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Nested sequences of sets, balls, Hausdorff Convergence

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Abstract. We discuss the behaviour of sequences of sets: we consider convergence of sequences, in particular of nested sequences, with respect to preservation of some properties.

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1 Introduction

Let X be a Banach space with origin o and $A \neq \emptyset$ be a bounded subset of X. We denote by ∂A, $\operatorname{cl} A$, and $\delta(A)$ the *boundary*, *closure*, and *diameter* of A, respectively. Moreover, we set

- $\gamma(A, x) = \sup\{\|x - a\| : a \in A\}$, $\forall x \in X$;
- $\gamma(A, B) = \inf\{\gamma(A, x) : x \in B\}$, $\forall B \subseteq X$;
- $\gamma'(A) = \sup\{\inf\{\|x - a\| : x \notin A\} : a \in A\}$ (*inner radius* of A).

The numbers $\gamma(A) := \gamma(A, X)$ and $\gamma(A, A)$ are called the *radius* and the *self radius* of A, respectively. Note that we always have

$$\gamma(A) \leq \gamma(A, A) \leq \delta(A) \leq 2\gamma(A) \leq 2\gamma(A, A). \tag{1.1}$$

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Also, for $x \in X$ and $\gamma \geq 0$, we denote by

$$B(x, \gamma) = \{y \in X : \ \|x - y\| \leq \gamma\}$$

the *ball centered at x having radius γ.*

We shall denote by c_0 and $C([0,1])$ the usual classical Banach spaces.

In this paper we study some classes of sets; we study nested sequences of sets in these classes and the convergence of sequences of sets in one of these classes. We present some properties that are preserved under these operations, considering in particular Kuratowski's and Hausdorff convergence.

Among the older papers concerning monotone sequences of convex sets, we recall the interesting one [6].

In Section 2 we consider nested sequences and Kuratowski's convergence; in Section 3, Hausdorff convergence; in Section 4, increasing sequences of sets. Section 5 is devoted to sequences of balls. Finally, in Section 6, we collect some less trivial examples.

For the sake of simplicity, we shall always assume that all sequences under consideration consist of closed, bounded sets.

2 Nested sequences of sets and Kuratowski's convergence

We recall the definition of Kuratowski's convergence. Given a sequence $\{A_n\}_{n=1}^{\infty}$ of (non-empty) sets, we say that it *converges to A in the sense of Kuratowski*, K-converges to A for short, and we write $A_n \xrightarrow{K} A$, if

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A,$$

where

$$\liminf_{n\to\infty} A_n = \{x \in X : \text{ there exists a sequence } \{x_n\}_{n=1}^{\infty},\ x_n \in A_n,\ \forall n \in \mathbb{N}, \text{ such that } \lim_{n\to\infty} x_n = x\}$$

and

$$\limsup_{n\to\infty} A_n = \{x \in X : \text{ there exist a subsequence } \{A_{n_k}\}_{k=1}^{\infty} \text{ of } \{A_n\}_{n=1}^{\infty}, \text{ and a sequence } \{x_{n_k}\}_{k=1}^{\infty},\ x_{n_k} \in A_{n_k},\ \forall k \in \mathbb{N}, \text{ such that } \lim_{k\to\infty} x_{n_k} = x\}.$$

Proposition 1. *If $\{A_n\}_{n=1}^{\infty}$ is a sequence of convex sets and*

$$\liminf_{n\to\infty} A_n = A,$$

then also A is a convex set.

Proof. Let x and y be two points in A and λ be an arbitrary number in $(0,1)$. Then there exist two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$x_n, y_n \in A_n, \ \forall n \in \mathbb{N}$$

and that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y.$$

Then

$$\begin{aligned}\lambda x + (1-\lambda)y &= \lambda \lim_{n\to\infty} x_n + (1-\lambda) \lim_{n\to\infty} y_n \\ &= \lim_{n\to\infty} (\lambda x_n + (1-\lambda) y_n).\end{aligned}$$

Since A_n is convex for each $n \in \mathbb{N}$, we have $\lambda x_n + (1-\lambda)y_n \in A_n$ for each $n \in \mathbb{N}$. Therefore $\lambda x + (1-\lambda)y \in A$. Thus A is convex. $\boxed{QED}$

Corollary 1. *If $\{A_n\}_{n=1}^{\infty}$ is a sequence of convex sets and $A_n \xrightarrow{K} A$, then also A is convex.*

We recall that if $\{A_n\}$ is a *nested sequence of sets*, i.e., if $A_{n+1} \subseteq A_n$, $\forall n \in \mathbb{N}$, then

$$A_k \xrightarrow{K} A = \bigcap_{n=1}^{\infty} A_n$$

(the last set is possibly empty), see [2, §5].

