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ON DERIVABLE BAER-ELATION PLANES

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1. **INTRODUCTION.** - In [5], Jha and Johnson introduce Baer-elation planes. These are finite translation planes of order q^2, $q=p^r$ which admit both Baer p-collineation groups and elation groups which normalize each other. By a result of Foulser [3], p=2.

Jha-Johnson consider, in particular, Baer-elation planes of order q^2 with kernel GF(q) of type (2,q) or type (q,2). That is, there is a Baer or elation group of order q. By the incompatibility results of Jha-Johnson [7],[8], the corresponding elation or Baer group has order ≤ 2.

Recently, Huang and Johnson [4] have determined all of the semifield Baer-elation planes of order 64, kernel GF(8) (of type (2.8)). Note that the Desarguesian plane of order 64 may be considered a Baer-elation plane of type (2,8). There are eight such Baer-elation planes of order 64 with the Desarguesian plane being the only one which is derivable.

In general the Desarguesian and Hall planes of even order q^2 and kernel $\geq$ GF(q) are the only known derivable Baer-elation planes of type (2,q) or type (q,2) and kernel $\geq$ GF(q). In [1], Biliotti and Menichetti study finite translation planes of order q^2 which admit affine elations and which may be derived from a semifield plane. The affine elations of the semifield plane which fix the derivable net in question inherit as Baer collineations of the derived plane. Hence, these derived planes of order q^2 are derivable

Baer-elation planes of type (q,2). However, Biliotti-Menichetti show that the *only* such plane with kernel GF(q) is the Hall plane.

So, a basic question is whether there are *derivable* Baer-elation planes of order q^2 kernel $\geq$GF(q) of type (2,q) or type (q,2).

Our main result answers this question.

THEOREM A.

Let π *be a finite translation plane of even order* q^2 *and kernel* $\geq$ GF(q).

(1) *Assume* π *admits an affine elation group* E *of order* q *and a nontrivial Baer 2-group* $\mathcal{B}$ *which normalize each other where* $\mathcal{B}$ *is in the linear translation complement. If the axis of* E *and an* E*-orbit of components defines a derivable net then* π *is Desarguesian.*

(2) *Assume* π *admits a Baer group* $\mathcal{B}$ *of order* q *and a nontrivial elation group* E. *If the net of degree* q+1 *containing the subplane pointwise fixed by* $\mathcal{B}$ *is derivable then* π *is Hall.*

2. BACKGROUND AND PROOF OF THEOREM A.

Let π be a Baer-elation plane which satisfies the assumption of Theorem A(1). Then π is of type (2,q) by Jha-Johnson [5] and

(2.1) LEMMA.

π may be represented in the following form:

$$\pi = \{(x_1,x_2,y_1,y_2) \mid x_i, y_i \in GF(q), \quad i = 1,2\} ,$$

and the components may be represented in the form $x = \mathcal{O}$,

$y = x \begin{bmatrix} u+v & , & f(v) \\ v & , & u \end{bmatrix}$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$, for all $u, v \in GF(q)$

and f is a 1-1 function $GF(q) \quad GF(q)$.

Proof. By Jha-Johnson [5] (3.7), we obtain the components in the form $x = \mathcal{O}$, $y = x \begin{bmatrix} u+v+m(v) & , & f(v)+m(u) \\ v & , & u \end{bmatrix}$ where $m, f : GF(q) \to GF(q)$. The elation group E has the form

$$E = \left\{ \begin{bmatrix} I & \begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix} \\ \mathcal{O} & I \end{bmatrix} \middle| \; u \in GF(q) \right\}$$

and the Baer group $\mathcal{B} = \langle\sigma\rangle$ is represented by $\sigma = \left[\begin{array}{cc|cc} 1 & 1 & & \\ 0 & 1 & \multicolumn{2}{c}{\mathcal{O}} \\ \hline & & 1 & 1 \\ \multicolumn{2}{c|}{\mathcal{O}} & 0 & 1 \end{array}\right]$.

