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On sets of type $(m, h)_2$ in $\mathrm{P}G(3, q)$ with $m \leq q$

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Abstract. A set of points of $\mathrm{P}G(3, q)$ of type $(m, h)_2$, with $m \leq q$, has size $k \geq m(q+1)$. In this paper, some characterization results of some sets of type $(m, h)_2$, $3 \leq m \leq q$, of minimal size $m(q+1)$ are given. Finally, sets of type $(3, h)_2$ in $\mathrm{P}G(3, q)$ are studied.

Keywords: projective space, linear code, intersection number, linear space

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1 Introduction

Let $\mathbb{P} = \mathrm{P}G(d, q)$ be the finite projective space of order q and dimension $d \geq 3$, $m_1, \ldots, m_s$ be s integers such that $0 \leq m_1 \leq \ldots \leq m_s$ and for any integer r, $1 \leq r \leq d-1$, let $\mathcal{P}_r$ denote the family of all the r–dimensional subspaces of $\mathbb{P}$.

A subset $\mathcal{K}$ of k points of $\mathbb{P}$ is of *type* $(m_1, \ldots, m_s)_r$ if $|\mathcal{K} \cap \pi| \in \{m_1, \ldots, m_s\}$ for any $\pi \in \mathcal{P}_r$, and for every $m_j \in \{m_1, \ldots, m_s\}$ there is at least one subspace $\pi \in \mathcal{P}_r$ such that $|\mathcal{K} \cap \pi| = m_j$. The non–negative integers $m_1, \ldots, m_s$ are the *intersection numbers* of $\mathcal{K}$ (with respect to $\mathcal{P}_r$).

Quadrics, algebraic varieties which are intersection of quadrics, subgeometries are subsets of points of $\mathbb{P}$ with few intersection numbers with respect to the lines, or to hyperplanes or to a given family of r–dimensonal subspaces, $2 \leq r \leq d-2$, so it arises in a natural way the problem of characterizing such classical objects using their intersection numbers with a prescribed family of subspaces. On the other hand, k–sets of $\mathbb{P}$ with exactly two intersection numbers with respect to hyperplanes give rise to two–weighted linear codes (cf e.g. [6]), and so it arises also an existence problem for sets of a given type. There is a wide literature devoted to the theory of k–sets of a given type, in the references the interested reader may find some papers on this topic.

The study on k–sets with two intersection numbers m and h, $m < h$, developes by increasing the values of m. There is a complete classification of k–sets

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of type $(m,h)_2$ for $m = 0,1$ [3, 7, 9, 10], $m = 2$ and $d = 3$ [4], for $m = 3$, $d = 3$ and $h = q+3$ [8]. In all such cases, except $m = 3, q = 2$ which is easily solved (cf e.g. [8]), one has $m \leq q$ and so it seems natural to study k–sets of type $(m,h)_2$ with $m \leq q$.

If $m \leq q$, a m–plane contains at least an external line thus counting the size of $\mathcal{K}$ via the planes on such external line gives $k \geq m(q+1)$. Some examples, such as subgeometries, sets of points of $m \leq q$ pairwise skew lines, have size $m(q+1)$ and so a characterization problem in the extremal case $k = m(q+1)$ arises.

Note that, already Beutelspacher [1] has considered a similar question for k–sets of $\mathbb{P}$ intersected by every hyperplane in at least m points, $m \leq q$, giving a complete description of those with minimal size $k = m(q+1)$ under the extra hypothesis $m \leq \sqrt{q}+1$.

In this paper, we will give some new characterizations of such sets of $PG(3,q)$ of minimal size and we also will study k–sets of type $(3,h)_2$ of $PG(3,q)$. We are going to prove the following results.

Theorem 1. *Let $\mathcal{K}$ be a k–set of points of* $\mathrm{PG}(3,q)$ *of type $(m,h)_2$, with $m \leq q$ and $k = m(q+1)$. If $s \geq m$ for every s–secant line with $s \geq 3$, then either $\mathcal{K}$ is a subset of points of a hyperbolic quadric $\mathcal{H}$ obtained by deleting $q+1-m$ pairwise skew lines of one of the two reguli of $\mathcal{H}$, or $m = \sqrt{q}+1$ and $\mathcal{K}$ is the subgeometry* $\mathrm{PG}(3,\sqrt{q})$ *or $m = 3$ and $\mathcal{K}$ is the set K_1 described in Example 1.1 below.*

Theorem 2. *Let $\mathcal{K}$ be a k–set of points of* $\mathrm{PG}(3,q)$ *of type $(m,h)_2$, with $m \leq q$, $k = m(q+1)$ and $q \geq 3$. If $\mathcal{K}$ contains at least $q-\sqrt{q}-1$ lines then $\mathcal{K}$ is the set of points of the union of m pairwise skew lines, or $m = q$ is a square, $k = q^2+q$ and $\mathcal{K}$ is the subgeometry* $\mathrm{PG}(3,\sqrt{q})$ *union the points of $q-\sqrt{q}-1$ lines.*

As remarked above, k–sets of $PG(3,q)$ of type $(3,q+3)_2$ have been completely classified. The next theorem concerns with the general case $(3,h)_2$.