As known, nonemptiness of nested sequences of closed, bounded, convex sets is connected with the reflexivity of the underlying space.

Kuratowski's convergence presents some defects. For example, none of the following equalities is true in general when $A_n \xrightarrow{K} A$:

(1) $\lim\limits_{n\to\infty} \gamma(A_n) = \gamma(A)$;

(2) $\lim\limits_{n\to\infty} \gamma'(A_n) = \gamma'(A)$;

(3) $\lim\limits_{n\to\infty} \delta(A_n) = \delta(A)$;

(4) $\lim\limits_{n\to\infty} \gamma(A_n, A_n) = \gamma(A, A)$.

Of course, for nested sequences we have

$$\lim_{n\to\infty} \gamma(A_n) \geq \gamma(A), \quad \lim_{n\to\infty} \gamma'(A_n) \geq \gamma'(A), \text{ and } \lim_{n\to\infty} \delta(A_n) \geq \delta(A).$$

We shall give examples (in Section 6) showing that also for nested sequences of convex sets, in general these are not equalities. Example 2 will show that the second inequality (concerning $\gamma'(\cdot)$) can be strict. Example 1 will show that the first and the third inequality can be strict; in this example, we also have $\lim_{n\to\infty} \gamma(A_n, A_n) > \gamma(A, A)$. The sequence $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$ does not necessarily converge (see Example 7). As said before, when it converges, it does not necessarily converge to $\gamma(A, A)$. Example 7 shows that $\gamma(A, A)$ can be larger than the limit and a) in Example 5 will show that $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$ can also be increasing.

It is known, and not difficult to prove (see, e.g., [5]), that the intersection of a nested sequence of balls is a (nonempty) ball.

Note that milder assumptions, based on the ratio between $\gamma'(A_n)$ and $\delta(A_n)$, imply the nonemptiness of the intersection of a nested sequence $\{A_n\}_{n=1}^{\infty}$ of sets (see [3]).

3 Hausdorff convergence

Given two nonempty sets A and B, set

$$d(A, B) = \inf\{\|a - b\| : \ a \in A, \, b \in B\}.$$

For any number $\varepsilon > 0$, set

$$A^{\varepsilon} = \{x \in X : \ d(A, \{x\}) \leq \varepsilon\}.$$

Trivially,

$$A \subseteq B \Rightarrow A^{\varepsilon} \subseteq B^{\varepsilon}. \tag{3.1}$$

We set

$$d_H(A, B) = \inf\{\varepsilon > 0 : \ A \subseteq B^{\varepsilon} \text{ and } B \subseteq A^{\varepsilon}\}.$$

The number $d_H(A, B)$ is called the *Hausdorff distance between A and B*.

We say that a sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty sets *converges to a (nonempty) set A in the sense of Hausdorff*, H-converges for short, and we write $A_n \xrightarrow{H} A$, if $\lim_{n\to\infty} d_H(A_n, A) = 0$. Hausdorff convergence is strictly stronger than Kuratowski's convergence (but they coincide when the underlying space is finite dimensional). For example, in general, a nested sequence $\{A_n\}_{n=1}^{\infty}$ of sets such that

$$\bigcap_{n=1}^{\infty} A_n = A \neq \emptyset$$

does not H-converge to A even if all these sets are convex, see Example 2.

As known, in contrast to Kuratowski's convergence, Hausdorff convergence preserves several sets classes (see, e.g., Proposition 4.1 in [2]).

We also recall the following fact (see [7]): if A and B are convex, then

$$d_H(A, B) = d_H(\partial A, \partial B). \tag{3.2}$$

As it is easy to see, this fact is not true in general if A and B are not convex. It was rediscovered recently, and discussed in details in [10].

The following simple result is known. We provide a proof for the sake of completeness.

