Envelopes of slant lines in the hyperbolic plane

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Abstract. In this paper we consider envelopes of families of equidistant curves and horocycles in the hyperbolic plane. As a special case, we consider a kind of evolutes as the envelope of normal equidistant families of a curve. The hyperbolic evolute of a curve is a special case. Moreover, a new notion of horocyclic evolutes of curves is induced. We investigate the singularities of such envelopes and introduce new invariants in the Lie algebra of the Lorentz group.

Keywords: slant geometry, Hyperbolic plane, horocycles, equidistant curves


1 Introduction

We consider the Poincaré disk model $D$ of the hyperbolic plane which is conformally equivalent to the Euclidean plane, so that a circle or a line in the Poincaré disk is also a circle or a line in the Euclidean plane. A geodesic in the Poincaré disk is a Euclidean circle or a line which is perpendicular to the ideal boundary (i.e., the unit circle). If we adopt geodesics as lines in the Poincaré disk, we have the model of the hyperbolic geometry. A horocycle is an Euclidean circle which is tangent to the ideal boundary. If we adopt horocycles as lines, we call this geometry a horocyclic geometry (a horospherical geometry for the higher dimensional case) [4, 6, 7, 8, 9]. We also have another kind of curves with the properties similar to those of Euclidean lines. A curve in the Poincaré disk is called an equidistant curve if it is a Euclidean circle or a Euclidean line whose intersection with the ideal boundary consists of two points. We define an equidistant curve depends on $\phi \in [0, \pi/2]$ whose angles with the ideal boundary
at the intersection points are \( \phi \) (cf., [10]). A geodesic is the special case with \( \phi = \pi/2 \) and a horocycle is the case with \( \phi = 0 \). Therefore, a geodesic is called a \textit{vertical pseudo-line} and a horocycle a \textit{horizontal pseudo-line}. For \( \phi \in (0, \pi/2] \), the corresponding pseudo-line is an equidistant curve, which we call a \( \phi \)\textit{-slant pseudo-line}. If we consider a \( \phi \)-slant pseudo-line as a line, we call this geometry a \textit{slant geometry} (cf., [1]).

In this paper we consider envelopes of families of \( \phi \)-slant pseudo-lines in the general setting. We investigate the singularities of such envelopes. Throughout the remainder of the paper, we adopt the Lorentz-Minkowski space model of the hyperbolic plane. For a \( 3 \times 3 \)-matrix \( A \), we say that \( A \) is a member of the Lorentz group \( SO_0(1,2) \) if \( \det A > 0 \) and the induced linear mapping preserves the Lorentz-Minkowski scalar product. The Lorentz group \( SO_0(1,2) \) canonically acts on the hyperbolic plane. It is well known that this action is transitive, so that the hyperbolic space is canonically identified with the homogeneous space \( SO_0(1,2)/SO(2) \). It follows that any point of the hyperbolic space can be identified with a matrix \( A \in SO_0(1,2) \) (cf., §3). Therefore, a one parameter family of \( \phi \)-slant pseudo-lines can be parametrized by using a curve in \( SO_0(1,2) \) (cf., §3 and 4). Then we apply the theory of unfoldings of function germs (cf., [2]) and obtain a classification of singularities of the envelopes of the families of \( \phi \)-slant pseudo-lines (cf., Theorem 5.6). The singularities of the envelopes are characterized by using invariants represented by the elements of Lie algebra \( \mathfrak{so}(1,2) \) of \( SO_0(1,2) \). In §6 we introduce the notion of \( \phi \)-slant evolutes of unit speed curves in the hyperbolic plane. If \( \phi = \pi/2 \), then the \( \phi \)-slant evolute is a hyperbolic evolute defined in [5]. Moreover, if \( \phi = 0 \), then the \( \phi \)-slant evolute is called a \textit{horocyclic evolute}. It means that the \( \phi \)-slant evolutes depending on \( \phi \) connects the hyperbolic evolute and the horocyclic evolute of the curve in the hyperbolic plane.

In [3] families of equal-angle envelopes in the Euclidean plane is investigated.

## 2 Basic concepts

We now present basic notions on Lorentz-Minkowski 3-space. Let \( \mathbb{R}^3 = \{(x_0, x_1, x_2)|x_i \in \mathbb{R}, i = 0, 1, 2\} \) be a 3-dimensional vector space. For any vectors \( \mathbf{x} = (x_0, x_1, x_2), \mathbf{y} = (y_0, y_1, y_2) \in \mathbb{R}^3 \), the \textit{pseudo scalar product} (or, the \textit{Lorentz-Minkowski scalar product}) of \( \mathbf{x} \) and \( \mathbf{y} \) is defined by \( \langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2 \). The space \( (\mathbb{R}^3, \langle \cdot, \cdot \rangle) \) is called \textit{Lorentz-Minkowski 3-space} which is denoted by \( \mathbb{R}_3^1 \). We assume that \( \mathbb{R}_3^1 \) is time-oriented and choose \( \mathbf{e}_0 = (1, 0, 0) \) as the \textit{future timelike vector}.

We say that a non-zero vector \( \mathbf{x} \) in \( \mathbb{R}_3^1 \) is \textit{spacelike}, \textit{lightlike} or \textit{timelike} if \( \langle \mathbf{x}, \mathbf{x} \rangle > 0, = 0 \) or \( < 0 \) respectively. The norm of the vector \( \mathbf{x} \in \mathbb{R}_3^1 \) is defined
by \( \| \mathbf{x} \| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \). Given a non-zero vector \( \mathbf{n} \in \mathbb{R}^3 \) and a real number \( c \), the plane with pseudo normal \( \mathbf{n} \) is given by

\[
P(\mathbf{n}, c) = \{ \mathbf{x} \in \mathbb{R}_1^3 | \langle \mathbf{x}, \mathbf{n} \rangle = c \}.
\]

We say that \( P(\mathbf{n}, c) \) is \textit{spacelike}, \textit{timelike} or \textit{lightlike} if \( \mathbf{n} \) is timelike, spacelike or lightlike respectively.

