Note di Matematica Vol. VII, 55-82(1987)

COMPLETING SEQUENCES AND SEMI-LB-SPACES (*)

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SUMMARY. - *Given a completing sequence in a locally convex space, we associate to it a Fréchet space and we use it to obtain localization results both in webbed spaces and semi-LB-spaces. Finally the fact that every convex webbed space is absolutely convex webbed is also proved.*

INTRODUCTION. - The vector spaces we shall use here are defined over the field $\mathbb{K}$ of real or complex numbers. The word "space" means "separated locally convex space". Given a space E, we denote by $\hat{E}$ its completion. $\mathbb{N}$ is the set of positive integers.

If A is a bounded, absolutely convex set in a space E, we denote by E_A the linear hull of A endowed with the norm of the Minkowski functional of A. A fundamental system of neighbourhoods of the origin in E_A is the family

$$\{ \frac{1}{n}A : n = 1,2,...\}.$$

It is said that A is a *Banach disc* when E_A is a Banach space. A space A is *unordered Baire-like* if, given any sequence (A_n) of closed and absolutely convex subsets of E convering E, there

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(*) This research was undertaken while the author visited the University of Lecce during the spring of 1985, at the invitation of Prof. V.B.Moscatelli.

(**)Supported in part by CAICYT (pr.83-2622).

is a positive integer p such that A_p is a neighbourhood of the origin [5]. As an immediate consequence, if (E_n) is a sequence of subspaces of an unordered Baire-like space E that covers E, there is a positive integer p such that E_p is unordered Baire-like and dense in E.

Following De Wilde [1] and [2], we define a *web* in a space E as a family

$$\mathscr{W} = \{C_{m_1, m_2, \ldots, m_n}\}$$

of subsets of E, where $n, m_1, m_2, \ldots, m_n$ are positive integers, and such that the following relations are satisfied:

$$E = \cup\{C_{m_1} : m_1 = 1,2,\ldots\} ,$$

$$C_{m_1, m_2, \ldots, m_n} = \cup\{C_{m_1, m_2, \ldots, m_n, m} : m = 1,2,\ldots\} , \quad n \geq 1 .$$

A web $\mathscr{W}$ is said to be *convex* (*absolutely convex*) if the sets defining it are convex (absolutely convex). A web $\mathscr{W}$ is *completing*, or a *$\mathscr{C}$-web*, if the following condition is satisfied: for every sequence (m_n) of positive integers there is a sequence (λ_n) of positive numbers such that for

$$x_n \in C_{m_1, m_2, \ldots m_n}, \quad 0 \leq |\mu_n| \leq \lambda_n, \quad \mu_n \in \mathbb{K}, \quad n=1,2,\ldots,$$

the series

$$\sum_{n=1}^{\infty} \mu_n \, x_n$$

converges in E. We shall say that a space E is a *convex (absolutely convex) webbed space* if it admits a convex (absolutely convex) $\mathscr{C}$-web.

We shall say that a sequence α_n in $\mathbb{N}^{\mathbb{N}}$, with

$$\alpha_n = (a_{n,p})_{p=1}^{\infty}, \quad n=1,2,\ldots,$$

is *semi-stationary* if, given any positive integer p, we have another positive integer q such that

$$a_{n,p} = a_{q,p}, \quad n \geq q.$$

A *semi-LB-representation* in a space F is a family of Banach discs

$$\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

verifying the following two conditions:

1. $\cup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = F$.
2. If (α_n) is a semi-stationary sequence in $\mathbb{N}^{\mathbb{N}}$, we have α in $\mathbb{N}^{\mathbb{N}}$ such that

$$A_{\alpha_n} \subset A_\alpha, \quad n = 1,2,\ldots .$$

We shall call a *semi-LB-space* a space admitting a semi-LB-representation.

1. ABSOLUTERY CONVEX $\mathscr{B}$-COMPLETING SEQUENCES.

In a space F, let $\mathscr{B}$ be a family of Banach discs that convers F and such that the finite union of members of $\mathscr{B}$ is contained in same member of $\mathcal{B}$. We shall say that a sequence (A_k) of subsets of F is *absolutely convex $\mathcal{B}$-completing* if it is a decreasing sequence, every A_k is absolutely convex, and there is a sequence (λ_k) of positive numbers such that given

$$0 \leq |\mu_k| \leq \lambda_k, \; x_k \in A_k, \quad k = 1,2,\ldots,$$

there is a B in $\mathcal{B}$ with

$$x_k \in F_B, \quad k = 1,2,\ldots,$$

and the series

$$\sum_{k=1}^{\infty} \mu_k x_k$$

converges in F_B. In what follows we shall suppose that

$$\lambda_1 = 1, \; \lambda_k \geq \lambda_{k+1}, \; \lambda_k \leq \frac{1}{2^k}, \; k = 2,3,\ldots,$$

which does not imply any loss of generality.