Proposition 2. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets H-converging to A. Then*

(1) $\lim\limits_{n\to\infty} \delta(A_n) = \delta(A)$*;*

(2) $\lim\limits_{n\to\infty} \gamma(A_n) = \gamma(A)$*;*

(3) $\lim\limits_{n\to\infty} \gamma(A_n, A_n) = \gamma(A, A)$*.*

Proof. (1). Almost trivial: see for example Proposition 4.1 in [8].

(2). Let $\gamma_n = \gamma(A_n)$ and $\gamma = \gamma(A)$. Given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $A_n \subseteq A^\varepsilon$ and $A \subseteq (A_n)^\varepsilon$ whenever $n \in \mathbb{N}$ is greater than n_ε.

Take $i \in \mathbb{N}$. Then there exists $c_i \in X$ such that

$$A \subseteq B\left(c_i, \gamma + \frac{1}{i}\right).$$

By (3.1), we have

$$A^\varepsilon \subseteq B\left(c_i, \gamma + \frac{1}{i}\right)^\varepsilon = B\left(c_i, \gamma + \frac{1}{i} + \varepsilon\right).$$

Therefore,

$$\gamma_n = \gamma(A_n) \leq \gamma(A^\varepsilon) \leq \gamma + \frac{1}{i} + \varepsilon,\ \forall n > n_\varepsilon.$$

Since i is arbitrary,

$$\gamma_n \leq \gamma + \varepsilon,\ \forall n > n_\varepsilon.$$

Also, for each $n \in \mathbb{N}$, there exists $c_i^n \in X$ such that

$$A_n \subseteq B\left(c_i^n, \gamma_n + \frac{1}{i}\right).$$

Thus we have (by (3.1) again),

$$\gamma = \gamma(A) \leq \gamma((A_n)^\varepsilon) \leq \gamma_n + \frac{1}{i} + \varepsilon,\ \forall n > n_\varepsilon.$$

It follows that

$$\gamma \leq \gamma_n + \varepsilon,\ \forall n > n_\varepsilon.$$

Thus,

$$|\gamma - \gamma_n| \leq \varepsilon,\ \forall n > n_\varepsilon.$$

It follows that

$$\lim_{n\to\infty} \gamma_n = \gamma.$$

(3). Let $\varepsilon > 0$; take n_ε as in the proof of previous part (2). For each $i \in \mathbb{N}$, there exists a point $a_i \in A$ such that

$$A \subseteq B\left(a_i, \gamma(A, A) + \frac{1}{i}\right).$$

From (3.1) it follows that

$$A^\varepsilon \subseteq B\left(a_i, \gamma(A, A) + \frac{1}{i} + \varepsilon\right).$$

Since $A \subseteq (A_n)^\varepsilon$, $\forall n > n_\varepsilon$, there exists a point $a_i^n \in A_n$ such that

$$\|a_i - a_i^n\| \leq \varepsilon.$$

Thus

$$\begin{aligned} A_n \subseteq A^\varepsilon &\subseteq B\left(a_i^n, \gamma(A, A) + \frac{1}{i} + \varepsilon\right) + (a_i - a_i^n) \\ &\subseteq B\left(a_i^n, \gamma(A, A) + \frac{1}{i} + 2\varepsilon\right),\ \forall n > n_\varepsilon. \end{aligned}$$

It follows that

$$\gamma(A_n, A_n) \leq \gamma(A, A) + 2\varepsilon,\ \forall n > n_\varepsilon.$$

By exchanging the role of A and A_n, we obtain

$$\gamma(A, A) \leq \gamma(A_n, A_n) + 2\varepsilon,\ \forall n > n_\varepsilon.$$

Thus

$$\lim_{n\to\infty} \gamma(A_n, A_n) = \gamma(A, A).$$

$\boxed{QED}$

Remark 1. A similar result in general is not true for $\gamma'(\cdot)$, as Example 4 in the last section shows. But it is true for convex sets as we are going to show.

This lemma will be useful to prove the next proposition.