For any vectors \( \mathbf{x} = (x_0, x_1, x_2), \mathbf{y} = (y_0, y_1, y_2) \in \mathbb{R}_1^3 \), \textit{pseudo exterior product} of \( \mathbf{x} \) and \( \mathbf{y} \) is defined to be

\[
\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix}
-e_0 & e_1 & e_2 \\
x_0 & x_1 & x_2 \\
y_0 & y_1 & y_2 \\
\end{vmatrix} = (-x_1y_2 - x_2y_1, x_2y_0 - x_0y_2, x_0y_1 - x_1y_0),
\]

where \( \{e_0, e_1, e_2\} \) is the canonical basis of \( \mathbb{R}_1^3 \). We also define \textit{Hyperbolic plane} by

\[
H_+^2(-1) = \{ \mathbf{x} \in \mathbb{R}_1^3 | \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 \geq 1 \},
\]

\textit{de Sitter 2-space} by

\[
S_1^2 = \{ \mathbf{x} \in \mathbb{R}_1^3 | \langle \mathbf{x}, \mathbf{x} \rangle = 1 \}
\]

and the (open) \textit{lightcone} at the origin by

\[
LC^* = \{ \mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}_1^3 | x_0 \neq 0, \langle \mathbf{x}, \mathbf{x} \rangle = 0 \}.
\]

We remark that \( H_+^2(-1) \) is a Riemannian manifold if we consider the induced metric from \( \mathbb{R}_1^3 \).

We now consider the plane defined by \( \mathbb{R}_0^2 = \{ (x_0, x_1, x_2) \in \mathbb{R}_1^3 | x_0 = 0 \} \). Since \( \langle \cdot, \cdot \rangle |_{\mathbb{R}_0^2} \) is the canonical Euclidean scalar product, we call it \textit{Euclidean plane}.

We adopt coordinates \( (x_1, x_2) \) of \( \mathbb{R}_0^2 \) instead of \( (0, x_1, x_2) \). On Euclidean plane \( \mathbb{R}_0^2 \), we have the \textit{Poincaré disc model} of the hyperbolic plane. We consider a unit open disc \( D = \{ \mathbf{x} \in \mathbb{R}_0^2 | \| \mathbf{x} \| < 1 \} \) and consider a Riemannian metric

\[
ds^2 = \frac{4(dx_1^2 + dx_2^2)}{1 - x_1^2 - x_2^2}.
\]

Define a mapping \( \Psi : H_+^2 \rightarrow D \) by

\[
\Psi(x_0, x_1, x_2) = \left( \frac{x_1}{x_0 + 1}, \frac{x_2}{x_0 + 1} \right).
\]

It is known that \( \Psi \) is an isometry. Moreover, the Poinaré disc model is conformally equivalent to the Euclidean plane.
3 Pseudo-lines in the hyperbolic plane

We consider a curve defined by the intersection of the hyperbolic plane with a plane in Lorentz-Minkowski 3-space, which is called a pseudo-circle if it is non-empty. The image of a pseudo-circle by the isometry $\Psi$ is a part of a Euclidean circle in the Poincaré disc $D$. Let $P(n, c)$ be a plane with a unit pseudo-normal $n$. We call $H^2_+(-1) \cap P(n, c)$ a circle, an equidistant curve and a horocycle if $n$ is timelike, spacelike or lightlike respectively. Moreover, if $n$ is spacelike and $c = 0$, then we call it a hyperbolic line (or, a geodesic). We remark that circles are compact and other pseudo-circles are non-compact. Therefore, equidistant curves or horocycles are called pseudo-lines.

We now consider a hyperbolic line

$$HL(n) = \{x \in H^2_+(-1) \mid \langle x, n \rangle = 0\}$$

and a horocycle

$$HC(\ell, -1) = \{x \in H^2_+(-1) \mid \langle x, \ell \rangle = -1\},$$

where $\ell$ is a lightlike vector. In general, a horocycle is defined by $\langle x, \ell \rangle = c$ for a lightlike vector $\ell$ and $c \neq 0$. However, if we choose $-\ell/c$ instead of $\ell$, then we have the above equation. We now consider parametrizations of a horocycle and a hyperbolic line respectively. For any $a_0 \in HC(\ell, -1)$, let $a_1$ be a unit tangent vector of $HC(\ell, -1)$ at $a_0$, so that $\langle a_1, \ell \rangle = 0$. We define $a_2 = a_0 \wedge a_1$. Then we have a pseudo orthonormal basis $\{a_0, a_1, a_2\}$ of $\mathbb{R}^3_1$ such that $\langle a_0, a_0 \rangle = -1$. We remark that $a_0$ is timelike and $a_1, a_2$ are spacelike. Since $\langle \ell - a_0, a_0 \rangle = \langle \ell, a_1 \rangle = 0$, we have $\pm a_2 = \ell - a_0$. We choose the direction of $a_1$ such that $a_2 = \ell - a_0$. It follows that $A = (a_0, a_1, a_2) \in SO_0(1, 2)$, where

$$SO_0(1, 2) = \left\{ A = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 \\ a_1^0 & a_1^1 & a_1^2 \\ a_2^0 & a_2^1 & a_2^2 \end{pmatrix} \mid 'A I_{1,2} A = I_{1,2}, \ a_0^0 \geq 1 \right\}$$

is the Lorentz group, where

$$I_{1,2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

For any $A = (a_0, a_1, a_2) \in SO_0(1, 2)$, $\{a_0, a_1, a_2\}$ is a pseudo orthonormal basis of $\mathbb{R}^3_1$. Then $\ell = a_0 + a_2$ is lightlike. It follows that we have $HC(\ell, -1) = HC(a_0 + a_2, -1)$ such that $a_0 \in HC(a_0 + a_2, -1)$ and $a_1$ is tangent to $HC(a_0 + a_2, -1)$ at $a_0$. Moreover, we have $a_0 \in HL(a_2)$ and $a_1$ is tangent to $HL(a_2)$ at $a_0$. Then we have the following lemma.
Lemma 3.1. With the above notation, we have

1. \( HC(\ell, -1) = \{x = a_0 + ra_1 + \frac{1}{2}r^2(a_0 + a_2) \mid r \in \mathbb{R}\} \).
2. \( HL(a_2) = \{\sqrt{r^2 + 1}a_0 + ra_1 \mid r \in \mathbb{R}\} \).

Proof. (1) For any \( x \in HC(\ell, -1) \), there exist \( \alpha, \beta, \gamma \in \mathbb{R} \) such that

\[ x = \alpha a_0 + \beta a_1 + \gamma a_2 \quad (\alpha \geq 1). \]

We put \( \beta = r \). Since \( \langle x, \ell \rangle = -\alpha + \gamma = -1 \), we have \( \alpha = \gamma + 1 \). Moreover, we also have \( \langle x, x \rangle = -\alpha^2 + \beta^2 + \gamma^2 = -(\gamma + 1)^2 + r^2 + \gamma^2 = -1 \), so that \( \gamma = \frac{1}{2}r^2 \).

Thus,

\[ x = a_0 + ra_1 + \frac{1}{2}r^2(a_0 + a_2) \]

holds. For the converse, we can easily show that \( \langle x, x \rangle = -1 \) and \( \langle x, \ell \rangle = -1 \) for the above vector.