When $\mathcal{B}$ is the family of all the Banach discs in F, the former concept coincides with the absolutely convex completing sequences of De Wilde (see [2, Proposition IV; 1.9]). We are going to consider the family $\mathcal{B}$ in order to obtain results that can be applied to the class of semi-LB-spaces.

We take a positive integer k and we write

$$B_k = \cup \{ \sum_{n=1}^{\infty} \lambda_n x_n : x_n \in A_{k+n-1}, \quad n = 1,2,\ldots \}.$$

It is immediate that B_k is absolutely convex and contains A_k.

Of course (B_n) is a decreasing sequence.

PROPOSITION 1. *If* W *is a neighbourhood if the origin in* F *, there are a positive integer* k *together with a positive number* λ *such that*

$$\lambda B_k \subset W.$$

Proof. It is not a restriction to assume that W is closed and absolutely convex. It is clear that the condition required for (B_k) is equivalent to the corresponding one with (A_k). But the latter is easy to prove. Let us suppose that the property does not hold. For every positive integer k there is a point x_k in A_k such that

$$\lambda_k \ x_k \notin W.$$

The series

$$\sum_{k=1}^{\infty} \lambda_k \ x_k$$

converges in F, consequently the sequence $(\lambda_k x_k)$ converges to the origin in F. So we have a positive integer p such that

$$\lambda_h \ x_k \in W \qquad \text{if} \quad k \geq p,$$

which is a contradiction. q.e.d.

Let G be a dense subspace of a metrizable space E. Let T be a linear mapping from G into F. We write

$$T^{-1}(A_k) = U_k \ , \ T^{-1}(B_k) = V_k .$$

$\bar{U}_k$ will be the closure of U_k in E and $\mathring{\bar{U}}_k$ the interior of $\bar{U}_k$ in the same space E. Let us suppose that $\bar{U}_k$ is a neighbourhood of the origin in E, $k=1,2,\ldots$.

PROPOSITION 2. *If the graph of* T *meets* $E \times F_B$ *in a closed subspace for every* B *in* $\mathscr{B}$, *we have that*

$$\mathring{\bar{U}}_k \subset V_k, \quad k = 1,2,\ldots.$$

Proof. We fix a positive integer k and we take any point x in $\mathring{\bar{U}}_k$. Let

$$\{W_n : n = 1,2,\ldots\}$$

be a fundamental system of neighbourhoods of the origin in E such that

$$W_n \subset \mathring{\bar{U}}_{n+k}, \quad n = 1,2.$$

We take x_1 in U_k such that

$$y_1 = x - x_1 \in \lambda_2 W_1.$$

Proceeding by recurrence, it is assumed that for a positive integer m we have found

$$y_m \in \lambda_{m+1} W_m.$$

We now determine

$$x_{m+1} \in U_{m+k}$$

such that

$$y_{m+1} = y_m - \lambda_{m+1} x_{m+1} \in \lambda_{m+2} W_{m+1}.$$

The sequence (y_n) obviously converges to the origin in E, and

$$y_n = x - x_1 - \lambda_2 x_2 - \dots - \lambda_n x_n$$

for every positive integer n. Consequently, we have in E

$$x = \sum_{n=1}^{\infty} \lambda_n x_n.$$

For every positive integer j,

$$Tx_j \in A_{k+j-1};$$

since (A_n) is $\mathscr{B}$-completing, we have a B in $\mathscr{B}$ such that

$$Tx_j \in F_B$$

and the series

$$\sum_{n=1}^{\infty} \lambda_n Tx_n$$

converges in F_B to a vector u that obviously belongs to B_k. The fact that Tx=u follows from the fact that the graph of T meets E x F_B in a closed set. Then x belongs to V_k and the proof is complete.

q.e.d.

PROPOSITION 3. *The set*

$$M := \cap \{A_k : k = 1,2,\dots\}$$

is contained in a Banach disc.