Lemma 1. *Let A and B be two closed convex sets satisfying*

$$\gamma' = \gamma'(A) > 0 \quad \textit{and} \quad \delta = d_H(A, B) < \frac{\gamma'}{2}.$$

Then

$$\gamma'(B) \geq \gamma' - \delta > 0. \tag{3.3}$$

Proof. Let ε be a number in $(0, (\gamma' - 2\delta)/3)$. Then there exists a point $c \in A$ such that $B(c, \gamma' - \varepsilon) \subseteq A$. Then $d(\{x\}, \partial A) > \delta$ holds for each point $x \in B(c, \gamma' - 2\varepsilon - \delta)$. Therefore (use formula (3))

$$B(c, \gamma' - 2\varepsilon - \delta) \cap \partial B = \emptyset.$$

Since

$$\gamma' - 2\varepsilon - \delta > \delta + \varepsilon \quad \text{and} \quad c \in A \subset (B)^{\delta+\varepsilon},$$

we have

$$B(c, \gamma' - 2\varepsilon - \delta) \cap B \neq \emptyset.$$

Since B is convex, this implies that

$$B(c, \gamma' - 2\varepsilon - \delta) \subseteq B.$$

Therefore $\gamma'(B) > \gamma' - 2\varepsilon - \delta$. Since ε is arbitrary, (3.3) holds. $\boxed{QED}$

Proposition 3. *If $A_n \xrightarrow{H} A$, where A_n is convex for each n, then*

$$\lim_{n\to\infty} \gamma'(A_n) = \gamma'(A).$$

Proof. First we consider the case when $\gamma'(A) = 0$. Suppose the contrary that $\{\gamma'(A_n)\}_{n=1}^{\infty}$ does not converge to 0. Then one of its subsequence, say $\{\gamma'(A_{n_k})\}_{k=1}^{\infty}$, converges to a number $\gamma_0' > 0$. Then there exists a sufficiently large integer k such that

$$d_H(A_{n_k}, A) < \frac{\gamma_0'}{4} = \frac{1}{2} \cdot \frac{\gamma_0'}{2} < \frac{1}{2} \cdot \gamma'(A_{n_k}).$$

Then, by Lemma 1,

$$\gamma'(A) > 0,$$

a contradiction. In the following we assume that $\gamma' = \gamma(A) > 0$. Set, for each $n \in \mathbb{N}$, $\gamma_n' = \gamma'(A_n)$.

Then, for each number $\varepsilon \in (0, \gamma'/3)$, there exists an integer n_ε such that $d_H(A_n, A) < \varepsilon$ holds for each $n > n_\varepsilon$. Thus

$$d_H(A_n, A) < \varepsilon < \frac{1}{3}\gamma' < \frac{1}{2}\gamma',\ \forall n > n_\varepsilon.$$

From Lemma 1 it follows that

$$\gamma'_n \geq \gamma' - \varepsilon,\ \forall n > n_\varepsilon.$$

We also have

$$d_H(A, A_n) < \varepsilon < \frac{1}{2}(\gamma' - \varepsilon) \leq \frac{1}{2}\gamma'_n,\ \forall n > n_\varepsilon.$$

By Lemma 1 again, we have

$$\gamma' \geq \gamma'_n - \varepsilon,\ \forall n > n_\varepsilon.$$

Hence

$$|\gamma' - \gamma'_n| \leq \varepsilon,\ \forall n > n_\varepsilon.$$

It follows that

$$\lim_{n\to\infty} \gamma'_n = \gamma'. \qquad \boxed{QED}$$

4 Increasing sequences of sets

Consider now *increasing sequences of sets*. Namely, sequences $\{A_n\}_{n=1}^\infty$ with the following property:

$$A_n \subseteq A_{n+1},\ \forall n \in \mathbb{N}.$$

Increasing, unbounded sequences of balls have been studied to characterize some geometric properties of Banach spaces (see [1] and [4]).

Here we will instead discuss the convergence of A_n to A (as $n \to \infty$) and the preservation of several properties, for increasing sequences satysfying the following condition:

$$\text{the set } A := \mathrm{cl}\left(\bigcup_{n=1}^\infty A_n\right) \text{ is bounded,}$$

which is equivalent to the existence of a ball containing all members of the sequence. In the following we say that a sequence of sets satisfying this condition is *uniformly bounded.*

We have (see [2], p. 146])

$$A_n \xrightarrow{K} \mathrm{cl}\left(\bigcup_{n=1}^{\infty} A_n\right);$$

but, in general,

$$A_n \not\xrightarrow{H} \mathrm{cl}\left(\bigcup_{n=1}^{\infty} A_n\right),$$

even if all the sets are convex (see Example 6).