(2) For any \( x \in HL(a_2) \), there exist \( \alpha, \beta, \gamma \in \mathbb{R} \) such that

\[ x = \alpha a_0 + \beta a_1 + \gamma a_2 \quad (\alpha \geq 1). \]

Since \( \langle x, a_2 \rangle = 0 \), \( \gamma = 0 \). If we put \( \beta = r \), then we have \( \langle x, x \rangle = -\alpha^2 + r^2 = -1 \), so that \( \alpha = \pm \sqrt{r^2 + 1} \). Since \( \alpha \geq 1 \), we have \( \alpha = \sqrt{r^2 + 1} \). By a straightforward calculation, the converse holds. \( \square \)

It is known that a horocycle \( \Psi(HC(a_0 + a_2, -1)) \) in the Poincaré disc \( D \) is a Euclidean circle tangent to the ideal boundary \( S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\} \). It is also known that a hyperbolic line \( \Psi(HL(a_2)) \) is a Euclidean circle or a Euclidean line orthogonal to the ideal boundary (cf., [11]). By these reasons, a horocycle is called a horizontal pseudo-line and a hyperbolic-line is called an orthogonal pseudo-line respectively. We now define a \( \phi \)-slant pseudo-line by

\[ SL(n_\phi, -\cos \phi) = \{x \in H^2_+(-1) \mid \langle x, n_\phi \rangle = -\cos \phi\}, \]

where \( n_\phi(t) = \cos \phi a_0 + a_2, \phi \in [0, \pi/2] \). Since \( \langle n_\phi, n_\phi \rangle = \sin^2 \phi > 0 \), \( n_\phi \) is spacelike. Thus, \( SL(n_\phi, -\cos \phi) = H^2_+(-1) \cap P(n_\phi, -\cos \phi) \) is an equidistant curve. Moreover, \( a_0 \in SL(n_\phi, -\cos \phi) \) and \( a_1 \) is tangent to \( SL(n_\phi, -\cos \phi) \) at \( a_0 \). Then \( SL(n_\phi, -\cos(\pi/2)) = HL(a_2) \) and \( SL(n_0, -\cos 0) = HC(a_0 + a_2, -1) \). We have the following parametrization of a \( \phi \)-slant pseudo-line.

Lemma 3.2. With the same notations as those in Lemma 3.1, we have

\[ SL(n_\phi, -\cos \phi) = \left\{ a_0 + ra_1 + \frac{\sqrt{r^2 \sin^2 \phi + 1} - 1}{\sin^2 \phi}(a_0 + \cos \phi a_2) \mid r \in \mathbb{R}\right\}. \]
Proof. We consider a point \( x \in SL(n_\phi(t), -\cos \phi) \). Since \( \{a_0, a_1, a_2\} \) is a pseudo-orthonormal basis of \( \mathbb{R}^3 \), there exist \( \alpha, \beta, \gamma \in \mathbb{R} \) such that \( x = \alpha a_0 + \beta a_1 + \gamma a_2, \ (\alpha \geq 1) \). Therefore, we have

\[
\langle x, n_\phi \rangle = \langle \alpha a_0 + \beta a_1 + \gamma a_2, \cos \phi a_0 + a_2 \rangle
= -\cos \phi \alpha + \gamma = -\cos \phi
\]

Thus, we have \( \gamma = \cos \phi (\alpha - 1) \). Moreover, \( \langle x, x \rangle = -\alpha^2 + \beta^2 + \gamma^2 = -\alpha^2 + \beta^2 + \cos^2 \phi (\alpha - 1)^2 = -1 \). It follows that

\[
\alpha = \frac{1}{\sin^2 \phi} (\pm \sqrt{\beta^2 \sin^2 \phi + 1 - \cos^2 \phi}).
\]

If we choose \( \alpha = -\frac{1}{\sin^2 \phi} (\sqrt{\beta^2 \sin^2 \phi + 1 + \cos^2 \phi}) \), then \( \alpha < 0 \). It contradicts to \( \alpha \geq 1 \). Hence, we have

\[
\alpha = \frac{1}{\sin^2 \phi} (\sqrt{\beta^2 \sin^2 \phi + 1 - \cos^2 \phi}), \quad \gamma = \frac{\cos \phi}{\sin^2 \phi} \left( \sqrt{\beta^2 \sin^2 \phi + 1 - 1} \right).
\]

We put \( \beta = r \). Then

\[
x = \frac{1}{\sin^2 \phi} (\sqrt{r^2 \sin^2 \phi + 1 - \cos^2 \phi}) a_0 + r a_1 + \frac{\cos \phi}{\sin^2 \phi} (\sqrt{r^2 \sin^2 \phi + 1 - 1}) a_2
= a_0 + r a_1 + \frac{1}{\sin^2 \phi} (\sqrt{r^2 \sin^2 \phi + 1 - 1}) (a_0 + \cos \phi a_2)
\]

For the converse, we have \( \langle x, x \rangle = -1, \langle x, n_\phi \rangle = -\cos \phi \) and \( (\sqrt{r^2 \sin^2 \phi + 1 - \cos^2 \phi}) / \sin^2 \phi \geq 1 \). Then \( x \in SL(n_\phi(t), -\cos \phi) \). \( \square \)

Remark 3.3. We can show \( \lim_{\phi \to 0} (\sqrt{r^2 \sin^2 \phi + 1 - 1}) / \sin^2 \phi = r^2 / 2 \). In [10] the third author showed that the angle between \( \Psi(SL(n_\phi)) \) and the ideal boundary \( S^1 \) of the Poincaré disc \( D \) at an intersection point is equal to \( \phi \). This is the reason why we call \( SL(n_\phi) \) the \( \phi \)-slant pseudo line.