Proof. If W is a neighbourhood of the origin in F, we apply Proposition 1 to obtain $\lambda > 0$ and a positive integer p such that

$$\lambda M \subset \lambda B_p \subset W$$

and thus M is a bounded subset of F. Let ψ be the canonical injection of F_M into F. We can extend ψ to a linear mapping $\hat{\psi}$ from the completion H of $\hat{F}_M$ into $\hat{F}$. Let G be equal to $\hat{\psi}^{-1}(F)$. If φ is the restriction of $\hat{\psi}$ to G, we have that the graph of φ is closed in HxF. If we denote by U_k the set $\varphi^{-1}(A_k)$ and by V_k the set $\varphi^{-1}(B_k)$, we have that the closure $\bar{U}_k$ of U_k in H is a neighbourhood of the origin in this space. Therefore, if we apply Proposition 2 we obtain that

$$\mathring{\bar{U}}_k \subset V_k ,$$

from which it follows that H=G. Consequently, the image through φ of the closed unit ball of H is a Banach disc in F containing the set M.

q.e.d.

Let us take v_k in A_k, k = 1,2,..., and let us denote by X_k the absolutely convex cover of

$$\{v_1, v_2, \ldots, v_k\} \cup A_k.$$

PROPOSITION 4. (X_k) *is an absolutely convex* $\mathcal{B}$*-completing sequence.*

Proof. Let us take x_k in X_k. There is y_k in A_k and

$$b_k, a_{kj} \in \mathbb{K}, \qquad j = 1,2,\dots,k,$$

such that

$$\sum_{j=1}^{k} |a_{kj}| + |b_k| \leq 1, \quad x_k = \sum_{j=1}^{k} a_{kj} v_j + b_k y_k .$$

If

$$0 \leq |\mu_k| \leq 2^{-k} \lambda_k$$

we have

$$\sum_{k=1}^{\infty} \mu_k x_k = \sum_{k=1}^{\infty} \mu_k \left(\sum_{j=1}^{k} a_{kj} v_j + b_k y_k \right)$$

$$= \sum_{j=1}^{\infty} \left(\sum_{k=j}^{\infty} \mu_k a_{kj} \right) v_j + \sum_{k=1}^{\infty} (\mu_k b_k) y_k .$$

Since

$$\left| \sum_{k=j}^{\infty} \mu_k a_{kj} \right| \leq \sum_{k=j}^{\infty} 2^{-k} \lambda_k \leq \lambda_j ,$$

$$|\mu_k b_k| \leq |\mu_k| \leq \lambda_k ,$$

it follows that the series

$$\sum_{k=1}^{\infty} \mu_k x_k$$

belongs to some F_B, $B \in \mathscr{B}$, and it converges in this space.

q.e.d.

PROPOSITION 5. *If*

$$v_k \in A_k,\ b_k \in \mathbb{K},\ k=1,2,\ldots,\ \text{and}\ \sum_{k=1}^{\infty} |b_k| < \infty \quad ,$$

then the series

$$\sum_{k=1}^{\infty} b_k v_k$$

converges in F *and the set*

$$A := \{ \sum_{k=1}^{\infty} a_k\ v_k : \sum_{k=1}^{\infty} |a_k| \leq 1 \}$$

is a Banach disc.

Proof. We write X_k to denote the absolutely convex cover of

$$\{ v_1, v_2, \ldots, v_k \} \cup A_k \quad .$$

According to the former proposition, (X_k) is an absolutely convex $\mathcal{B}$-completing sequence. We know that

$$\cap\{X_k : k = 1,2,\ldots\}$$

is contained in a Banach disc P by Proposition 3. Let us observe that

$$v_k \in P, \quad k = 1,2,\ldots$$

and the conclusion now is obvious.

q.e.d.

The former proposition ensures that the following sets are well defined:

$$C_k = \{ \sum_{j=1}^{\infty} a_j x_j : x_j \in A_{k+j-1}, \ a_j \in \mathbb{K}, \ j=1,2,\ldots, \sum_{j=1}^{\infty} |a_j| \leq 1 \},$$

$$k = 1,2,\ldots .$$

We write D_k for the linear hull of C_k. We set

$$F^{(A_k)} = \cap \{D_r : r = 1,2,\ldots\} .$$

According to Proposition 1, the family

$$\frac{1}{r}(F^{(A_k)} \cap C_r), \quad r = 1,2,\ldots,$$

is a fundamental system of neighbourhoods of the origin in $F^{(A_k)}$ for a locally convex and metrizable topology finer than the topology induced by F on $F^{(A_k)}$. Let us suppose that $F^{(A_k)}$ is endowed with this metrizable topology.