It is clear that the sequences $\{\delta(A_n)\}_{n=1}^{\infty}$, $\{\gamma(A_n)\}_{n=1}^{\infty}$, and $\{\gamma'(A_n)\}_{n=1}^{\infty}$ are non-decreasing. Example 6 shows that in general this is not true for $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$. More precisely, $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$ not always has a limit: it can be not monotone, or also be strictly decreasing, see also Example 5 b).

Proposition 4. *For a uniformly bounded increasing sequence $\{A_n\}_{n=1}^{\infty}$, set*

$$A = \mathrm{cl}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

We always have

(1) $\delta(A) = \lim_{n\to\infty} \delta(A_n)$;

(2) $\gamma(A, A) \geq \limsup_{n\to\infty} \gamma(A_n, A_n)$ (and equality not always holds);

(3) $\gamma(A) \geq \lim_{n\to\infty} \gamma(A_n)$ (and equality not always holds);

(4) $\gamma'(A) \geq \lim_{n\to\infty} \gamma'(A_n)$ (and equality not always holds).

Moreover, we can have strict inequality in the last three statements also for sequences of convex sets.

Proof. (1). Let $\delta = \lim_{n\to\infty} \delta(A_n)$. It is clear that $\delta \leq \delta(A)$. We prove the converse.

For two arbitrary points $x, y \in \bigcup_{n=1}^{\infty} A_n$, $x, y \in A_n$ for sufficiently large n. Thus

$$\delta(A) = \delta\left(\mathrm{cl}\left(\bigcup_{n=1}^{\infty} A_n\right)\right) \leq \lim_{n\to\infty} \delta(A_n) = \delta.$$

(2). Given $\varepsilon > 0$, there exists a point $x \in A$ such that

$$\gamma(A, x) < \gamma(A, A) + \varepsilon.$$

Since $x \in A$, there exists a sufficiently large $n_\varepsilon \in \mathbb{N}$ such that there exists a point $x_n \in A_n,\ \forall n > n_\varepsilon,$ satisfying

$$\|x - x_n\| \le \varepsilon.$$

Then, $\forall n > n_\varepsilon$, we have

$$\gamma(A_n, A_n) \le \gamma(A, A_n) \le \gamma(A, x_n) \le \gamma(A, x) + \varepsilon \le \gamma(A, A) + 2\varepsilon.$$

It follows that

$$\limsup_{n\to\infty} \gamma(A_n, A_n) \le \gamma(A, A) + 2\varepsilon.$$

Therefore

$$\limsup_{n\to\infty} \gamma(A_n, A_n) \le \gamma(A, A).$$

(3) and (4). Clearly, since $A_n \subseteq A,\ \ \forall n \in \mathbb{N}$, we have

$$\gamma(A) \ge \gamma(A_n) \text{ and } \gamma'(A) \ge \gamma'(A_n).$$

Thus the conclusion.

Now we will show that, even for increasing sequences of convex sets, inequalities in the last three cases are possible.

Consider in c_0 the sets

$$A_n = \{(\alpha_1, \alpha_2, \cdots) \in c_0 :\ \alpha_i \in [0,1],\ \forall i \le n;\ \alpha_i = 0,\ \forall i > n\},\ \forall n \in \mathbb{N}.$$

Then $A = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} A_n\right)$ contains all sequences with components in $[0,1]$. So

$$\gamma(A) = \gamma(A, A) = 1 \text{ and } \gamma(A_n) = \gamma(A_n, A_n) = \frac{1}{2},\ \forall n \in \mathbb{N}.$$

This proves the results for (2) and (3).

Now consider, for $n \in \mathbb{N}$, the convex sets

$$A_n = \{(\alpha_1, \alpha_2, \cdots) \in c_0 :\ \alpha_i \in [-1,1],\ \forall i \le n;\ \alpha_i \in [-1/2, 1/2],\ \forall i > n\}.$$

Then $A = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} A_n\right)$ is the set of sequences with components in $[-1,1]$. Thus (see (4)):

$$\gamma'(A_n) = 1/2,\ \forall n \in \mathbb{N};\ \gamma'(A) = 1. \qquad \boxed{QED}$$

Remark 2. $\lim\limits_{n\to\infty} \gamma(A_n, A_n)$ does not always exist. When it exists, it can be different from $\gamma(A, A)$ (see Example 6). Also, Example 5 a), will show that $\{\gamma(A_n, A_n\}_{n=1}^{\infty}$ can be also decreasing.