(See also [5] (3.2).)

The nontrivial E-orbits Γ_v of components are

$$\left\{ y = x \begin{bmatrix} u+v+m(v) & , & f(v)+m(u) \\ v & , & u \end{bmatrix} \right\} = \Gamma_v$$

for each fixed v in $GF(q)$ and for all $u \in GF(q)$. Suppose one of these orbits union $(x = \mathcal{O})$ is derivable. Then by Jha-Johnson [9], *each* orbit Γ_v union $(x = \mathcal{O})$ is a derivable net. In particular, $\Gamma_0 \cup (x = \mathcal{O})$ is derivable. So $x = \mathcal{O}$, $y = x \begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix}$ for all $u \in GF(q)$ is a derivable partial spread.

By Foulser [2], this derivable partial spread is Desarguesian and is a regulus with respect to some field $L \cong GF(q)$. Hence, the set of matrices $\left\{\begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix} \mid u \in GF(q)\right\}$ form a field $\cong GF(q)$. This implies that $vm(u) = um(v)$ for all $u,v \in GF(q)$. For $v=1$, $m(u) = um(1)$ for all $u \in GF(q)$. However, if we initially assume that $x = y$ is a component of the subplane pointwise fixed by σ then $m(1) = 0$. So, we have the proof to (2.1).

(2.2) LEMMA.

Let $\Sigma = PG(3,q)$ and represent Σ by homogeneous coordinates (x_0,x_1,x_2,x_3). Let $\mathscr{C}$ be the quadratic cone in Σ with vertex $(0,0,0,1)$ with equation $x_0x_1 = x_2^2$. If $x = \mathcal{O}$, $y = x\begin{bmatrix} u+v, & f(v) \\ v, & u \end{bmatrix}$ represents the translation plane π then let $\rho_{\bar{v}}$ be the projective plane of Σ whose equation is $vx_0-f(v)x_1+vx_2+x_3=0$. Then $\{\rho_v \cap \mathscr{C} \mid v \in GF(q) = \mathscr{F}\}$ is a set of ovals (conics) which are mutually disjoint and which partition $\mathscr{C} - \{(0,0,0,1)\}$. That is, $\mathscr{F}$ is a flock of the quadratic cone.

Proof. This is part of a more general theorem which will appear elsewhere. However, with the conditions on the matrix spread set

$$\begin{bmatrix} u+v & , & f(v) \\ v & , & u \end{bmatrix} - \begin{bmatrix} s+t & , & f(t) \\ t & , & s \end{bmatrix}$$

being nonsingular on zero, it is straightforward to verify that $\{\rho_v \cap \mathscr{C} \mid v \in GF(q)\}$ is a flock. We obtain

$$\det \begin{bmatrix} (u+v)-(s+t) & , & f(v)-f(t) \\ v-t & , & u-s \end{bmatrix} = 0$$

if and only if

$$((u-s)+(v-t))(u-s) - (v-t)(f(v)-f(t)) = 0$$

if and only if

$$(u-s)^2 + (u-s)(v-t) - (v-t)(f(v)-f(t))$$

iff

$$(u-s)^2(v-t)^2 + (u-s)(v-t)^3 - (v-t)^3(f(v)-f(t))$$

iff for $(u-s)(v-t) = z$.

$$z^2 + z(v-t)^2 - (v-t)^3(f(v-f(t))$$

iff for $v \neq t$

$$\left(\frac{z}{v-t}\right)^2 + \left(\frac{z}{v-t}\right)(v-t) - (v-t)(f(v)-f(t))$$

iff $x^2 + x(v-t) - (v-t)(f(v)-f(t))$ is irreducible for all $v \neq t$.