4 One-parameter families of pseudo-lines

In this section we consider one-parameter families of pseudo-lines. By Lemmas 3.1 and 3.2, we consider a one-parameter family of pseudo-orthonormal bases of \( \mathbb{R}^3 \). Let \( A : J \to SO_0(1, 2) \) be a \( C^\infty \)-mapping. If we write \( A(t) = (t a_0(t) \ t a_1(t) \ t a_2(t)) \), then \( \{a_0(t), a_1(t), a_2(t)\} \) is a one-parameter family of pseudo-orthonormal bases of \( \mathbb{R}^3 \). We call it a pseudo-orthonormal moving frame.
of \( \mathbb{R}^3 \). By the standard arguments, we can show the following Frenet-Serret type formulae for the pseudo-orthonormal moving frame \( \{ a_0(t), a_1(t), a_2(t) \} \):

\[
\begin{pmatrix}
a_0'(t) \\
a_1'(t) \\
a_2'(t)
\end{pmatrix} = \begin{pmatrix}
0 & c_1(t) & c_2(t) \\
c_1(t) & 0 & c_3(t) \\
c_2(t) & -c_3(t) & 0
\end{pmatrix} \begin{pmatrix}
a_0(t) \\
a_1(t) \\
a_2(t)
\end{pmatrix}.
\]

Here,

\[
\begin{align*}
c_1(t) &= \langle a_0'(t), a_1(t) \rangle \\
c_2(t) &= \langle a_0'(t), a_2(t) \rangle \\
c_3(t) &= \langle a_1'(t), a_2(t) \rangle
\end{align*}
\]

Then, the matrix \( C(t) = \begin{pmatrix}
0 & c_1(t) & c_2(t) \\
c_1(t) & 0 & c_3(t) \\
c_2(t) & -c_3(t) & 0
\end{pmatrix} \) is an element of Lie algebra \( \mathfrak{so}(1,2) \) of the Lorentz group \( SO_0(1,2) \). The above Frenet-Serret type formulae are written by \( A'(t)A^{-1}(t) = C(t) \). For any \( C^\infty \)-mapping \( C : I \to \mathfrak{so}(1,2) \) and \( A_0 \in SO_0(1,2) \), we can apply the unique existence theorem for systems of linear ordinary differential equations, so that there exists a unique \( A(t) \in SO_0(1,2) \) such that \( A(0) = A_0 \) and \( A'(t)A^{-1}(t) = C(t) \).

We now consider a mapping \( g_\phi : I \times J \to H^2_+ \), where \( g_\phi(r,t) \) is defined by

\[
\begin{align*}
a_0(t) + ra_1(t) + \frac{\sqrt{r^2 \sin^2 \phi + 1} - 1}{\sin^2 \phi} (a_0(t) + \cos \phi a_2(t)) & \quad \text{if } \phi \neq 0, \\
a_0(t) + ra_1(t) + \frac{r^2}{2} (a_0(t) + a_2(t)) & \quad \text{if } \phi = 0,
\end{align*}
\]

where \( I, J \subset \mathbb{R} \) are intervals. Then we have \( SL(n_\phi(t), -\cos \phi) = \{ g_\phi(I \times \{ t \}) \mid t \in J \} \), for \( n_\phi(t) = \cos \phi a_0(t) + a_2(t) \). Thus \( g_\phi \) is a one-parameter family of \( \phi \)-slant pseudo-lines. Moreover, \( g_\phi \) is a one-parameter family of horocycles and \( g_{\phi/2} \) is a one-parameter family of hyperbolic lines.

## 5 Height functions

For a one parameter family of \( \phi \)-slant pseudo-lines \( g_\phi \), we define a family of height functions \( H : J \times H^2_+ \to \mathbb{R} \) by \( H(t, x) = \langle x, n_\phi(t) \rangle + \cos \phi \). Then we have the following proposition.

**Proposition 5.1.** For \( g_\phi : I \times J \to H^2_+ \), we have the following:

1. \( H(t, x) = 0 \) if and only if there exists \( r \in I \) such that \( x = g_\phi(r,t) \),

2. \( H(t, x) = \frac{\partial H}{\partial t}(t, x) = 0 \) if and only if there exists \( r \in I \) such that \( x = g_\phi(r,t) \)

and

\[-\sqrt{r^2 \sin^2 \phi + 1} c_2(t) + r (\cos \phi c_1(t) - c_3(t)) = 0.\]
Proof. (1) If $H(t, x) = 0$, then $\langle x, n_\phi(t) \rangle = -\cos \phi$, $x \in H_+^2(-1)$. Thus, there exists $r \in I$ such that $x = g_\phi(r, t)$. The converse also holds.
(2) Since $H(t, x) = 0$, there exists $r \in I$ such that $x = g_\phi(r, t)$. Suppose that $\phi \neq 0$. Then we have the following classification (cf., [2]).

$$\frac{\partial H}{\partial t}(t, x) = \langle x, n'_\phi(t) \rangle$$

$$= \left( \frac{\sqrt{r^2 \sin^2 \phi + 1 - \cos^2 \phi}}{\sin^2 \phi} \alpha_0(t) + r \alpha_1(t) + \frac{\cos \phi (\sqrt{r^2 \sin^2 \phi + 1 - 1})}{\sin^2 \phi} \alpha_2(t), n'_\phi(t) \right)$$

$$= -\sqrt{r^2 \sin^2 \phi + 1}c_2(t) + r (\cos \phi c_1(t) - c_3(t)).$$

If $\phi = 0$, then

$$\frac{\partial H}{\partial t}(t, x) = \langle x, n'_\phi(t) \rangle = -c_2(t) + r(c_1(t) - c_3(t)).$$

This completes the proof. \qed

We now review some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [2]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call $F$ an $r$-parameter unfolding of $f$, where $f(s) = F_{x_0}(s, x_0)$. We say that $f$ has an $A_k$-singularity at $s_0$ if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that $f$ has an $A_k$-singularity at $s_0$ if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let $F$ be an unfolding of $f$ and $f(s)$ has an $A_k$-singularity ($k \geq 1$) at $s_0$. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial s}$ at $s_0$ by $f^{(k-1)}(\frac{\partial F}{\partial s}(s, x_0))(s_0) = \sum_{i=0}^{k-1} \alpha_{ji}(s - s_0)^j$ for $i = 1, \ldots, r$. Then $F$ is called an $\mathcal{R}$-versal unfolding if the $k \times r$ matrix of coefficients $(\alpha_{ji})_{i=0 \ldots k-1, j=1 \ldots r}$ has rank $k$ ($k \leq r$). We introduce an important set concerning the unfoldings relative to the above notions. The discriminant set of $F$ is the set

$$\mathcal{D}_F = \left\{ x \in \mathbb{R}^r \mid \text{there exists } s \text{ such that } F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0 \right\}.$$  

Then we have the following classification (cf., [2]).

Theorem 5.2. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an $r$-parameter unfolding of $f(s)$ which has an $A_k$ singularity at $s_0$ ($k = 1, 2$). Suppose that $F$ is an $\mathcal{R}$-versal unfolding.

(1) If $k = 1$, then $\mathcal{D}_F$ is locally diffeomorphic to $\mathbb{R}^{r-1}$.
(2) If $k = 2$, then $D_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$. Here, $C = \{(x_1, x_2) \in (\mathbb{R}^2, 0) \mid x_1 = t^2, x_2 = t^3, t \in (\mathbb{R}, 0) \}$ is the ordinary cusp.