We conclude this section with a simple statement.

Proposition 5. *If $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of convex sets that is uniformly bounded then $A = \operatorname{cl}\left(\bigcup\limits_{n=1}^{\infty} A_n\right)$ is also convex.*

Proof. We only need to show that $A' = \bigcup\limits_{n=1}^{\infty} A_n$ is convex. For arbitrary $x, y \in A'$ and arbitrary $\lambda \in (0, 1)$, there exists a sufficiently large n such that $x, y \in A_n$. Since A_n is convex,

$$\lambda x + (1 - \lambda)y \in A_n \subseteq A'.$$

Thus A' is convex. $\boxed{QED}$

5 Some facts concerning balls

We indicate a few more results concerning sequences of balls. Suppose that $\{B_n\}_{n=1}^{\infty}$ is a sequence of balls such that $B_n \xrightarrow{K} B \neq \emptyset$. In general B is not a ball (see Example 3). We noticed (in the final part of Section 2) that if $\{B_n\}_{n=1}^{\infty}$ is a nested sequence of balls, then

$$B = \bigcap_{n=1}^{\infty} B_n$$

is a (nonempty) ball. The usual proofs of this result also show that

$$B_n \xrightarrow{H} B \text{ and } \gamma(B) = \lim_{n\to\infty} \gamma(B_n).$$

As known, balls are characterized by the following equivalent properties (see [9, Section 3]): a non-empty closed set D is a ball if and only if

$$\gamma'(D) = \frac{\delta(D)}{2} \Leftrightarrow \gamma(D) = \gamma'(D) \Leftrightarrow \gamma(D, D) = \gamma'(D). \tag{5.1}$$

We have

Proposition 6. *If $\{A_n\}_{n=1}^{\infty}$ is a sequence of balls H-converging to A, then A is a ball.*

Proof. Using Proposition 2, Proposition 3, and (5.1) we obtain

$$\delta(A) = \lim_{n\to\infty} \delta(A_n) = 2 \lim_{n\to\infty} \gamma'(A_n) = 2\gamma'(A).$$

Therefore A is a ball. $\boxed{QED}$

Proposition 7. *Let $\{B_n\}_{n=1}^{\infty}$ be a uniformly bounded increasing sequence of balls. Then $B = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} B_n\right)$ is a ball and $\{B_n\}_{n=1}^{\infty}$ H-converges to B.*

Proof. By Proposition 4 and (5.1) we have

$$\delta(B) = \delta\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \delta(B_n) = \lim_{n\to\infty} 2\gamma'(B_n) \leq 2\gamma'(B),$$

which implies that $\delta(B) = 2\gamma'(B)$ (we used the trivial fact that $2\gamma'(B) \leq \delta(B)$). Thus B is a ball. Moreover,

$$\delta(B) = 2\gamma(B) \geq 2 \lim_{n\to\infty} \gamma(B_n) = \lim_{n\to\infty} \delta(B_n) = \delta(B).$$

So

$$\gamma(B) = \lim_{n\to\infty} \gamma(B_n). \tag{5.2}$$

Let ε be a positive number. Then

$$B_n \subseteq B \subseteq B^{\varepsilon},\ \forall n \in \mathbb{N}.$$

Since B is a ball, there exists a point $x \in X$ such that $B = B(x, \gamma(B))$. By (5.2), for sufficiently large $n \in \mathbb{N}$, we have

$$\gamma(B_n) > \gamma(B) - \frac{\varepsilon}{2}.$$

Denote by x_n the center of B_n. Then

$$x - x_n \in B(o, \gamma(B) - \gamma(B_n))$$

and

$$\begin{aligned} B = B(x, \gamma(B)) &= B(x_n, \gamma(B)) + x - x_n \\ &\subseteq B(x_n, 2\gamma(B) - \gamma(B_n)) \\ &\subseteq (B(x_n, \gamma(B_n)))^{\varepsilon} = (B_n)^{\varepsilon}. \end{aligned}$$

