Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **35** (2015) no. 1, 125–133. doi:10.1285/i15900932v35n1p125

Generalized Lie's Series

Antonio Avantaggiati
Dipartimento di Scienze di Base ed Applicate per l'Ingegneria, Universitá di Roma 'La Sapienza'

Eduardo Pascali
Dipartimento di Matematica e Fisica "Ennio De Giorgi", Universitá del Salento; corresponding author e-mail eduardo.pascali@unisalento.it

Received: 28.1.2015; accepted: 16.3.2015.

Abstract. Starting from a well know definition of Lie's series, we define a new type of series which permit to obtain (or re-obtain) existence and representation of solutions for some types of particular differential equations.

Keywords: Lie's Series; Evolution equations; Representation of solutions.

MSC 2000 classification: 35A01; 35C10

1 Introduction

For many evolution problems one of the most (and beautiful) thecnical instrument are the Lie's series. In the book ([3]) there is a sistematic study of such series and applications. In many interesting paper the Lie's series are applied to the study of celestial mechanics, laser physics, neutron transport and so (see [7], [2], [8] for a list of references). Many paper deal also the numerical aspect of this type of series ([1]).

Lie's series are related to the well know Taylor's series for the analitical functions:

$$F(z+t) = \sum_{0}^{+\infty} \frac{t^n}{n!} F^{(n)}(z). \tag{1}$$

In fact they are of the form

$$\sum_{0}^{+\infty} \frac{t^n}{n!} D^{(n)} f(z) \tag{2}$$

where D is a differential linear operator with $D^{(n)}f = D(D^{(n-1)}f)$ and $D^0 f = f$.

An investigation of the role of the sequence of functions $(\frac{t^n}{n!})_n$ in the general theory of Lie's series, give the possibility to think a generalization arguing as follows.

Let X and $Y \subset X$ be real vector spaces and let $L : X \to Y$ be linear and let (φ_n) be a sequence of functions such that $\varphi_n \in C^1(R, R) \quad \forall n \in N$.

We define "generalized Lie's serie" a formal series of the following type

$$\sum_{0}^{+\infty} \varphi_n(t) L^{(n)}(f) \tag{3}$$

if the sequence (φ_n) verify

$$\dot{\varphi}_n(t) = \varphi_{n-1}(t) \quad \forall n \geq 1 \quad \forall t \tag{4}$$

or similar conditions (see next sections).

We are able to give some results for these series and to present some applications. In particular we are able to give existence and representation results for the solution of some differential equation. Moreover in these representations the role of the data in the equation is clear.

2 A first abstract result

We consider now only the formal aspect of the question. So we assume that all series given in the sequel converge in a such way that all step of the proof are true.

The following result is analogous of the fundamental theorem given in ([3]).

Theorem 2. Let X and $Y \subset X$ be normed (or Banach) vector real spaces of real functions; let $L : X \to Y$ be a linear continuous trasformation. Let $f \in Y$, $\varphi_0 \in C^1(R, R)$ and consider the following differential equation

$$\dot{u}(t) = Lu(t) + \dot{\varphi}_0(t) f. \tag{5}$$

We consider a sequence of functions (φ_n) as follows (starting from φ_0)

$$\dot{\varphi}_n = \varphi_{n-1} \quad \forall n \geq 1 \quad \forall t. \tag{6}$$

Then the function

$$u(t) = \varphi_0(t) f + \sum_{1}^{+\infty} \varphi_n(t) L^n(f) \tag{7}$$

is a solution of the equation (5).

Proof. Let $u_p(t) = \varphi_0(t)f + \sum_1^p \varphi_n(t)L^n(f)$ and derive with respect to t

$$\dot{u}_p(t) = \dot{\varphi}_0(t)f + \sum_1^p \dot{\varphi}_n(t)L^n(f) = \dot{\varphi}_0(t)f + \sum_1^p \varphi_{n-1}(t)L^n(f) =$$

$$= \dot{\varphi}_0(t)f + L(\sum_1^p \varphi_{n-1}(t)L^{n-1}(f)) = \dot{\varphi}_0(t)f + L(\sum_0^{p-1} \varphi_n(t)L^n(f)).$$

Then there exists the limit of the sequence $(\dot{u}_p)_p$ and

$$lim_{p\to+\infty}\dot{u}_p(t) = \dot{\varphi}_0(t)f + L(\sum_0^{+\infty} \varphi_n(t)L^n(f)) = \dot{\varphi}_0(t)f + Lu(t).$$

