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DIRICHLET PROBLEM FOR THE LAPLACE OPERATOR IN A RECTANGLE AND IN A STRIP

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Summary. *Bounds for the solution of the Dirichlet problem for the Laplace operator in a rectangle and in a strip are given by means of the solution of a "symmetrized" problem.*

INTRODUCTION. Many authors have considered the effect of Schwartz symmetrization on elliptic (see, e.g., [10], [9], [11]) and parabolic (see [3], [2]) problems.

The principal aim of these papers is to obtain some optimal bounds for the solutions of these problems.

A typical case in this setting is the following (see [9]).

Consider the problem

$$\begin{cases} -\Delta u = f & \text{in } G \\ u = 0 & \text{on } \partial G \end{cases}$$

and look for the

$$\sup \frac{\|u\|_p}{\|f\|_q} \qquad p,q \text{ suitable}$$

where the supremum is taken when f ranges through the functions with a fixed rearrangement and G ranges through the domains of R^n with fixed measure.

This least upper bound is attained for the solution of the problem

$$\begin{cases} -\Delta w = f^{\#} & \text{in } G^{\#} \\ w = 0 & \text{on } \partial G^{\#} \end{cases}$$

where $f^{\#}$ is the spherically symmetric rearrangement of f in the sense of Hardy-Littlewood and $G^{\#}$ is the ball centered at the origin with same measure of G.

In fact, this is a useful point of view because now we deal with majorization formulas for symmetric and then simpler problems. Moreover it is well-known that symmetrization results are of particularly relevant interest in many fields of Mathematical Physics (see, e.g., the classical book of Polya-Szegõ [12]).

In this paper, following the previous point of view, we study the effect of a Steiner symmetrization on the following problems. Consider the problem

$$(0.1)\qquad \begin{cases} -\Delta u(x_1,x_2) = f(x_1,x_2) & \text{in } G \\ u(x_1,x_2) = 0 & \text{on } \partial G, \end{cases}$$

where G = (-a,a) x (0,b) with a, b>0 and Δ is the Laplacian operator. Our aim is to give some bounds for $u(x_1,x_2)$ by using the solution of a "symmetrized" problem of the type

$$(0.1)^{\#}\begin{cases} -\Delta U(x_1,x_2) = f^{\#}(x_1,x_2) & \text{in } G \\ U(x_1,x_2) = 0 & \text{on } \partial G, \end{cases}$$

where, for any fixed $x_2 \in [0,b]$, $f^{\#}(\cdot\,,x_2)$ is the symmetrically decreasing rearrangement of $f(\cdot\,,x_2)$ as defined by Hardy and Littlewood.(1)

(1) $\forall x_2 \in [0,b]$, $f^{\#}(x_1,x_2)=f^{*}(C_1|x_1|,x_2)$ where $f^{*}(t,x_2)$ $(t\geq 0)$ denotes the deacresing rearrangement of $f(\cdot,x_2)$ in $[0,+\infty)$, $C_1=2$. For more details, see [3] and [9].

We suppose f and $f^{\#}$ sufficiently smooth so that there is existence and uniqueness for the solutions of problems (0.1), $(0.1)^{\#}$ in $C^2(G) \cap C^{\circ}(\bar{G})$.

In section 1 we will give, for any $x_2 \in [0,b]$, a bound for the L^1 - norm of $u(\cdot,x_2)$ in terms of the L^1 - norm of $U(\cdot,x_2)$ and then we will obtain also a bound of the L^1 - norm of u in terms of the L^1 - norm of U in all G.

Consider then the problem

$$(0.2) \quad \begin{cases} -\Delta u(x_1,x_2) = f(x_1,x_2) & \text{in} \quad G \\ u(x_1,x_2) = 0 & \text{on} \quad \partial G, \end{cases}$$

where $G = (-\infty,+\infty) \times (0,b)$ with $b>0$, and the "symmetrized" problem

$$(0.2)^{\#} \begin{cases} -\Delta U(x_1,x_2) = f^{\#}(x_1,x_2) & \text{in} \quad G \\ U(x_1,x_2) = 0 & \text{on} \quad \partial G, \end{cases}$$

with f and then $f^{\#}$ belonging to $L^2(G)$.

It is well known that there exist unique solutions to (0.2) and $(0.2)^{\#}$ in $W_o^{1,2}(G) \cap W_{loc}^{2,2}(G)$; see [5].