Thus B_n H-converges to B. $\boxed{QED}$

6 Examples

Example 1. Consider in c_0 the convex sets

$$A_n = \{(\alpha_1, \alpha_2, \cdots) : \ \alpha_i = 0, \ \forall i < n; \ |\alpha_i| \leq 1, \ \forall i \geq n\}, \ \forall n \in \mathbb{N}.$$

They form a nested sequence such that

$$\bigcap_{n=1}^{\infty} A_n = \{o\} =: A.$$

Then

$$\lim_{n\to\infty} \delta(A_n) \neq \delta(A), \ \lim_{n\to\infty} \gamma(A_n) \neq \gamma(A), \ \lim_{n\to\infty} \gamma(A_n, A_n) \neq \gamma(A, A).$$

In fact,

$$\delta(A_n) = 2 \ \text{and} \ \gamma(A_n) = \gamma(A_n, A_n) = 1, \ \forall n \in \mathbb{N}.$$

Example 2. Consider in c_0 the following sets. For each $n \in \mathbb{N}$, put

$$A_n = \{(\alpha_1, \alpha_2, \cdots) \in B(o, 2) : \ \alpha_i \in [0, 2], \ \forall i \leq n\}.$$

Then $\{A_n\}_{n=1}^{\infty}$ is a nested sequence of convex sets, and

$$\bigcap_{n=1}^{\infty} A_n =: A = \{(\alpha_1, \alpha_2, \cdots) \in c_0 : \ \alpha_i \in [0, 2], \ \forall i \in \mathbb{N}\}.$$

We have

$$\lim_{n\to\infty} \gamma'(A_n) > 0 = \gamma'(A).$$

Note that by Proposition 3,

$$A_n \overset{H}{\not\longrightarrow} A.$$

Example 3. Consider in c_0 the balls

$$B_n = B\left(\sum_{i=1}^{n} e_i, 1\right).$$

Clearly,

$$\frac{\delta(B_n)}{2} = \gamma(B_n) = \gamma'(B_n) = \gamma(B_n, B_n) = 1, \ \forall n \in \mathbb{N},$$

and $B_n \xrightarrow{K} A$, where

$$A = \{(\alpha_1, \alpha_2, \cdots) \in c_0 : \ \alpha_i \in [0, 2], \ \forall i \in \mathbb{N}\} \neq \emptyset,$$

which is not a ball. Moreover

$$\gamma(A) = \gamma(A, A) = 2 = \delta(A) \ \text{and} \ \gamma'(A) = 0 \neq \lim_{n\to\infty} \gamma'(B_n).$$

Example 4. (See Remark 1). Consider, in the Euclidean plane, the increasing sequence

$$A_n = B(o,1) \setminus \left\{ (\alpha, \beta) : 0 < \alpha < \frac{1}{n+1} \right\}.$$

We have

$$A_n \xrightarrow{H} A = B(o,1), \quad \gamma'(A_n) = \frac{1}{2}, \ \forall n \in \mathbb{N},$$

and

$$\gamma'(A) \neq \lim_{n\to\infty} \gamma'(A_n).$$

Example 5. a) This example shows that for a strictly decreasing sequence $\{A_n\}_{n=1}^{\infty}$, the sequence $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$ can also be increasing.

Consider in $X = C([0,1])$ the following sets:

$$A_n = \left\{ f \in X : \ f(\alpha) \in [0,1], \ \forall \alpha \in [0,1]; \ f\left(\frac{1}{2}\right) \leq \frac{1}{n+1} \right\}, \ \forall n \in \mathbb{N}.$$

Put

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Then

$$\gamma(A_n, A_n) = \frac{n}{n+1},$$

which shows that $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$ is increasing, and

$$\lim_{n\to\infty} \gamma(A_n, A_n) = 1 = \gamma(A, A).$$

b) This example shows that for a strictly increasing sequence $\{A_n\}_{n=1}^{\infty}$, the sequence $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$ can also be decreasing.