So the function u has derivative with respect to t and

$$\dot{u}(t) = \dot{\varphi}_0(t)f + Lu(t).$$

Remark 1. If there exists $n^* \in N$ such that $L^{n*}f = 0$ (and so $L^n f = 0$ for every $n \geq n^*$), the equation (5) has, of course, the solution

$$u(t) = \varphi_0(t)f + \sum_1^{n^*-1} \varphi_n(t)L^n(f).$$

Remark 2. The condition of linearity of L can be replaced with

$$lim_{p\to+\infty}[L(\sum_0^{p-1} \varphi_n(t)L^n(f)) - \sum_0^{p-1} \varphi_n(t)L^{n+1}(f)] = 0$$

3 Other abstract results

With a natural modification of the proof of the theorem 2, we can obtain some existence theorem and a representation formula for solutions of some special differential equations.

As in previous section we consider only the formal aspect of the proof.

Let L be a continuous linear transformation between the normed vector real spaces X and $Y \subset X$. The following facts hold.

Proposition 6. Consider the following equation

$$\dot{u}(t) = k(t)Lu(t) + \dot{\varphi}_0(t)f \tag{8}$$

where $k \in C^1(R, R)$. Then the function

$$u(t) = \varphi_0(t)f + \sum_{1}^{+\infty} \varphi_n(t)L^n(f) \tag{9}$$

is a solution of (8) when $(\varphi_n)_n$ (starting from φ_0) is defined, for every $n \in N$, by

$$\dot{\varphi}_n(t) = k(t)\varphi_{n-1}(t). \tag{10}$$

The proof is similar to the proof of Theorem 2.

Proposition 7. Consider the following equation

$$\dot{W}(t) = \frac{1}{h(t)}LW(t) + \frac{\dot{h}(t)}{h(t)}W(t) + h(t)\varphi_0(t)L(f) \tag{11}$$

where $\varphi_0, h \in C^1(R,R)$, $h \neq 0$. Then the function

$$w(t) = h(t)\sum_{1}^{+\infty} \varphi_n(t)L^n(f) \tag{12}$$

is a solution of (11), when $\dot{\varphi}_n(t) = \varphi_{n-1}(t)$.

The proof of the last proposition follows from the fact that the function

$$u(t) = \varphi_0(t)f + h(t)\sum_{1}^{+\infty} \varphi_n(t)L^n(f)$$

satisfy the following equation

$$\frac{d}{dt}[u(t) - \varphi_0(t)f] = \frac{\dot{h}(t)}{h(t)}]u(t) - \varphi_0(t)f] + h(t)\varphi_0(t)L(f) - \frac{1}{h(t)}[L(u(t) - \varphi_0(t)f)]$$

so that $W(t) = u(t) - \varphi_0(t)f$ is a solution of the equation (11).
Now we can also prove this proposition.

Proposition 8. Consider the following equation

$$\dot{W}(t) = \varphi_0(t)[LW(t) + L(f)]. \tag{13}$$

Then there exists a solution of the equation (13).

The easy proof is related to the following remark. If we consider the function

$$u(t) = \varphi_0(t)f + \sum_{1}^{+\infty} \int_0^t \varphi_0(t_1)\int_0^{t_1} \varphi_0(t_2)....\int_0^{t_{n-1}} \varphi_0(t_n)dt_n....dt_1 L^n(f),$$

then such a function is a solution of the equation

$$\frac{d}{dt}[u(t) - \varphi_0(t)f] = \varphi_0(t)L(u(t) - \varphi_0(t)f) + \varphi_0(t)L(f).$$

So, $W(t) = u(t) - \varphi_0(t)f$ satisfy (13).

Finally we have

Proposition 9. Consider the following equation

$$\dot{W}(t) = a\varphi_0(at)LW(t) + [\frac{d}{dt}log|h(t)|]W(t) + a\varphi_0(at)h(t)L(f)], \tag{14}$$

with $a \in R$. Then there exists a solution of equation (14)

In such a situation we consider the function

$$u(t) = \varphi_0(at)f + h(t)\sum_{1}^{+\infty}\varphi_n(at)L^n(f).$$

It is easy to prove that function

$$W(t) = u(t) - \varphi_0(at)f \equiv h(t)\sum_{1}^{+\infty}\varphi_n(at)L^n(f)$$

is a solution of (14)

Remark 3. It is remarkable that in all the representation formula the coefficient-function of the equation are inserted and partecipate explicitely to the representation. So it seem that appropriate information about the solution can be investigated starting from particular property of coefficient-function.

The greater difference with the application of the classical Lie's series related on the fact that the system $(\frac{t^n}{n!})_n$ play for all equation while for the generalized Lie's series for all equation a different system can play.