The plane $(\rho_v \cap \rho_t)(0,0,0,1)$ of Σ is given by $(v-t)x_0-(f(v)-f(t))x_1 + (v-t)x_2 = 0$. This plane contains a point on the cone $\mathscr{C}$ $(x_0x_1 = x_2^2)$ if and only if

$$(v-t)\bar{x}_0\bar{x}_1 - (f(v)-f(t))\bar{x}_1^2 + (v-t)\bar{x}_1\bar{x}_2 = 0$$

for some $\bar{x}_0, \bar{x}_1, \bar{x}_2$. This is true iff

$$(v-t)\bar{x}_2^2 - (f(v)-f(t))\bar{x}_1^2 + (v-t)\bar{x}_1\bar{x}_2 = 0$$

and since clearly $\bar{x}_1\bar{x}_2 \neq 0$, and

$$(v-t)^3\bar{x}_2^2 - (v-t)^2(f(v)-f(t))\bar{x}_1^2 + (v-t)^3\bar{x}_1\bar{x}_2 = 0,$$

write $(v-t)\frac{\bar{x}_2}{\bar{x}_1} = Z$ so that

$$(v-t)Z^2 + (v-t)^3 Z - (v-t)^2(f(v)-f(t)) = 0$$

iff for $v \neq t$

$$x^2 + x(v-t) - (v-t)(f(v)-f(t)) = 0.$$

So, $(\rho_v \cap \rho_t)(0,0,0,1)$ *cannot* contain a point of $\mathscr{C}$ and we obtain a *flock* of $\mathscr{C}$.

(2.3) LEMMA.

With the assumptions of (2.2), the planes ρ_v of the flock all contain a common point.

Proof. Note that (1,0,1,0) satisfies

$$vx_0 - f(v)x_1 + vx_2 + x_3 = 0$$

for all $v \in GF(q)$.

Now by a recent result of Thas [10] (1.5.6) if the planes of a flock of a quadratic cone in PG(3,q) contain a common point and q is even then the flock is *linear*. This means the set of planes $\{\rho_v\}$ is the set of planes containing some line $\mathcal{L}$ of PG(3,q) (except for the plane $\mathcal{L}\cdot(0,0,0,1)$). So

$$\rho_v : vx_0 - f(v)x_1 + vx_2 + x_3$$

and

$$\rho_t : tx_0 -(f(t)x_1 + tx_2 + x_3$$

contain the common line $\mathcal{L}$ contained in the plane

$$(v-t)x_0 -(f(v)-f(t))x_1 + (v-t)x_2 = 0.$$

If $x_0 = x_2 = 0$ then for $t = 0$ and $v \neq 0$, $f(v)x_1 = 0$ iff $x_1=0$ as f is 1-1. So we may assume x_0 or $x_2 \neq 0$. Silimarly, if $x_0+x_2=0$ then we obtain $f(v)x_1 = 0$ and the common point (1,0,1,0). Hence, we may assume that $x_0 + x_2 \neq 0$ and hence $v(f(v))^{-1} = \frac{x_1}{x_0+x_2}$ for some point (x_0,x_1,x_2,x_3) different than (0,0,0,1) or (1,0,1,0). But, this says that $v(f(v))^{-1} = w(f(w))^{-1}$ for all nonzero v,w of GF(q). So vf(w) = wf(v) for all nonzero v,w in GF(q) and hence, f(v) = vf(1).

Note that the translation plane now is represented by components $x = \mathcal{O}$, $y = x \begin{bmatrix} u+v, & vf(1) \\ v & u \end{bmatrix}$ which clearly represents the Desargue-sian plane. Hence, we have the proof to Theorem A(1).

We now assume the conditions of Theorem A(2).

If the elation group E does not leave the net containing the

Baer axis *invariant* then there are at least two such Baer groups of order q and the plane is Hall by Jha-Johnson [6].

If the elation group does leave the net (derivable) invariant then the axis of E must be a component of the net and in the derived plane, we obtain the hypotheses of Theorem A(1). Hence, in either case, π must be the Hall plane.

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