Consider in $X = C([0,1])$ the following sets:

$$A_n = \left\{ f \in X : \ f(\alpha) \in [0,1], \ \forall \alpha \in [0,1]; \ f\left(\frac{1}{2}\right) \leq \frac{n}{2n+1} \right\}, \forall n \in \mathbb{N}.$$

Put

$$A = \mathrm{cl}\,(\bigcup_{n=1}^{\infty} A_n).$$

Then

$$\gamma(A_n, A_n) = \frac{n+1}{2n+1},$$

which shows that $\{\gamma(A_n, A_n)\}_{n=1}^{\infty}$ is decreasing, and

$$\lim_{n\to\infty} \gamma(A_n, A_n) = \frac{1}{2} = \gamma(A, A).$$

Example 6. This example clarifies the behaviour of $\gamma(A_n, A_n)$ for an increasing sequence of sets (see point (2) in Proposition 4).

Let $X = C([0,1])$. Consider in $(0,1]$ an increasing sequence $\{\alpha_n\}_{n=1}^{\infty}$ converging to 1. For each $n \in \mathbb{N}$, denote by A_{2n} the set of functions $f \in X$ satisfying:

(1) $f(\alpha) \in [0,1]$, $\forall \alpha \in [0, \alpha_n)$;

(2) $f(\alpha) \in [0, \frac{1}{2}]$, $\forall \alpha \in [\alpha_n, \alpha_{n+1})$;

(3) $f(\alpha) = 0$, $\forall \alpha \in [\alpha_{n+1}, 1]$.

And, by A_{2n-1} the set of functions $f \in A_{2n}$ so that $f(\alpha_n) = 0$. Then the sequence $\{A_n\}_{n=1}^{\infty}$ is increasing,

$$\gamma(A_{2n}, A_{2n}) = \frac{1}{2},\ \gamma(A_{2n-1}, A_{2n-1}) = 1 \text{ and } \ \delta(A_n) = 1,\ \forall n \in \mathbb{N}.$$

Therefore,

$$\liminf_{n\to\infty} \gamma(A_n, A_n) = \frac{1}{2} < \limsup_{n\to\infty} \gamma(A_n, A_n) = 1.$$

A function $f \in \bigcup\limits_{n=1}^{\infty} A_n$ must be 0 in some interval $[\alpha_0, 1]$ for some number $\alpha_0 < 1$. Moreover, $f \in A = \operatorname{cl}\left(\bigcup\limits_{n=1}^{\infty} A_n\right)$ if

$$f(1) = 0 \text{ and } f(\alpha) \in [0,1]\,,\ \forall \alpha \in [0,1).$$

Therefore

$$\delta(A) = 1 = \gamma(A, A).$$

From Proposition 2, it follows that $A_n \overset{H}{\not\longrightarrow} A$.

Example 7. Next example is somehow similar to the previous one.

Let $X = C([0,1])$. Consider a strictly decreasing sequence $\{\alpha_n\}_{n=1}^{\infty}$ contained in $(\frac{1}{2}, 1]$, converging to $\frac{1}{2}$. For each $n \in \mathbb{N}$, let A_{2n-1} be the set of functions $f \in X$ such that:

(1) $f(\alpha) \in [0,1],\ \forall \alpha \in [0, \alpha_{n+1})$,

(2) $f(\alpha) \in [0, \frac{1}{2}],\ \forall \alpha \in [\alpha_{n+1}, \alpha_n)$,

(3) $f(\alpha) = 0,\ \forall \alpha \in [\alpha_n, 1]$.

Also, let A_{2n} be the set of functions $f \in A_{2n-1}$ such that $f(\alpha_{n+1}) = 0$.

Let A be the set of functions $f \in X$ such that:

(1) $f(\alpha) \in [0,1], \ \forall \alpha \in [0, \frac{1}{2})$;

(2) $f(\alpha) = 0, \ \forall \alpha \in [\frac{1}{2}, 1]$.

Then $\{A_n\}_{n=1}^{\infty}$ is a nested sequence, and therefore

$$A_n \xrightarrow{K} \bigcap_{n=1}^{\infty} A_n = A.$$

Moreover, for each $n \in \mathbb{N}$, we have

$$\gamma(A_{2n-1}, A_{2n-1}) = \frac{1}{2}; \quad \gamma(A_{2n}, A_{2n}) = 1 = \gamma(A, A).$$

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