PROPOSITION 6. $F^{(A_k)}$ *is a Fréchet space.*

Proof. Let (y_r) be a Cauchy sequence in $F^{(A_k)}$. We select a subsequence (z_r) of (y_r) such that

$$2^{2r}(z_{r+1} - z_r) \in C_r .$$

Then we have

$$x_{jr} \in A_{r+j-1}, \ a_{jr} \in \mathbb{K}, \ j = 1,2,\ldots, \ \sum_{j=1}^{\infty} |a_{jr}| \leq 1,$$

such that

$$2^{2r}(z_{r+1} - z_r) = \sum_{j=1}^{\infty} a_{jr} x_{jr} .$$

We fix a positive intergers. Then

$$\sum_{r=s}^{\infty}(z_{r+1}-z_r) = \sum_{r=s}^{\infty}\sum_{j=1}^{\infty}\frac{a_{jr}}{2^{2r}}x_{jr} = \sum_{m=s}^{\infty}\sum_{r=s}^{m}\frac{a_{(m-r+1)r}}{2^{2r}}x_{(m-r+1)r}.$$

We put

$$\nu_m = \sum_{r=1}^{m}\left|\frac{a_{(m-r+1)r}}{2^{2r}}\right|, \quad m = s,\ s+1, \ldots,$$

and $y_m = 0$ if $\nu_m = 0$,

$$y_m = \sum_{r=s}^{m}\frac{a_{(m-r+1)r}}{2^{2r}\nu_m}x_{(m-r+1)r} \quad \text{if} \quad \nu_m \neq 0 .$$

Clearly, y_m belongs to A_m and

$$\sum_{r=s}^{\infty}(z_{r+1} - z_r) = \sum_{m=s}^{\infty}\nu_m y_m \tag{1}$$

On the ther hand,

$$\sum_{m=s}^{\infty}\nu_m = \sum_{m=s}^{\infty}\sum_{r=s}^{m}\left|\frac{a_{(m-r+1)r}}{2^{2r}}\right| =$$

$$= \sum_{r=s}^{\infty}\sum_{j=1}^{\infty}\frac{|a_{jr}|}{2^{2r}} \leq \sum_{r=s}^{\infty}\frac{1}{2^{2r}} < \frac{1}{2^s} .$$

Consequently, the series (1) is convergent in F and its sum belongs to $\frac{1}{2^s}C_s$. Therefore, if

$$\sum_{r=1}^{\infty}(z_{r+1}-z_r) = u$$

in F, we have (z_r) converging to $u - z_1$ in F. On the other hand,

$$\sum_{r=s}^{\infty} (z_{r+1} - z_r) = u - z_1 - z_s \in \frac{1}{2^s} C_s,$$

from which it follows that

$$u \in D_s, \quad s = 1,2,\ldots$$

and this

$$u \in F^{(A_k)}.$$

It also follows from (1) that (z_s) converges to $u-z_1$ in $F^{(A_k)}$. Finally, it is obvious that (y_r) also converges to $u-z_1$ in $F^{(A_k)}$.

q.e.d.

THEOREM 1. *Let* f *be a linear mapping from a metrizable space* E *into* F *such that the graph of* f *meets* $E \times F_B$ *in a closed set for every* B *of* $\mathcal{B}$. *If the closure of* $f^{-1}(A_k)$ *in* E *is a neighbourhood of the origin, then* $f(E) \subset F^{(A_k)}$ *and* $f : E \to F^{(A_k)}$ *is continuous.*

Proof. We fix a positive integer k. According to Proposition 2, $f^{-1}(B_k)$ is a neighbourhood of the origin in E and, consequently, f(E) is contained in the linear hull of B_k. From the definitions, it is clear that $2C_k$ contains B_k. Thus we have $f(E) \subset D_k$ and so

$$f(E) \subset F^{(A_k)}.$$

If (x_n) is a sequence in E converging to the origin and r is a positive integer, there is another positive integer p such that

$$x_n \in \frac{1}{2^r} B_r, \quad n \geq p.$$

Then

$$f(x_n) \in \frac{1}{r}(F^{A_k)} \cap C_r), \quad n \geq p,$$

from which the continuity of f follows.

q.e.d.

PROPOSITION 7. *Let* f *be a continuous and injective linear mapping from a space* E *into* F. *If the closure* M_k *of* $f^{-1}(A_k)$ *in* E *is a neighbourhood of the origin, then the family*

$$\{\frac{1}{k} M_k : k = 1,2,\ldots\}$$

is a fundamental system of neighbourhoods of the origin for a metrizable locally convex topology on E.