By Proposition 5.1, the discriminant set $D_H$ of $H$ is

$$D_H = \left\{ g_\phi(r, t) \mid -\sqrt{r^2 \sin^2 \phi + 1c_2(t) + r (\cos \phi c_1(t) - c_3(t))} = 0 \right\}.$$

Suppose $c_2(t) \neq 0, \cos \phi c_1(t) - c_3(t) \neq 0$. If

$$-\sqrt{r^2 \sin^2 \phi + 1c_2(t) + r (\cos \phi c_1(t) - c_3(t))} = 0,$$

then we have

$$r = \pm \frac{c_2(t)}{\sqrt{(\cos \phi c_1(t) - c_3(t))^2 - (\sin \phi c_2(t))^2}}.$$

If $r = -c_2(t)/\sqrt{(\cos \phi c_1(t) - c_3(t))^2 - (\sin \phi c_2(t))^2}$, then

$$-\sqrt{r^2 \sin^2 \phi + 1c_2(t) + r (\cos \phi c_1(t) - c_3(t))} \neq 0,$$

so that

$$r = \frac{c_2(t)}{\sqrt{(\cos \phi c_1(t) - c_3(t))^2 - (\sin \phi c_2(t))^2}}.$$

For $\phi = 0$, we can also choose $r = c_2(t)/(c_1(t) - c_3(t))$. Therefore, if $\phi \neq 0$, then

$$D_H = \left\{ g_\phi(r, t) \mid r = \frac{c_2(t)}{\sqrt{(\cos \phi c_1(t) - c_3(t))^2 - (\sin \phi c_2(t))^2}}, c_2(t) \neq 0, \right\},$$

$$\cos \phi c_1(t) - c_3(t) \neq 0.$$

Under the assumptions that $c_2(t) \neq 0$ and $\cos \phi c_1(t) - c_3(t) \neq 0$, we have a $g[\phi] : J \rightarrow H^2_2$, where $g[\phi](t)$ is defined by

$$\begin{align*}
\begin{cases}
\frac{a_0(t) + r(t)a_1(t) + \sqrt{r(t)^2 \sin^2 \phi + 1 - 1}}{\sin^2 \phi} (a_0(t) + \cos \phi a_2(t)) & \text{if } \phi \neq 0, \\
\frac{a_0(t) + r(t)a_1(t) + \frac{r(t)^2}{2}(a_0(t) + a_2(t))}{2} & \text{if } \phi = 0.
\end{cases}
\end{align*}$$

Here

$$r(t) = \begin{cases}
\frac{c_2(t)}{\sqrt{(\cos \phi c_1(t) - c_3(t))^2 - (\sin \phi c_2(t))^2}} & \text{if } \phi \neq 0, \\
\frac{c_2(t)}{c_1(t) - c_3(t)} & \text{if } \phi = 0.
\end{cases}$$
Then \( g[\phi](t) \) is a parametrization of \( D_H \) and it is the envelope of the family of \( \phi \)-slant pseudo lines \( \{ SL(n_\phi(t), -\cos \phi) \}_{t \in \mathcal{J}} \).

In order to classify the singularities of \( g[\phi] \), we apply the theory of unfoldings to \( H \). For any \((r_0, t_0) \in I \times J\), we put \( x_0 = g_\phi(r_0, t_0) \) and consider the function germ \( h_{x_0} : (J, t_0) \rightarrow (\mathbb{R}, 0) \) defined by

\[
h_{x_0}(t_0) = H(t_0, x_0) = \langle x_0, n_\phi(t_0) \rangle + \cos \phi.
\]

Then the germ of \( H \) at \((t_0, x_0)\) is a two-dimensional unfolding of \( h_{x_0} \). We now try to search the conditions for \( h_{x_0} \) has the \( A_k \)-singularity, \((k = 1, 2)\). If \( \phi \neq 0 \), then we define two invariants

\[
\begin{aligned}
\delta[\phi]_1(t) &= -\sqrt{r^2 \sin^2 \phi + 1} c_2 + r (\cos \phi c_1' - c_2') + r c_2 (c_1 - \cos \phi c_3) \\
&\quad - c_1 (\cos \phi c_1 - c_3) - \cos \phi c_2^2 \\
&\quad + \frac{\sqrt{r^2 \sin^2 \phi + 1}}{\sin^2 \phi} (\cos \phi c_1 - c_3)(\cos \phi c_3 - c_1), \\
\delta[\phi]_2(t) &= -\sqrt{r^2 \sin^2 \phi + 1} c_2' \\
&\quad + r (\cos \phi c_1' - c_3') + r (c_2 (c_1' - \cos \phi c_3') + 2 c_2' (c_1 - \cos \phi c_3)) \\
&\quad - 2 c_1 (\cos \phi c_1' - c_3') - c_1 (\cos \phi c_1 - c_3) - 3 \cos \phi c_2 c_2' \\
&\quad + \frac{\sqrt{r^2 \sin^2 \phi + 1}}{\sin^2 \phi} (\cos \phi c_1 - c_3)(\cos \phi c_3 - c_1' \\
&\quad + 2 (\cos \phi c_1' - c_3')(\cos \phi c_3 - c_1)),
\end{aligned}
\]

where \( c_i = c_i(t), \ (i = 1, 2, 3) \) and \( r = r(t) \). For \( \phi = 0 \), we also define

\[
\begin{aligned}
\delta[0]_1(t) &= -c_2' + r (c_1' - c_3') - c_1 (c_1 - c_3) - \frac{1}{2} c_2^2, \\
\delta[0]_2(t) &= -c_2' + r (c_1'' - c_3'') + r c_2 (c_1' - c_3') - 2 c_1 (c_1' - c_3') \\
&\quad - c_1 (c_1 - c_3) - c_2 c_2' + \frac{3}{2} r c_2 (c_3' - c_1').
\end{aligned}
\]

We remark that \( \lim_{\phi \to 0} \delta[\phi]_1(t) = \delta[0]_1(t) \) and \( \lim_{\phi \to 0} \delta[\phi]_2(t) = \delta[0]_2(t) \). We expect that \( \delta[\phi]_1(t) = \delta[\phi]_2(t) = 0 \) if and only if \( \delta[\phi]_1(t) = \delta[\phi]_1(t) = 0 \). However, we can only show this relation for a special case (cf., §6).

**Proposition 5.3.** We have the following assertions:

1. \( h'_{x_0}(t_0) = 0 \) always holds,
2. \( h''_{x_0}(t_0) = 0 \) if and only if \( \delta[\phi]_1(t_0) = 0 \),
3. \( h'''_{x_0}(t_0) = h''_{x_0}(t_0) = 0 \) if and only if \( \delta[\phi]_1(t_0) = \delta[\phi]_2(t_0) = 0 \).