4 Some Applications and Remarks

In many applications, of course, cannot be easy to investigate the convergence of the formal series.

In the following examples (see [6] for many of them) we are in a situation for which the formal series reduce to a finite sum. Also in this semplified situation we seem to have interesting results.

PB1: ***There is or not a solution of the following equation***

$$\frac{\partial u(x,t)}{\partial t} = t\frac{\partial u(x,t)}{\partial x} - x^2 sent \tag{15}$$

such that $u(x,\pi) = x^2$ ***and*** $u(x,-\pi) = -x^2$ ***?***

Assume $f(x) = x^2$, $\varphi_0(t) = cost$, $\dot{\varphi}_i(t) = t\varphi_{i-1}(t)$ and $L = \frac{\partial}{\partial x}$. Because $L^3(f) = 0$, we have that the function

$$u(x,t) = x^2 cost + 2x\int_{-\pi}^{t} \tau cos\tau d\tau + 2\int_{-\pi}^{t} \tau \int_{-\pi}^{\tau} \eta cos\eta d\eta d\tau$$

is a solution of (15) satisfying the required conditions.

Now we consider a problem as to the previous.

PB2: ***There is or not a solution of the following equation***

$$\frac{\partial u(x,t)}{\partial t} = t\frac{\partial^2 u(x,t)}{\partial x^2} - x^2 sent \tag{16}$$

such that $u(x,-\pi) = -x^2$ ***and*** $u(x,-\pi) = -x^2$ ***?***

In such situation it suffice, with respect to the previous problem, to consider $L = \frac{\partial^2}{\partial x^2}$ obtaining

$$u(x,t) = x^2 cost + 2x\int_{-\pi}^{t} \tau cos\tau d\tau$$

as a solution of the equation (16) verifying the required conditions.

Then it easy to think that for all type of the equations of the form

$$\frac{\partial u(x,t)}{\partial t} = h(t)\frac{\partial u(x,t)}{\partial x} - P(x)sent \tag{17}$$

$$\frac{\partial u(x,t)}{\partial t} = h(t)\frac{\partial^2 u(x,t)}{\partial x^2} - P(x)sent \tag{18}$$

where $h \in C^1$ and P is a polinomial function in the x variable, it is possible to give a solution.

Moreover we can consider other type of equations.

PB3: ***Give a solution of the following equation***

$$\frac{\partial u(x,t)}{\partial t} = A\frac{\partial^2 u(x,t)}{\partial x^2} + B\frac{\partial u(x,t)}{\partial x} + x^2\dot{\varphi}_0(t) \tag{19}$$

where A, B ***are real number.***

This equation is a generalization of a well known equation, namely the equation of the random walk with drift and source. In such situation there is a solution of the following type

$$u(x,t) = \varphi_0(t)x^2 + 2(A + Bx)\varphi_1(t) + 2B^2\varphi_2(t)$$

where we assume $Lu = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x}$ and $\varphi'_{i+1} = \varphi_i$. Now it is certainly true that such solution is periodic in the variable t if the same holds for all φ_i.

PB4: ***Give a solution for the following equation***

$$\frac{\partial u(x,t)}{\partial t} = A\frac{\partial^2 u(x,t)}{\partial x^2} + B\frac{\partial u(x,t)}{\partial x} + Cu(x,t) + x\dot{\varphi}_0(t) \tag{20}$$

where A, B, C ***are real number.***

The previous equation is related to the problem of diffusion with transport, reaction and source. If we consider $Lu = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x} + Cu$ and still $\varphi'_{i+1} = \varphi_i$, a solution of (20) is

$$u(x,t) = x[\varphi_0(t) + \sum_1^\infty C^i\varphi_i(t)] + BC^2\sum_1^\infty i\varphi_i(t)$$

PB5: ***Give a solution of the following equation***

$$\frac{\partial u(x,t)}{\partial t} = A\frac{\partial^2 u(x,t)}{\partial x^2} + B\frac{\partial u(x,t)}{\partial x} + Cu(x,t) + x^2\dot{\varphi}_0(t) \tag{21}$$

where A, B, C ***are real number.***

Assuming L as in the previous problem, $\varphi'_{i+1} = \varphi_i$, we remark that $L^n(x^2) = C^n x^2 + nBC^{n-1}x + (2nAC^{n-1} + n(n-1)B^2C^{n-2})$; so a solution of (21) is

$$u(x,t) = x^2[\varphi_0(t) + \sum_1^\infty C^i\varphi_i(t)] + 2\frac{Bx+A}{C}\sum_1^\infty iC^i\varphi_i(t) + \frac{B^2}{C^2}\sum_1^\infty i(i-1)C^i\varphi_i(t).$$

Moreover we consider also the following problem.