In section 2 we will obtain for a strip a result which is similar to that obtained for a rectangle; moreover we will give, for any $x_2 \in [0,b]$, a bound for $\sup\limits_{-\infty<x_1<+\infty} |u(x_1,x_2)|$ in terms of $\sup\limits_{-\infty<x_1<+\infty} U(x_1,x_2)$ and also a bound for $\sup\limits_{\bar{G}} |u(x_1,x_2)|$ in terms of $\sup\limits_{\bar{G}} U(x_1,x_2)$.

To obtain these results we will use a technique developed by

C. Bandle in her treatment of parabolic operators (see [2]). In the same framework, we can quote the work of C. Borell, see [4], where a symmetrization like that of Steiner is used.

SECTION 1.

Consider the boundary problem

$$(1.1)\quad \begin{cases} -\Delta u(x_1,x_2) = f(x_1,x_2) & \text{in} \quad G \\ u(x_1,x_2) = 0 & \text{on} \quad \partial G \end{cases}$$

where $G = (-a,a) \times (0,b)$, $a,b>0$.

We suppose f smooth enough to guarantee existence and uniqueness of the solution of (1.1) in $C^2(G)\cap C^\circ(\bar{G})$ (see [6], th. 4.3, pag. 55 and [7], th. 3.1, pag. 328).

Moreover we suppose f non-negative and so, by the maximum principle, u is positive in G. The assumption $f(x_1,x_2) \geq 0$ is not restrictive for our aims, since, for an arbitrary f, the modulus of the solution relative to f is less than, or equal to, the solution relative to $|f|$.

Put

$$(1.2)\qquad H(s,x_2) = \int_{-s}^{s} u(x_1,x_2)dx_1$$

with $s\in[0,a]$ and $x_2\in[0,b]$.

LEMMA 1.1. *The following differential inequality holds:*

$$(1.3)\quad \frac{\partial^2 H}{\partial s^2}(s,x_2) + \frac{\partial^2 H}{\partial x_2^2}(s,x_2) + \int_0^{2s} f^*(s',x_2)ds' \geq 0,$$

in $(0,a)\times(0,b)$.

Moreover we have:

$$(1.4)\qquad \begin{cases} H(0,x_2) = 0, \\ H(s,0) = H(s,b) = 0, \\ \dfrac{\partial H}{\partial s}(a,x_2) = 0, \end{cases}$$

with $s \in [0,a]$, $x_2 \in [0,b]$.

Proof. Fix $x_2 \in (0,b)$; we obtain from (1.1) by integration

$$(1.5)\qquad -\int_{-s}^{s} \frac{\partial^2 u}{\partial x_1^2}\,dx_1 - \int_{-s}^{s} \frac{\partial^2 u}{\partial x_2^2}\,dx_1 = \int_{-s}^{s} f\,dx_1,$$

for every $s \in (0,a)$. We obtain easily:

$$\int_{-s}^{s} \frac{\partial^2 u}{\partial x_1^2}dx_1 = -\frac{\partial^2 H}{\partial s^2}(s,x_2)$$

and then, observing that from Hardy's inequality (2)

$$\int_{-s}^{s} f\,dx_1 \le \int_{0}^{2s} f^*(s',x_2)ds',$$

from (1.5) we obtain:

$$\frac{\partial^2 H}{\partial s^2}(s,x_2) + \frac{\partial^2 H}{\partial x_2^2}(s,x_2) + \int_{0}^{2s} f^*(s',x_2)ds' \ge 0,$$

in $(0,a)\times(0,b)$, that is (1.3).

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(2) The Hardy's inequality is (see [9])

$$\int_{-r}^{r} fg\,dx \le \int_{0}^{2r} f^*\,g^*\,ds.$$

The equalities (1.4) are easily obtained.

Consider now the "symmetrized" problem

$$(1.1)^{\#} \quad \begin{cases} -\Delta U(x_1,x_2) = f^{\#}(x_1,x_2) & \text{in} \quad G \\ U(x_1,x_2) = 0 & \text{on} \quad \partial G . \end{cases}$$

We suppose $f^{\#}$ sufficiently smooth so that problem $(1.1)^{\#}$ has a unique solution $U \in C^2(G) \cap C^\circ(\bar{G})$. For example, if $f>0$ in G, $f=0$ on ∂G and $f \in C^{0,1}_{loc}(G)$, then also $f^{\#} \in C^{0,1}_{loc}(G)$ (see [2], proposition 1.2) and problems (1.1) and $(1.1)^{\#}$ have unique solutions u,U respectively in $C^2(G) \cap C^\circ(\bar{G})$.

The function $f^{\#}$ is non-negative in G, and so U is positive in G.