Proof. We must show that

$$\bigcap_{k=1}^{\infty} \frac{1}{k} M_k = \{0\} .$$

Let us take a point x in E, x≠0. We find a neighbourhood of the origin U in F, closed and absolutely convex and such that

$$f(x) \notin U.$$

Then

$$x \in f^{-1}(U).$$

According to Proposition 1 there is a positive integer k such that

$$\frac{1}{k} A_k \subset U$$

and, therefore,

$$\frac{1}{k} M_k \subset f^{-1}(U),$$

showing that x does not belong to $\frac{1}{k}M_k$.

q.e.d.

THEOREM 2. *Let* f *be a linear mapping with closed graph from a space* E *into* F. *Let us suppose that for every positive integer* k *, the closure of* $f^{-1}(A_k)$ *in* E *is a neighbourhood of the origin. Then we have*

$$f(E) \subset F^{(A_k)} \text{ and } f : E \to F^{(A_k)} \text{ is continuous.}$$

Proof. Since the graph of f is closed, there is a Hausdorff and locally convex topology $\mathscr{V}$ on F, coarser than the original one, and such that

$$f : E \to F[\mathscr{V}]$$

is continuous, (cf. [3] and [4]). The sequence (A_k) is also a $\mathscr{B}$-completing sequence of absolutely convex subsets in $F[\mathscr{V}]$ and $f^{-1}(0)$ is closed in E. Let φ be the canonical mapping from E onto $G := E/f^{-1}(0)$ and ψ the canonical injection from G into F, with

$$f = \psi \circ \varphi .$$

According to the former proposition, and denoting by M_k the closure of $\psi^{-1}(A_k)$ in G, k=1,2,..., we obtain the family

$$\{ \frac{1}{k} M_k : k = 1,2,\dots \}$$

as a fundamental system of neighbourhoods of the origin in G for a metrizable and locally convex topology $\mathcal{U}$ on G. Then the closure of $\psi^{-1}(A_k)$ in $G[\mathcal{U}]$ coincides with M_k and, therefore, it is a neighbourhood of the origin in this space. Now the conclusion follows applying Theorem 1.

q.e.d.

2. ABSOLUTELY CONVEX WEBBED SPACES

In all this section

$$W = \{C_{m_1, m_2, \dots, m_n}\} \tag{2}$$

will be an absolutely convex and completing web in a space E. If $\alpha = (a_n)$ is an element of $\mathbb{N}^{\mathbb{N}}$, we have an absolutely convex and completing sequence

$$(C_{a_1, a_2, \dots, a_k})_{k=1}^{\infty}.$$

We shall write E_α to denote the Fréchet space $E_{(C_{a_1, a_2, \dots, a_k})}$ and we say that

$$\{E_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

is the family of Fréchet spaces associated to the web (2).

THEOREM 3. *Let* f *be a linear mapping from a metrizable and unordered Baire-like space* F *into the space* E. *If the graph of* f *meets* $F \times E_B$ *in a closed subspace for every Banach disc* B *of* E, *there is* α *in* $\mathbb{N}^{\mathbb{N}}$ *such that* $f(F) \subset E_\alpha$ *and* $f : F \to E_\alpha$ *is continuous.*

Proof. Given a sequence (p_n) of positive integers, we denote by $L_{p_1,p_2,\ldots,p_n}$ the linear hull of $f^{-1}(C_{p_1,p_2,\ldots,p_n})$ in F, $n = 1,2,\ldots$. We have

$$F = \bigcap_{n=1}^{\infty} L_n,$$

from which it follows that for a positive integer m_1 the space L_{m_1} is unordered Baire-like and dense in F. Proceeding by recurrence, let us suppose that the positive integers $m_1,m_2,\ldots,m_p$ have been obtained in such a way that the space $L_{m_1,m_2,\ldots,m_p}$ is unordered Baire-like and dense in F. We have

$$L_{m_1,m_2,\ldots,m_p} = \bigcup_{m=1}^{\infty} L_{m_1,m_2,\ldots,m_p,m}$$

from which we have again a positive integer m_{p+1} such that the space $L_{m_1,m_2,\ldots,m_{p+1}}$ is unordered Baire-like and dense in F.

Obviously, the closures in F of $f^{-1}(C_{m_1,m_2,\ldots,m_k})$, $k=1,2,\ldots$, are neighbourhood of the origin in F. Therefore according to Theorem

1 we obtain for $\alpha = (a_k)$ that $f(F) \subset E$ and $f : F \to E_\alpha$ is continuous.

q.e.d.

