$ it has to be shown that the tensor-map

$$\otimes : E \times F \to E \otimes_\mu F$$

is (b,b)-hypocontinuous. So, by symmetry, it is enough to find, for every zero-neighbourhood W of $E \otimes_\mu F$ and $A\in b(E)$, a zero-neighbourhood V of F such that $A \otimes V \subset W$.

Take $x_o \in E$ and $\varphi_o \in E'$ such that $\langle\varphi_o, x_o\rangle = 1$. Then

$$C := \{\varphi_o \otimes y \mid y \in A\} \subset L(E,E)$$

is equicontinuous and hence $C \otimes \{id_F\}$ is equicontinuous $E \otimes_\mu F \to E \otimes_\mu F$ as well. Denoting by J the canonical continuous map $\{x_o\} \times F \to E \otimes_\mu F$ (property (1) of tensortopologies), it follows that $(C \otimes \{id_F\}) \circ J$ is equicontinuous, whence there is a $V \in \mathcal{U}_F(0)$ such that

$$W \supset (C \otimes \{id_F\}) \circ J(x_o, V) = A \otimes V.$$

The result implies that, in the category of all locally convex spaces, tensortopologies are not uniform if they are not coarser than $\beta$ - such as $\iota$ or, e.g., the (c,c)-hypocontinuous topology (c the compact, absolutely convex sets). Though this is unfortunate, the situation improves on subclasses: e.g., on barrelled spaces, where $\iota = \beta$ always holds - the statement of the proposition is meaningless in this case.

For a more interesting example of a somehow better situation, take the category DUAL of duals of locally convex spaces (with the strong topology) and the dual mappings as morphisms. If e is the cover-prescription of all absolutely convex, equicontinuous sets, then $(G,F) \rightsquigarrow F \otimes_{e,b} F$ is a uniform tensortopology on DUAL X LOC, the uniform mapping property interpreted as follows: If $C \subset L(E_2,E_1)$ and $D \in L(F_1,F_2)$ are equicontinuous, then $C' \otimes D \subset L((E_1)'_b \otimes_{e,b} F_1, (E_2)'_b \otimes_{e,b} F_2)$ is equicontinuous.

Now, taking for $U^o \subset E'_b$ equicontinuous the set

$$C := \{\varphi \otimes x_o \mid \varphi \in U^o\} \subset L(E,E)$$

as in the proof of the proposition, yields:

*The* (e,b)*-hypocontinuous tensortopology is the finest uniform tensortopology on* DUAL x LOC.

The (e-b)- hypocontinuous topology was used, for example, in [2] to obtain a Radon-Nikodym-theorem for operator-valued measures. Again the same proof shows that the (e,e)-hypocontinuous tensortopology is the finest uniform tensortopology on DUALxDUAL- with the appropriate interpretation of the uniform mapping property.

4. In his thesis Grothendieck ([4], chap. I, p.93-95) mentioned another condition in order to study "interesting" tensortopologies $\mu$; his condition is equivalent to:

(G) If $\Phi \in (E \otimes_\mu F)'$ then

$$\Phi^1 \otimes id_F : E \otimes_\mu F \to F'_b \otimes_\iota F$$

$$id_E \otimes \Phi^2 : E \otimes_\mu F \to E \otimes_\iota E'_b$$

are continuous.

($\Phi^1 : E \to F'_b$ and $\Phi^2 : F \to E'_b$ the linear maps associated with $\Phi$). Since the trace-functional tr is $\iota$-continuous, the formula

$$\langle tr_F, \Phi^1 \otimes id_F \rangle = \langle tr_E, id_E \otimes \Phi^2 \rangle = \Phi$$

yields that the continuity of one of the mappings in (G) implies that $\Phi \in (E \otimes_\mu F)'$.

Taking for E a space which is not quasibarrelled (i.e., $E \hookrightarrow E_b''$ is not continuous), the map $\Phi := tr\epsilon(E \otimes_\iota E')'$ shows that the inductive topology $\iota$ does not satisfy (G). The condition seems only to be interesting for barrelled spaces: Grothendieck ([4], chap. I, p. 95) states that $\iota$, $\pi$ and $\varepsilon$ satisfy it for barrelled spaces. More generally:

**PROPOSITION.** *If $\alpha$ is a finitely generated tensornorm and E and F are barrelled, then the tensornorm-topology $\alpha$ on $E \otimes F$ satisfies* (G).