**Proof.** Assertion (1) holds by Proposition 5.1, (2).
(2) Since
\[ n''_{\phi} = (n'_\phi)' = c'_2a_0 + (\cos \phi c'_1 - c'_3)a_1 + \cos \phi c'_2a_2 + (c_1(\cos \phi c_1 - c_3) + \cos \phi c'_2)a_0 + c_2(c_1 - \cos \phi c_3)a_1 + (c_3(\cos \phi c_1 - c_3) + c'_2)a_2, \]
we have
\[
\frac{\partial^2 H}{\partial t^2}(t, x) = \langle x, n''_{\phi} \rangle = -\sqrt{s^2 \sin^2 \phi + 1}c'_2 + s(\cos \phi c'_1 - c'_3)
+ \frac{(\cos^2 \phi - \sqrt{s^2 \sin^2 \phi + 1}) (c_1(\cos \phi c_1 - c_3) + \cos \phi c'_2)}{\sin^2 \phi}
+ \frac{\cos \phi(\sqrt{s^2 \sin^2 \phi + 1} - 1) (c_3(\cos \phi c_1 - c_3) + c'_2)}{\sin^2 \phi}
\]
\[
= -\sqrt{s^2 \sin^2 \phi + 1}c'_2 + s(\cos \phi c'_1 - c'_3) - c_1(\cos \phi c_1 - c_3) - \cos \phi c'_2
+ \frac{(1 - \sqrt{s^2 \sin^2 \phi + 1}) (c_1(\cos \phi c_1 - c_3) + \cos \phi c'_2)}{\sin^2 \phi} + sc_2(c_1 - \cos \phi c_3)
\]
\[
+ \frac{\cos \phi(\sqrt{s^2 \sin^2 \phi + 1} - 1) (c_3(\cos \phi c_1 - c_3) + c'_2)}{\sin^2 \phi}
\]
\[
= -\sqrt{s^2 \sin^2 \phi + 1}c'_2 + s(\cos \phi c'_1 - c'_3) + sc_2(c_1 - \cos \phi c_3)
- c_1(\cos \phi c_1 - c_3) - \cos \phi c'_2 + \frac{(\sqrt{s^2 \sin^2 \phi + 1} - 1)(\cos \phi c_1 - c_3)(\cos \phi c_3 - c_1)}{\sin^2 \phi}
= \delta[\phi](t).
\]

(3) We have
\[ n'''_{\phi} = c'_2a'_0 + (\cos \phi c'_1 - c'_3)a'_1 + \cos \phi c'_2a'_2 + 2(c'_2a'_0 + (\cos \phi c'_1 - c'_3)a'_1 + \cos \phi c'_2a'_2) + c'_2a''_0 + (\cos \phi c_1 - c_3)a''_1 + \cos \phi c_2a''_2. \]
Here, \( a'_0 = c_1a_1 + c_2a_2, \ a'_1 = c_1a_0 + c_3a_2, \ a'_2 = c_2a_0 - c_3a_1. \) Then
\[
\begin{align*}
   a''_0 &= (c'_2 + c'_3)a_0 + (c'_1 - c_2c_3)a_1 + (c'_2 + c_1c_3)a_2, \\
   a''_1 &= (c'_1 + c_2c_3)a_0 + (c'_2 - c_3^2)a_1 + (c'_3 + c_1c_2)a_2, \\
   a''_2 &= (c'_2 - c_1c_3)a_0 + (c_1c_2 - c'_3)a_1 + (c'_2 - c_3^2)a_2.
\end{align*}
\]
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Therefore,
\[
\begin{align*}
n''''_\phi &= c''_2 a_0 + (\cos \phi c''_2 - c''_3) a_1 + \cos \phi c''_2 a_2 \\
&+ (c_2(c''_1^2 + c''_2^2 - c''_3^2) + 2c_1(\cos \phi c'_1 - c'_3) + c'_1(\cos \phi c_1 - c_3) + 3 \cos \phi c''_2) a_0 \\
&+ ((\cos \phi c_1 - c_3)(c''_1^2 + c''_2^2) + c_2(c'_1 - \cos \phi c'_3) + 2c_2'(c_1 - \cos \phi c_3)) a_1 \\
&+ (\cos \phi c_2(c''_1^2 + c''_2^2 - c''_3^2) + 2c_3(\cos \phi c'_1 - c'_3) + c'_3(\cos \phi c_1 - c_3) + 3c_2' a_2.
\end{align*}
\]

By the calculation similar to case (2), we have
\[
\frac{\partial^3 H}{\partial t^3}(t, x) = \delta[\phi][2](t).
\]

For the case \( \phi = 0 \), we also have the similar arguments to the above case.
This completes the proof. \( \Box \)

We have the following corollary.

**Corollary 5.4.** For \( h_{x_0} \) as the above proposition, we have the following:

1. \( h_{x_0} \) has the \( A_1 \)-singularity at \( t = t_0 \) if and only if \( \delta[\phi][1](t_0) \neq 0 \).
2. \( h_{x_0} \) has the \( A_2 \)-singularity at \( t = t_0 \) if and only if \( \delta[\phi][1](t_0) = 0, \delta[\phi][2](t_0) \neq 0 \).

Then we have the following proposition.

**Proposition 5.5.** For \( h_{x_0} \) as the above proposition, we have the following:

1. If \( h_{x_0} \) has the \( A_1 \)-singularity, then \( H \) is a \( R \)-versal unfolding of \( h_{x_0} \).
2. If \( h_{x_0} \) has the \( A_2 \)-singularity, then \( H \) is a \( R \)-versal unfolding of \( h_{x_0} \).

**Proof.** We consider a parametrization of \( H^2_+ \) defined by
\[
\psi(x_2, x_3) = \left( \sqrt{x_2^2 + x_3^2 + 1}, x_2, x_3 \right).
\]

Then we have
\[
H(t, x_2, x_3) = H(t, \psi(x_2, x_3)) = \langle \psi(x_2, x_3), n_\phi(t) \rangle + \cos \phi.
\]

We write \( n_\phi(t) = (n_{\phi 1}(t), n_{\phi 2}(t), n_{\phi 3}(t)) \) and have
\[
\frac{\partial H}{\partial x_i}(t, x_2, x_3) = n_{\phi i}(t) - \frac{x_i}{\sqrt{x_2^2 + x_3^2 + 1}} n_{\phi 1}(t) \quad (i = 2, 3).
\]

Moreover, we have
\[
\frac{\partial}{\partial t} \frac{\partial H}{\partial x_i}(t, x_2, x_3) = n'_{\phi i}(t) - \frac{x_i}{\sqrt{x_2^2 + x_3^2 + 1}} n''_{\phi 1}(t).
\]
We write that $x_0 = (x_{01}, x_{02}, x_{03})$. Then the 1-jet of $(\partial H/\partial x_i)(t, x_{02}, x_{03})$ at $t = t_0$ is
\[
\frac{\partial H}{\partial x_i}(t, x_{02}, x_{03}) = \frac{\partial H}{\partial x_i}(t_0, x_{02}, x_{03}) + \frac{1}{2} \frac{\partial^2 H}{\partial x_i \partial x_j}(t_0, x_{02}, x_{03})(t - t_0).
\]
From now on, we remove $(t_0)$ for abbreviation.