THEOREM 4. *If* f *is a linear mapping with closed graph from an unordered Baire-like space* F *into the space* E, *then there exists* α *in* $\mathbb{N}^{\mathbb{N}}$ *such that* $f(F) \subset E_\alpha$ *and* $f : F \to E_\alpha$ *is continuous.*

Proof. Proceeding as we have done in the former theorem we can obtain $\alpha = (a_k)$ in $\mathbb{N}^{\mathbb{N}}$ such that $f^{-1}(C_{m_1,m_2,\ldots,m_k})$ is a neighbourhood of the origin in F, k=1,2,.... The conclusion now follows applying Theorem 2.

q.e.d.

COROLLARY. *Every continuous linear mapping from an unordered Baire-like space* F *into* E *can be extended to a continuous linear mapping from* $\hat{F}$ *into* E.

3. SEMI-LB-SPACES.

Let

$$\{ A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \} \tag{3}$$

be a semi-LB-representation in a space E. Given positive integers k, $m_1, m_2, \ldots, m_k$, we write

$$M_{m_1,m_2,\ldots,m_k} = \cup\{A_\alpha : \alpha = (a_n) \in \mathbb{N}^{\mathbb{N}},\ a_n = m_n,\ n=1,2,\ldots,k\}.$$

Let $C_{m_1,m_2,\ldots,m_k}$ be the absolutely convex cover of M_{m_1,m_2,m_k}.

We denote by $\mathscr{B}$ the family (3) of Banach discs.

PROPOSITION 8. *Given* (m_k) *in* $\mathbb{N}^{\mathbb{N}}$, *the sequence*

$$C(_{m_1,m_2,\ldots,m_k})$$

is absolutely convex and $\mathscr{B}$*-completing.*

Proof. Let x_k be a vector in $C_{m_1,m_2,\ldots,m_k}$, $k=1,2,\ldots$. There are

$$x_{kj} \in M_{m_1,m_2,\ldots,m_k},\ a_{kj} \in \mathbb{K},\ j=1,2,\ldots,p(k)$$

such that

$$x_k = \sum_{j=1}^{p(k)} a_{kj}\, x_{kj},\quad \sum_{j=1}^{p(k)} |a_{kj}| \leq 1 .$$

Let

$$\alpha_{kj} = (a_{n,kj}) \in \mathbb{N}^{\mathbb{N}},\ a_{n,kj} = m_n,\ n = 1,2,\ldots,k ,$$

and

$$x_{kj} \in A_{\alpha_{kj}} ,\quad j = 1,2,\ldots,p(k).$$

The sequence

$$\alpha_{11},\alpha_{12},\ldots,\alpha_{1p(1)},\alpha_{21},\alpha_{22},\ldots,\alpha_{2p(2)},\ldots,\alpha_{k1},\alpha_{k2},\ldots,\alpha_{kp(k)},$$

obviously is semi-stationary; therefore, we have α in $\mathbb{N}^{\mathbb{N}}$ such that

$$A_{\alpha_{kj}} \subset A_\alpha ,\ j = 1,2,\ldots,p(k),\ k = 1,2,\ldots .$$

Consequently,

$$x_k \in A_\alpha \ , \quad k = 1,2,\ldots,$$

and if

$$b_k \in \mathbb{K},\ k = 1,2,\ldots, \text{ and } \sum_{k=1}^{\infty} |b_k| \leq 1,$$

the series

$$\sum_{k=1}^{\infty} b_k\, x_k$$

converges in E_{A_α} .

q.e.d.

If $\alpha = (m_k) \in \mathbb{N}^{\mathbb{N}}$ we denote by E_α the Fréchet space $E^{(C_{m_1,m_2,\ldots,m_k})}$ and we shall say that

$$\{E_\alpha \ : \ \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

in the family of Fréchet spaces associated to the semi-LB-represen tation (3).

The following two theorems are proved using Theorem 1 and Theorem 2 respectively.

THEOREM 5. *Let* f *be a linear mapping from a metrizable Baire space* F *into the space* E. *If the graph of* f *meets* $F \times E_{A_\beta}$ *in a closed subspace for every* β *in* $\mathbb{N}^{\mathbb{N}}$ *there is* α *in* $\mathbb{N}^{\mathbb{N}}$ *such that* $f(F) \subset E_\alpha$ *and* $f : F \to E_\alpha$ *is continuous.*

THEOREM 6. *If* f *is a linear mapping with closed graph from a Baire space* F *into the space* E , *there is* α *in* $\mathbb{N}^{\mathbb{N}}$ *such that*

$f(F) \subset E_\alpha$ *and* $f : F \to E_\alpha$ *is continuous.*

In the set $\mathbb{N}^{\mathbb{N}}$ we consider the following order relation "$\leq$": for $\alpha = (a_n)$ and $\beta = (b_n)$ in $\mathbb{N}^{\mathbb{N}}$ we say that $\alpha \leq \beta$ if and only if $a_n \leq b_n$ for every positive integer n.