PB6: ***Give a solution of the equation***

$$\frac{\partial u(x,t)}{\partial t} = \dot{g}(t)[\dot{\varphi}_0(g(t))f(x) + L(u(x,t))] \tag{22}$$

Arguing as previous, a solution is

$$u(x,t) = \varphi_0(g(t))f(x) + \sum_1^{+\infty}\varphi_i(g(t))L^i(f(x)). \tag{23}$$

In particular if $g(t) = \sqrt{t}$, (23) give a solution for the equation

$$2\sqrt{t}\frac{\partial u(x,t)}{\partial t} = \dot{\varphi}_0(\sqrt{t})f(x) + L(u(x,t)).$$

Otherwhise, if $g(t) = \frac{1}{1-\alpha}t^{1-\alpha}$ we have that a solution for the equation

$$t^\alpha\frac{\partial u(x,t)}{\partial t} = \dot{\varphi}_0(\frac{1}{1-\alpha}t^{1-\alpha})f(x) + L(u(x,t))$$

is the following function

$$u(x,t) = \varphi_0(\frac{1}{1-\alpha}t^{1-\alpha})f(x) + \sum_1^{+\infty} \varphi_i(\frac{1}{1-\alpha}t^{1-\alpha})L^i(f(x)).$$

If $g(t) = \frac{1}{t}$, for the equation

$$t\frac{\partial u(x,t)}{\partial t} = \dot{\varphi}_0(logt)f(x) + L(u(x,t))$$

a solution is

$$u(x,t) = \varphi_0(logt)f(x) + \sum_1^{+\infty} \varphi_i(logt)L^i(f(x))$$

We remark that if $L(f) = 0$ our procedure fails.

Remark 4. A simple abstract situation for which $\sum_0^{+\infty} \varphi_{n,j}(t)L_j^n(f_j)(x)$ is convergent can be considered as follows.

Let $X = C([c,d];R)$ the Banach space w.r.t. norm $||f||_X = sup_{x\in[c,d]}|f(x)|$; assume $L : X \longrightarrow X$ be a linear, continuous operator such that

$$\exists\alpha \in [0,1[\quad ||L|| \le \alpha \tag{24}$$

where, of course, $||L|| = sup_{||f||_X\le 1}||Lf||$. Moreover assume that

$$\exists M \ge 0 \quad |\varphi_i(t)| \le M \quad \forall(x,t) \in [c,d] \times [a,b]. \tag{25}$$

Then for every $f \in X$ the serie $\sum_0^{+\infty} \varphi_{n,j}(t)L_j^n(f_j)(x)$ is convergent w.r.t. the norm

$$||u(.,.)|| = sup_{(x,t)\in[c,d]\times[a,b}|u(x,t)|. \tag{26}$$

Remark 5. From a very abstract point of view, it seem interesting to study the limit for formulas of the following type

$$\varphi_{0,j}(t)f_j + \sum_1^{+\infty} \varphi_{n,j}(t)L_j^n(f_j) \tag{27}$$

where (L_j) is a sequence of continuous linear transformations, (f_j) a sequence of functions and, for every j, $(\varphi_{n,j})$ is a sequence such that $\dot{\varphi}_{n,j} = \varphi_{n-i,j}$.

References

[1] S. Filippi: *A new Lie series method for the numerical integration of O.D.E. with application to the restricted problem of three bodies* (NASA)

[2] J.M. Finn: *Lie series: a prespective, local and global methods of nonlinear Dynamics*, Lect.Notes in Physics **252** (1984), pp.63-86

[3] W. Grobner: *Serie di Lie e loro applicazioni*; ed. Cremonese, Roma (1973)

[4] A. Hanslmeier and D. Dvorak: *Numerical integration with Lie series*, Astronomy and Astrophysics, **132**, (1984), pp. 203-207

[5] R. Hermann: *Differential Geometry and the Calculus of the Variation*, Mathematics in Science and Engineering, Vol. **49**, Academic Press, New York-London 1968 x+440 pp.

[6] S. Salsa: *Equazioni a derivate parziali*, Springer (2010)

[7] S. Steinberg: *Lie series and nonlinear ordinary differential equations*, JMAA vol **101** june 1984, pp. 39-63

[8] Willi-Hans Steeb: *Lie Series tecnique, ordinary differential equations and dynamical integration*; International School for scientific computing, Z, Naturforsch, **59a** (2004) pp.349-352