*Proof.* If $\Phi \in (E \otimes_\alpha F)'$, then there are zero-neighbourhoods U and V and $\Phi_o \in (\tilde{E}_U \otimes_\alpha \tilde{E}_V)'$ such that

$$\Phi = \Phi_o \circ (\kappa_U \otimes \kappa_V) .$$

A folklore result (see e.g. [7], p.410) says that

$$\Phi^1 \otimes id_G : \tilde{E}_U \otimes_\alpha G \to (\tilde{F}_V)' \otimes_\pi G$$

is continuous for every Banach-space G and hence for every locally convex space G (by the very definition of the tensornorm-topologies). Using now that $\iota = \pi$ on the tensorproduct of a Banach- and a barrelled space and the mapping property for $\iota$ and $\pi$, it follows that

$$\Phi^1 \otimes id_F : E \otimes_\alpha F \to \tilde{E}_U \otimes_\alpha F \to (\tilde{F}_V)' \otimes_\pi F = (\tilde{F}_V)' \otimes_\iota F \to F_b' \otimes_\iota F$$

is continuous. The continuity of $id_E \otimes \Phi^2$ follows from this applied to the transposed tensornorm $\alpha^t$ on $F \otimes E$.

Since $E \otimes_\beta F \to G$ is continuous if and only if all

$$[\![A]\!] \otimes_\pi F \to G \qquad A \in b(E)$$

$$E \otimes_\pi [\![B]\!] \to G \qquad B \in b(F)$$

are continuous (see 2.(e)), the proposition implies as well that $\beta$ satisfies (G) for barrelled spaces; note that $\iota = \beta$ for barrelled spaces.

**PROPOSITION:** *Neither* $\iota, \beta, \pi, \varepsilon$ *nor any tensornorm-topology* $\alpha$ *(for finitely generated* $\alpha$ *) satisfies condition* (G) *on the class of all locally convex spaces.*

*Proof.* For the inductive topology $\iota$ this was shown before. Let $\alpha$ be a finitely generated tensornorm, $(G, \|\cdot\|)$ a Banach-space and $T : G' \to G'$ a nuclear operator with infinite-dimensional range. Define

$$E := (G', \|\cdot\|), \; F := (G, \|\cdot\|) \otimes (G, \sigma(G, G'))$$

and $\Phi \in (E \otimes_\varepsilon F)' \subset (E \otimes_\alpha F)'$ by

$$\Phi(\varphi \otimes (x,y)) := \langle T\varphi, x \rangle_{G', G} \quad .$$

Obviously $\Phi^1 = I_1 \circ T$, where $I_1 : E = G' \to G' \otimes G' = F'$ is the embedding on the first component. If $\Phi^1 \otimes id_F : E \otimes_\alpha F \to F'_b \otimes_\iota F$ were continuous, then

$$\psi : E \otimes_\pi F \xrightarrow{id} E \otimes_\alpha F \xrightarrow{\Phi^1 \otimes id_F} F'_b \otimes_\iota F \longrightarrow \mathbb{K}$$
$$(\varphi, \psi) \otimes (x, y) \rightsquigarrow \langle \varphi, y \rangle$$

would be continuous as well, which means that there are $\varphi_1, \ldots, \varphi_n \in G'$ with

$$|\langle T\eta, y\rangle| = |\langle \psi, \eta \otimes (o,y)\rangle| \leq \|\eta\|_{G'} \max_{i=1,\ldots,n} |\langle \varphi_i, y\rangle| ,$$

hence $T(G') \subset \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$. This is impossible. So $\alpha$ does not satisfy (G).

Since on the tensorproduct of a Banach- and arbitrary locally convex space $\beta$ and $\pi$ coincide and the counter-example was of this type, $\beta$ does not satisfy (G) as well.

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Ricevuto il 3/10/1985

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