A *quasi-LB-representation* in a space G is a family

$$\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

of Banach discs satisfying the following conditions:

1. $\cup \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = G$.

2. If $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ and $\alpha \leq \beta$, then $B_\alpha \subset B_\beta$

We say that a space admitting a quasi-LB-representation is a *quasi-LB-space.*

It is obvious that a quasi-LB-representation is a semi-LB-representation, and thus, a quasi-LB-space is a semi-LB-space.

Lifting theorems have been proved in [6] for quasi-LB-representations. These results can be formulated with some minor modifications for semi-LB-representations.

4. CONVEX WEBBED SPACES

Let

$$\mathcal{V} = \{ L_{n_1, n_2, \ldots, n_k} \}$$

be a convex $\mathscr{C}$-web in a space E. if $M_{n_1,n_2,\ldots,n_k}$ is the convex cover of

$$\{0\} \cup L_{n_1,n_2,\ldots,n_k} ,$$

we write

$$A_{n_1,n_2,\ldots,n_k} = M_{n_1,n_2,\ldots,n_k} - M_{n_1,n_2,\ldots,n_k} .$$

We denote by T an injective mapping from $\mathbb{N}^2$ onto $\mathbb{N}$, When (p_1,r_1) belongs to $\mathbb{N}^2$ and $T(p_1,r_1) = n_1$, we put

$$B_{n_1} = p_1 A_{r_1}.$$

Proceeding by recurrence, let us suppose that for a positive integer $k > 1$ we have constructed the subsets

$$B_{n_1,n_2,\ldots,n_{k-1}} ,$$

where $n_1,n_2,\ldots,n_{k-1}$ are arbitrary positive integers. Given positive integers $p_1,r_1,p_2,r_2,\ldots,p_k,r_k$ we write

$$P_{n_1,n_2,\ldots,n_k} = p_1 p_2,\ldots,p_k A_{r_1,r_2,\ldots,r_k} ,$$

$$B_{n_1,n_2,\ldots,n_k} = P_{n_1,n_2,\ldots,n_k} \cap B_{n_1,n_2,\ldots,n_{k-1}} ,$$

where

$$T(p_j,r_j) = n_j, \; j = 1,2,\ldots,k .$$

Since $p_1 A_{r_1}$ contains L_{r_1}, it follows that

$$\cup \{B_{n_1} : n_1 = 1, 2, \ldots \} = E.$$

Let us now take a point x in $B_{n_1, n_2, \ldots, n_k}$; then x belongs to $P_{n_1, n_2, \ldots, n_k}$ and therefore there exist two points y and z in

$$p_1 p_2 \ldots p_k \quad L_{r_1, r_2, \ldots, r_k}$$

together with two numbers α and β, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, such that

$$x = \alpha y - \beta x.$$

If y coincides with z, there is a positive integer r_{k+1} such that

$$y = z \in p_1 p_2 \ldots p_k \ L_{r_1, r_2, \ldots, r_k, r_{k+1}}$$

and so

$$x \in p_1 p_2 \ \ldots \ p_k \ A_{r_1, r_2, \ldots, r_{k+1}} .$$

If y is not equal to z, we take

$$H = \{\lambda z + (1-\lambda) y \ : \ 0 \leq \lambda \leq 1\} .$$

Since H is contained in

$$p_1 p_2 \ldots p_k \ L_{r_1, r_2, \ldots, r_k},$$

there exists a positive integer r_{k+1} such that

$$p_1 p_2 \dots p_k \, L_{r_1, r_2, \dots, r_{k+1}}$$

meets H in two points at least. A positive integer p_{k+1} can be determined such that

$$2H \subset p_1 p_2 \dots p_k \, p_{k+1} \, A_{r_1, r_2, \dots, r_k, r_{k+1}}$$

and therefore

$$x \in p_1 p_2 \dots p_k \, p_{k+1} \, A_{r_1, r_2, \dots, r_k, r_{k+1}} \tag{5}$$

Consequently, (5) holds in the two cases considered. If

$$T(p_{k+1}, r_{k+1}) = n_{k+1} ,$$

it follows that

$$x \in P_{n_1, n_2, \dots, n_{k+1}} \cap B_{n_1, n_2, \dots, n_k} = B_{n_1, n_2, \dots, n_{k+1}}$$

from which we have

$$B_{n_1, n_2, \dots, n_k} = \cup \{B_{n_1, n_2, \dots, n_k, n_{k+1}} : n_{k+1} = 1, 2, \dots\}, \tag{6}$$

PROPOSITION 9. *The family*

$$\mathcal{U} = \{B_{n_1, n_2, \dots, n_k}\}$$

is a completing web in E.