Put

$$(1.6) \qquad \tilde{H}(s,x_2) = \int_{-s}^{s} U(x_1,x_2)\, dx_1$$

with $s \in [0,a]$, and $x_2 \in [0,b]$.

LEMMA 1.2. *The following equality holds:*

$$(1.7) \qquad \frac{\partial^2 \tilde{H}}{\partial s^2}(s,x_2) + \frac{\partial^2 \tilde{H}}{\partial x_2^2}(s,x_2) + \int_0^{2s} f^*(s',x_2)\,ds' = 0,$$

in $(0,a) \times (0,b)$.

Moreover we have:

$$(1.8)\qquad \begin{cases} \tilde{H}(0,x_2) = 0 , \\ \tilde{H}(s,0) = \tilde{H}(s,b) = 0 , \\ \dfrac{\partial \tilde{H}}{\partial s}(a,x_2) = 0, \end{cases}$$

with $s\epsilon[0,a]$ *and* $x_2\epsilon[0,b]$.

Proof. The proof of (1.7) is similar to that of (1.3), lemma 1.1, if we note that

$$\int_{-s}^{s} f^{\#}(x_1,x_2)dx_1 = \int_{0}^{2s} f^{*}(s',x_2)ds'.$$

The equalities (1.8) are obvious.

Put now

$$(1.9)\qquad d(s,x_2) = H(s,x_2) - \tilde{H}(s,x_2) ,$$

with $s\epsilon[0,a]$ and $x_2\epsilon[0,b]$.

From lemmas 1.1 and 1.2 we have

$$(1.10)\qquad \frac{\partial^2 d}{\partial s^2} + \frac{\partial^2 d}{\partial s_2^2} \equiv \Delta d \geq 0$$

in $(0,a) \times (0,b)$ with

$$(1.11)\qquad \begin{cases} d(0,x_2) = 0 \\ d(s,0) = d(s,b) = 0, \\ \dfrac{\partial d}{\partial s}(a,x_2) = 0, \end{cases}$$

for $s\epsilon[0,a]$ and $x_2\epsilon[0,b]$.

Now we prove the following

LEMMA 1.3. *If* $d(s,x_2)\not\equiv 0$, *then* $d(s,x_2)<0$ *in* $(0,a]\times(0,b)$.

Proof. We have $d(s,x_2)\not\equiv 0$ and $\Delta d \geq 0$ in $(0,a)\times(0,b)$, and so by the maximum principle, the maximum of $d(s,x_2)$ must be attained on the sides of the rectangle $(0,a)\times(0,b)$ of $\mathbb{R}^2$.

But the maximum cannot be attained on $\{s=a,\ 0<x_2<b\}$; in fact $\frac{\partial d}{\partial s}(a,x_2)=0$ for $x_2\in(0,b)$, while if (a,x_2) is a maximum point, we would have $\frac{\partial d}{\partial s}(a,x_2)>0$ (see th.7 of [8], pag. 65). Then it follows from the first two relations of (1.11) that $\max d(s,x_2)=0$.

So we proved that $d(s,x_2)<0$ in $(0,a]\times(0,b)$.

We can now prove:

THEOREM 1.1. *Let* u *be the solution of problem* (1.1) *and* U *the solution of problem* $(1.1)^{\#}$. *Then, for every* $x_2\in[0,b]$,

(1.12) $$\|u(\cdot,x_2)\|_{L^1((-a,a)\times\{x_2\})} \leq \|U(\cdot,x_2)\|_{L^1((-a,a)\times\{x_2\})},$$

and then

(1.13) $$\|u\|_{L^1(G)} \leq \|U\|_{L^1(G)} .$$

Proof. From lemma 1.3 and equalities (1.11) we have

(1.14) $$d(s,x_2) \leq 0 ,$$

in $[0,a]\times[0,b]$.

By (1.9) and (1.14) we have

(1.15) $$H(s,x_2) \leq \tilde{H}(s,x_2),$$

that is

$$(1.16) \qquad \int_{-s}^{s} u(x_1,x_2)dx_1 \leq \int_{-s}^{s} U(x_1,x_2)dx_1 .$$

Then, for every $x_2 \in [0,b]$:

$$\int_{-a}^{a} u(x_1,x_2)\, dx_1 \leq \int_{-a}^{a} U(x_1,x_2)dx_1 ,$$

that is (1.12).

SECTION 2.