1. Since $h_{x_0}$ has the $A_1$-singularity, we show that the rank of the matrix
\[
\begin{pmatrix}
    n_{\phi 2} - \frac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} & n_{\phi 3} - \frac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1}
\end{pmatrix}
\]
is equal to one. If the rank is zero, then
\[
n_{\phi 2} - \frac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} = - \frac{\cos \phi x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} = 0
\]
\[
n_{\phi 3} - \frac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} = 1 - \frac{\cos \phi x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} = 0
\]
Thus, we have the sum of the power of the both equations
\[
0 = \frac{\cos^2 \phi (x_{02}^2 + x_{03}^2) - (x_{02}^2 + x_{03}^2 + 1)}{x_{02}^2 + x_{03}^2 + 1}
\]
\[
= - \frac{\sin^2 \phi (x_{02}^2 + x_{03}^2 + 1)}{x_{02}^2 + x_{03}^2 + 1} \neq 0.
\]
This is a contradiction.

2. Since $h_{x_0}$ has the $A_2$-singularity, we show that the rank of the matrix
\[
B = \begin{pmatrix}
    n_{\phi 2} - \frac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} & n_{\phi 3} - \frac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} \\
    n'_{\phi 2} - \frac{x_{02}'}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1} & n'_{\phi 3} - \frac{x_{03}'}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1}
\end{pmatrix}
\]
is equal to two. Since $n'_{\phi} = c_2 a_0 + (\cos \phi c_1 - c_3) a_1 + \cos \phi c_2 a_2$,
\[
\det B = \begin{vmatrix}
    n_{\phi 2} - \frac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} & n_{\phi 3} - \frac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} \\
    n'_{\phi 2} - \frac{x_{02}'}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1} & n'_{\phi 3} - \frac{x_{03}'}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1}
\end{vmatrix}
\]
\[
= \frac{1}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} |x_0, n_{\phi}, n'_{\phi}| 
\]
\[
= - (\cos \phi c_1 - c_3) \sqrt{s^2 \sin^2 \phi + 1} + s \sin^2 \phi c_2 
\]
\[
= - \frac{(\cos \phi c_1 - c_3) \sqrt{s^2 \sin^2 \phi + 1} + s \sin^2 \phi c_2}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} \neq 0
\]
This means that the rank of $B$ is two. \hfill \Box

It follows from Theorem 5.2 and Proposition 5.5, we have shown the following theorem.

**Theorem 5.6.** Let $\{a_0(t), a_1(t), a_2(t)\}_{t \in J}$ be pseudo-orthonormal moving frame of $\mathbb{R}^3$. Suppose $c_2(t) \neq 0$ and $\cos \phi_1(t) - c_3(t) \neq 0$. Then we have the following:

1. The envelope $g[\phi]$ of the family of $\phi$-slant pseudo-lines $SL(n_\phi, -\cos \phi)$ is regular at a point $t = t_0$ if and only if $\delta[\phi]_1(t_0) \neq 0$.
2. The envelope $g[\phi]$ of the family of $\phi$-slant pseudo-lines $SL(n_\phi, -\cos \phi)$ at a point $t = t_0$ is locally diffeomorphic to the cusp $C$ if and only if $\delta[\phi]_1(t_0) = 0$, $\delta[\phi]_2(t_0) \neq 0$.

6 **Slant evolutes of hyperbolic plane curves**

There is the notion of hyperbolic evolutes of hyperbolic plane curves [5]. Let $\gamma : J \rightarrow H^2_+$ be a unit speed curve, where we use the parameter $s \in J$ instead of $t$. We call $t(s) = \gamma'(s)$ a unit tangent vector of $\gamma$ at $s$. Since $\langle \gamma(s), \gamma(s) \rangle = -1$ we have $\langle \gamma(s), t(s) \rangle = 0$. We define $e(s) = \gamma(s) \wedge t(s)$, which is called a unit binormal vector of $\gamma$ at $s \in J$. Then we have $\langle e(s), e(s) \rangle = \langle \gamma(s) \wedge t(s), \gamma(s) \wedge t(s) \rangle = -\langle \gamma(s), \gamma(s) \rangle \langle t(s), t(s) \rangle + \langle \gamma(s), t(s) \rangle^2 = 1$. Therefore, we have a pseudo-orthonormal moving frame $\{\gamma(s), -e(s), t(s)\}$ of $\mathbb{R}^3$, which is called a hyperbolic Sabban frame along $\gamma$.

$a_0(s) = \gamma(s)$, $a_1(s) = -e(s)$, $a_2(s) = t(s)$

Then we have the following Frenet-Serret type formulae:

\[
\begin{aligned}
\gamma'(s) &= t(s) \\
t'(s) &= \gamma(s) + \kappa_g(s)e(s) \\
e'(s) &= -\kappa_g(s)t(s),
\end{aligned}
\]

where $\kappa_g(s) = |\gamma(s), \gamma'(s), \gamma''(s)|$ is called the geodesic curvature of $\gamma$. Since $a_0(s) = \gamma(s)$, $a_1(s) = -e(s)$, $a_2(s) = t(s)$, we have $c_1(s) = 0$, $c_2(s) = 1$ and $c_3(s) = -\langle a_1(s), a_2'(s) \rangle = \langle e(s), t'(s) \rangle - \langle \gamma(s) \wedge t(s), t'(s) \rangle = |\gamma(s), t(s), t'(s)| = |\gamma(s), \gamma'(s), \gamma''(s)| = \kappa_g(s)$. In this case, the family of $\phi$-slant pseudo-lines $g_\phi : I \times J \rightarrow H^2_+(-1)$ is

\[
g_\phi(r, s) = \begin{cases}
\gamma(s) - r e(s) + \frac{\sqrt{r^2 \sin^2 \phi + 1} - 1}{\sin^2 \phi} (\gamma(s) + \cos \phi t(s)) & \text{if } \phi \neq 0, \\
\gamma(s) - r e(s) + \frac{r^2}{2} (\gamma(s) + t(s)) & \text{if } \phi = 0.
\end{cases}
\]
Therefore, the envelope \( g[\phi] : J \rightarrow H_1^2 \) of \( g_\phi \) is

\[
g[\phi](s) = \begin{cases} 
\gamma(s) - r(s) e(s) + \frac{\sqrt{r(s)^2 \sin^2 \phi + 1}}{\sin \phi} (\gamma(s) + \cos \phi t(s)) & \text{if } \phi \neq 0, \\
\gamma(s) + \frac{1}{\kappa_g(s)} e(s) + \frac{1}{2\kappa_g^2(s)} (\gamma(s) + t(s)) & \text{if } \phi = 0,
\end{cases}
\]

where

\[r(s) = \frac{1}{\sqrt{\kappa_g^2(s) - \sin^2 \phi}}.\]