Proof. From (4) and (6) we know that $\mathcal{U}$ is a web in E. Given a sequence of positive integers (r_k) we determine a sequence

(λ_k) of positive numbers such that the series

$$\sum_{k=1}^{\infty} \mu_k x_k$$

converges in E whenever

$$0 \leq \mu_k \leq \lambda_k, \ x_k \in L_{r_1,r_2,\ldots,r_k}, \quad k = 1,2,\ldots .$$

Let us now suppose that for the sequence (n_j) in $\mathbb{N}^{\mathbb{N}}$ we have

$$T^{-1}(n_j) = (p_j,r_j), \quad j = 1,2,\ldots .$$

If we take z_k in $B_{n_1,n_2,\ldots,n_k}$ we have

$$z_k \in p_1 p_2 \ldots p_k \, A_{r_1,r_2,\ldots,r_k}$$

and we can find u_k and v_k in $L_{r_1,r_2,\ldots,r_k}$ together with $0 \leq \alpha_k \leq 1$, $0 \leq \beta_k \leq 1$, such that

$$z_k = p_1 p_2 \ldots p_k(\alpha_k u_k - \beta_k v_k).$$

Let us now take

$$0 \leq \mu_k \leq (p_1 p_2 \ldots p_k)^{-1} \lambda_k$$

and we have the convergent series

$$\sum_{k=1}^{\infty} \mu_k p_1 p_2 \ldots p_k \alpha_k u_k \text{ and } \sum_{k=1}^{\infty} \mu_k p_1 p_2 \ldots p_k \beta_k v_k$$

from which it follows that the series

$$\sum_{k=1}^{\infty} \mu_k z_k$$

also converges in E.

q.e.d.

When E is a real space we write

$$C_{n_1,n_2,\ldots,n_k} = B_{n_1,n_2,\ldots,n_k}$$

and in case of E being a complex space we write

$$C_{n_1,n_2,\ldots,n_k} = B_{n_1,n_2,\ldots,n_k} \cap i\, B_{n_1,n_2,\ldots,n_k}$$

whenever k, $n_1,n_2,\ldots,n_k$ are positive integers.

PROPOSITION 10. *The family*

$$W = \{ C_{n_1,n_2,\ldots,n_k} \}$$

is an absolutely convex and completing web in E.

Proof. The result is obvious when E is a real space. Let us now suppose that E is a complex space. If x is any point of E there are two positive integers p_1 and r_1 such that the strongt line with and-points in x and ix is contained in $p_1 A_{r_2}$. It now follows that both x and ix are in B_{n_1}, where $n_1 = T(p_1,r_1)$. Thus we have

$$\cup \{C_{n_1} : n_1 = 1,2,\ldots\} = E .$$

If x is any point in $C_{n_1,n_2,\ldots,n_k}$, we know that x and ix belong to $B_{n_1,n_2,\ldots,n_k}$. We put $(p_1,r_1) = T^{-1}(n_j)$, $j=1,2,\ldots$.

Then we have

$$x, ix \in p_1p_2\ldots p_k\, A_{r_1,r_2,\ldots,r_k}$$

and therefore there are two positive integers p_{k+1} and r_{k+1} such that

$$x, ix \in p_1p_2\ldots p_{k+1}\, A_{r_1,r_2,\ldots,r_{k+1}}$$

Consecuently, we have

$$x, ix \in B_{n_1,n_2,\ldots,n_{k+1}},$$

where $T(p_{k+1},r_{k+1}) = n_{k+1}$ and so

$$x \in C_{n_1,n_2,\ldots,n_{k+1}} .$$

Thus

$$\cup\{C_{n_1,n_2,\ldots,n_{k+1}} : n_{k+1} = 1,2,\ldots\} = C_{n_1,n_2,\ldots,n_k}$$

and hence W is a web in E. Finally it is clear that W is absolutely convex and completing.

q.e.d.

The following theorem is now clear:

THEOREM 7. *If* F *is a convex webbed space, then* F *is an absolutely convex webbed space.*

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