Consider the boundary problem:

$$(2.1) \qquad \begin{cases} -\Delta u(x_1,x_2) = f(x_1,x_2) & \text{in} \quad G \\ u(x_1,x_2) = 0 & \text{on} \quad \partial G, \end{cases}$$

with $G = (-\infty,+\infty) \times (0,b)$, $b>0$, and the "symmetrized" one:

$$(2.1)^{\#} \qquad \begin{cases} -\Delta U(x_1,x_2) = f^{\#}(x_1,x_2) & \text{in} \quad G \\ U(x_1,x_2) = 0 & \text{on} \quad \partial G . \end{cases}$$

We suppose $f \in L^2(G)$ and consequently $f^{\#} \in L^2(G)$. Then, by theorem 5.4 of [5], pag. 632, problems (2.1) and $(2.1)^{\#}$ have unique weak solutions u, U respectively, belonging to $W_o^{1,2}(G) \cap W_{loc}^{2,2}(G)$. Moreover we take f, $f^{\#}$ in $C^{o,\lambda}(\bar{G})$, with $0 < \lambda \leq 1$, so that $u, U \in C^2(G) \cap C^{\circ}(\bar{G})$ and finally we suppose $f \geq 0$ and so, by the maximum principle, we have $u \geq 0$ in G.

Also we obtain $U \geq 0$ in G, since $f^{\#} \geq 0$ in G.

As in section 1, we put

(2.2) $$H(s,x_2) = \int_{-s}^{s} u(x_1,x_2)dx_1,$$

(2.3) $$\tilde{H}(s,x_2) = \int_{-s}^{s} U(x_1,x_2)dx_1,$$

(2.4) $$d(s,x_2) = H(s,x_2) - \tilde{H}(s,x_2)$$

with $s \geq 0$ and $x_2 \in [0,b]$.

We obtain, as in lemmas 1.1 and 1.2,

(2.5) $$\Delta d \geq 0, \quad \text{in} \quad (0,+\infty) \times (0,b),$$

and

(2.6) $$\begin{cases} d(0,x_2) = 0, & x_2 \in [0,b] \\ d(s,0) = d(s,b) = 0, & s \geq 0. \end{cases}$$

We will obtain the following

LEMMA 2.1. $d(s,x_2) \leq 0$ *in* $[0,+\infty) \times [0,b]$.

In order to prove lemma 2.1 we will use the following

THEOREM (Phragmèn-Lindelöf) - *Let* (r,θ) *be polar coordinates such that the polar semiaxis is coincident with the positive* x_1 *axis and let* $V = \{ (r,\theta),\ r>0,\ -\frac{\pi}{2\alpha} < \theta < \frac{\pi}{2\alpha} \}$, *be the open sector of the angle* $\frac{\pi}{\alpha}$.

Let v *be a function in* $C^2(V) \cap C^\circ(\bar{V})$ *such that*

$$\Delta v \geq 0 \qquad \text{in } V,$$

and assume $v \leq M$ *on the boundary* $\theta = \pm \frac{\pi}{2\alpha}$ *and*

(2.7) $$\min \lim_{R \to +\infty} (R^{-\alpha} \max_{r=R} v(r,\theta)) \leq 0.$$

Then $v \leq M$ *on* V.

REMARK 2.1. Let V' be an open subset of V, and v a function in $C^2(V') \cap C^\circ(\bar{V}')$ such that $\Delta v \geq 0$ in V', $v \leq M$ on $\partial V'$ and suppose that (2.7) holds relatively to V'. Then it is easily seen that $v \leq M$ in V'.(3)

Proof of lemma 2.1. Since (2.6) hold, it is sufficient to prove that $d(s,x_2) \leq 0$ in $(0,+\infty) \times (0,b)$.

Let $\alpha = 1$, $V' = (0,+\infty) \times (0,b)$, $v \equiv d$.

By (2.5), (2.6), Phragmèn-Lindelöf theorem and remark 2.1, it is sufficient to prove that (2.7) holds in $(0,+\infty) \times (0,b)$.

We will prove that:

(2.8) $$\lim_{R \to +\infty} \left(\frac{\max_{(s,x_2) \in \Gamma_R} d(s,x_2)}{R} \right) = 0 ,$$

where Γ_R is the arc of circle centered in $(0,0)$ and with radius R, contained in $(0,+\infty) x (0,b)$.