We call \( g[\pi/2] \) a hyperbolic evolute and \( g[0] \) a horocyclic evolute of \( \gamma \), respectively.

For \( s_0 \in J \), we define

\[
\sigma[\phi]_1(s_0) = \kappa_g'(s_0) + \frac{\cos \phi(\kappa_g(s_0)^2 - \sin^2 \phi)}{\sqrt{\kappa_g^2(s_0) - \sin^2 \phi - \kappa_g(s_0)}},
\]

\[
\sigma[\phi]_2(s_0) = \kappa_g''(s_0) + \cos \phi \kappa_g'(s_0) \left( \frac{\sqrt{\kappa_g^2(s_0) - \sin^2 \phi + 2\kappa_g(s_0)}}{\sqrt{\kappa_g^2(s_0) - \sin^2 \phi - \kappa_g(s_0)}} \right).
\]

In this case, by a straightforward calculation, we can show that \( \sigma[\phi]'_1(s) = \sigma[\phi]'_2(s) \). Moreover, we can show that \( \delta[\phi]_1(s) = \delta[\phi]_2(s) = 0 \) if and only if \( \sigma[\phi]'_1(s) = \sigma[\phi]'_1(s) = 0 \). As special cases, we have

\[\sigma[0]'_1(s_0) = \kappa_g'(s) - \frac{1}{2} \kappa_g(s), \quad \sigma[\pi/2]'_1(s_0) = \kappa_g'(s).\]

As a corollary of Theorem 5.6, we have the following theorem.

**Theorem 6.1.** Let \( \gamma : J \rightarrow H_1^2(-1) \) be a unit speed curve with \( \kappa_g(s)^2 - \sin^2 \phi > 0 \). Then we have the following:

1. \( g[\phi] \) is a regular curve at \( s = s_0 \) if and only if \( \sigma[\phi]'_1(s_0) \neq 0 \).
2. \( g[\phi] \) is locally diffeomorphic to the cusp \( C \) at \( s = s_0 \) if and only if

\[\sigma[\phi]'_1(s_0) = 0 \quad \text{and} \quad \sigma[\phi]'_1(s_0) \neq 0.\]

As a special case, we have the following corollary.

**Corollary 6.2.** Let \( \gamma : J \rightarrow H_1^2(-1) \) be a unit speed curve.

(A) Suppose \( \kappa_g'^2 > 1 \). Then we have the following (cf., [5]):

1. The hyperbolic evolute \( g[\pi/2] \) is a regular curve at \( s = s_0 \) if and only if \( \kappa_g'(s) \neq 0 \).
2. The hyperbolic evolute \( g[\pi/2] \) is locally diffeomorphic to the cusp \( C \) at \( s = s_0 \) if and only if \( \kappa_g'(s_0) \neq 0 \) and \( \kappa_g''(s_0) \neq 0 \).
(B) Suppose $\kappa_g \neq 0$. Then we have the following:

1. The horocyclic evolute $g[0]$ is a regular curve at $s = s_0$ if and only if $\kappa'_g(s) - \frac{1}{2} \kappa_g(s) \neq 0$.
2. The horocyclic evolute $g[0]$ is locally diffeomorphic to the cusp $C$ at $s = s_0$ if and only if

$$\kappa'_g(s_0) - \frac{1}{2} \kappa_g(s_0) = 0 \text{ and } \kappa''_g(s_0) - \frac{1}{2} \kappa'_g(s_0) \neq 0.$$ 

The hyperbolic evolute is given by

$$g[\pi/2](s) = \begin{cases} 
-1 & (\kappa_g(s) \gamma(s) + e(s)) \text{ if } \kappa_g(s) < -1, \\
\sqrt{\kappa_g^2(s) - 1} (\kappa_g(s) \gamma(s) - e(s)) & \text{if } \kappa_g(s) > 1 
\end{cases}$$

and the horocyclic evolute is

$$g[0](s) = \gamma(s) + \frac{1}{\kappa_g(s)} e(s) + \frac{1}{2 \kappa_g^2(s)} (\gamma(s) + t(s)).$$

In [5] hyperbolic evolutes was introduced and the classified the singularities. Moreover, a de Sitter evolute of $\gamma$ was introduced in [5], which is located in the de Sitter 2-space. It corresponds to points of $\gamma(s)$ with $\kappa_g^2(s) < 1$. Here we only consider families of hyperbolic lines, so that we do not consider de Sitter evolutes. It is also shown in [5] that $g[\pi/2](s)$ is a constant point if and only if $\gamma$ is a part of a circle. This condition is also equivalent to $\kappa'_g(s) \equiv 0$. We have a natural question what is $\gamma$ when $g[0](s)$ is a constant point. Of course it is equivalent to

$$\kappa'_g(s) - \frac{1}{2} \kappa_g(s) \equiv 0.$$ 

The solution of the above differential equation is $\kappa_g(s) = ce^{s/2}$ for a constant real number $c$. The curvature tends to infinity, so that $\gamma$ is a kind of spirals in $H^2_+(1)$. If $c = 1/2$, the curve with the curvature $\frac{1}{2} e^{s/2}$ in the Euclidean plane is called a *Nielsen spiral*. So we call $\gamma$ with $\kappa_g(s) = \frac{1}{2} e^{s/2}$ a *hyperbolic Nielsen spiral*. We have two open problems as follows:

1. What is $\gamma$ with $\sigma[\phi]_1(s) \equiv 0$?
2. For a general one-parameter family of pseudo-lines, is it always true that $\delta[\phi]_1(t) = \delta[\phi]_2(t) = 0$ if and only if $\delta[\phi]_1(t) = \delta[\phi]_1'(t) = 0$?
Envelopes of slant lines

References