In fact, for any $(s,x_2) \in \Gamma_R$,

(3) For the proof of Phragmèn-Lindelöf theorem and of remark 2.1, see [8], pp. 93-96.

$$(2.9)\qquad |d(s,x_2)| \le \int_{-s}^{s} u(x_1,x_2)dx_1 + \int_{-s}^{s} U(x_1,x_2)dx_1 .$$

Then, by the Schwartz-Hölder inequality,

$$(2.10)\qquad \int_{-s}^{s} u(x_1,x_2)dx_1 \le (2s)^{\frac{1}{2}}(\int_{-s}^{s} u(x_1,x_2)^2dx_1)^{\frac{1}{2}} \le$$

$$\le (2s)^{\frac{1}{2}}(\int_{-\infty}^{+\infty} u(x_1,x_2)^2dx_1)^{\frac{1}{2}} .$$

Moreover, by the embedding of $W^{1,2}(G)$ in $L^2((-\infty,+\infty)\times\{x_2\})$ (see [1], lemma 5.19), a positive constant C exists, which is independent of u and x_2 and such that

$$(2.11)\qquad (\int_{-\infty}^{+\infty} u(x_1,x_2)^2dx_1)^{\frac{1}{2}} \le C\|u\|_{W^{1,2}(G)} .$$

From (2.10) and (2.11), we obtain

$$(2.12)\qquad \int_{-s}^{s} u(x_1,x_2)dx_1 \le Ks^{\frac{1}{2}},$$

where $K = 2^{\frac{1}{2}} C\|u\|_{W^{1,2}(G)}$.

In the same way we obtain for U:

$$(2.13)\qquad \int_{-s}^{s} U(x_1,x_2)dx_1 \le K's^{\frac{1}{2}} ,$$

where $K' = 2^{\frac{1}{2}}C\|U\|_{W^{1,2}(G)}$.

From (2.9), (2.12), and (2.13), it follows

$$|d(s,x_2)| \leq C' s^{\frac{1}{2}},$$

where $C' = K+K'$ is independent of s and x_2.

Then we easily obtain

$$|\max_{\Gamma_R} d(s,x_2)| \leq C' R^{\frac{1}{2}}$$

and so (2.8) follows.

From lemma 2.1 and in the same way as in the proof of theorem 1.1, we obtain

THEOREM 2.1. *Let* u *be the solution of problem* (2.1) *and* U *the solution of problem* $(2.1)^{\#}$. *Then,* $\forall x_2 [0,b]$,

$$(2.14) \qquad \|u(\cdot,x_2)\|_{L^1((-\infty,+\infty)\times\{x_2\})} \leq \|U(\cdot,x_2)\|_{L^1((-\infty,+\infty)\times\{x_2\})},$$

and so also

$$(2.15) \qquad \|u\|_{L^1(G)} \leq \|U\|_{L^1(G)} .$$

Moreover the following theorem holds:

THEOREM 2.2. *Let* u,U *the solutions of problems* (2.1) *and* $(2.1)^{\#}$ *respectively. Then,* $\forall x_2 \in [0,b]$,

$$(2.16) \qquad \sup_{-\infty<x_1<+\infty} u(x_1,x_2) \leq \sup_{-\infty<x_1<+\infty} U(x_1,x_2) ,$$

and so also

$$(2.17) \qquad \sup_{\bar{G}} u(x_1,x_2) \leq \sup_{\bar{G}} U(x_1,x_2).$$

Proof. Fix $x_2 \in (0,b)$; from lemma 2.1 and the first of (2.6) we obtain

$$\frac{\partial d}{\partial s}(0,x_2) \leq 0. \tag{2.18}$$

Since $\frac{\partial H}{\partial s}(0,x_2) = 2u(0,x_2)$ and $\frac{\partial \tilde{H}}{\partial s}(0,x_2) = 2U(0,x_2)$, we have $\frac{\partial d}{\partial s}(0,x_2) = 2(u(0,x_2)-U(0,x_2))$ and then, from (2.18),

$$u(0,x_2) \leq U(0,x_2). \tag{2.19}$$

Now, let (k,x_2) be a point of G, with $k \in \mathbb{R}$, at height x_2. We write $f_{|k|}(x_1,x_2)$ for the translation of $f(x_1,x_2)$ of the quantity $|k|$, in the direction of the positive x_1-axis if $k<0$, and in the direction of the negative x_1-axis if $k \geq 0$. If we consider $f_{|k|}$ instead of f, then also u will be translated. We write $u_{|k|}$ for this translation.

Now, $\forall x_2 \in (0,b)$, $f^{\#}(x_1,x_2)=(f_{|k|})^{\#}(x_1,x_2)$.

Then, by (2.19),

$$u(k,x_2) = u_{|k|}(0,x_2) \leq U(0,x_2) \leq \sup_{-\infty<x_1<+\infty} U(x_1,x_2),$$

and then (2.16) follows.

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