Theorem 3. *Let $\mathcal{K}$ be a k–set of* $\mathrm{PG}(3,q)$ *of type $(3,h)_2$ and let b_i, $0 \leq i \leq q+1$ denote the number of lines intersecting $\mathcal{K}$ in exactly i points. Then, $h-3|q$ and one the following holds be true.*

(j) *$h = q+3$ and $\mathcal{K}$ is determined.*

(jj) *$h < q+3$, every line has at most three points in common with $\mathcal{K}$, $b_i > 0$ for every $i \in \{0,1,2,3\}$ and either*

(jj1) *$q = 8$, $h = 7$ and $k = 39$ and all the planes containing a 3–line are h–planes*[1]*.*

[1]As remarked in the review of the paper [8] on Mathematical Reviews (MathSciNet), an

or

(jj2) on each 3*–line there is at least one* 3*–plane,* $k \leq 3+q(h-3)$ *and the number* c_3 *of* 3*–planes is*

$$c_3 = b_2\frac{(q+1)h-k-2q}{3(h-3)} + b_3\frac{(q+1)h-k-3q}{(h-3)},$$

where

$$b_2 = \frac{[(q+1)h-k][6(q^2+1)-k(k-1)]+3kq(k-3)}{2q}$$

$$b_3 = \frac{[(q+1)h-k][k(k-1)-6(q^2+1)]-2kq(k-4)}{6q}.$$

Moreover, if $h > 13$ *then*

$$qh+3-\frac{7}{2}q < (q+1)h - \frac{7}{2}q \leq k \leq qh-3q+3.$$

1.1 Some previous results on k–sets of type $(m,h)_2$ in P$G(3,q)$

In this section, we recall some known results on sets of points of P$G(3,q)$ with two intersection numbers with respect to the planes.

Theorem A ([10], 1973) *A set* $\mathcal{K}$ *of points of* P$G(d,q)$*,* $d > 2$*, of type* $(1,h)_2$ *is either a line or an ovoid of* P$G(3,q)$*,* $q > 2$*, or the elliptic quadric or* P$G(3,2)$*.*

Theorem B ([4], 2006) *A set* $\mathcal{K}$ *of points of* P$G(3,q)$*,* $r > 2$*, of type* $(2,h)_2$ *is the set of points of the union of two skew lines.*

For $m = 3$, sets of points of P$G(3,3)$ of type $(3,h)_2$ which are not the pointsets of the union of three lines occur (cf [8]). In the next example, they are descibed via the homogeneous coordinates (x,y,z,t).

EXAMPLE 1.1. *Sets of type* $(3,6)_2$ *in* P$G(3,3)$ *containing no line.*

$K_1 = \{A(1,0,0,0), B(0,1,0,0), C(0,1,1,1), D(0,0,1,0), E(0,1,0,1),$
$F(0,0,0,1), G(1,0,0,1), H(1,1,0,1), I(1,0,2,0), L(1,2,2,0),$
$M(1,0,2,1), N(0,1,1,0)\}.$

$K_2 = \{A(1,1,2,1), B(1,0,0,0), C(0,1,0,0), D(0,0,1,0), E(0,0,0,1),$

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example of a 39–set of type $(3,7)_2$ in P$G(3,8)$ exists. It is associated with a quasy–cyclic $GF(8)$ linear 2 weight code of length 39, dimension 4 and weights 32 and 36.

$F(0,0,1,2), G(1,1,1,1), H(1,1,1,2), I(1,0,2,0), L(1,2,2,0),$
$M(0,1,2,2), N(0,1,1,0), O(1,0,2,2), P(1,2,1,1), Q(1,2,1,2)\}.$

$K_3 = \{A(1,0,0,0), B(0,1,1,0), C(0,1,0,0), D(0,0,1,0), E(0,0,0,1),$
$F(1,1,2,1), G(1,1,1,1), H(1,0,1,2), I(1,1,1,2), L(1,2,2,0),$
$M(0,1,2,2), N(1,1,2,2), O(0,1,2,1), P(1,0,1,1), Q(1,0,2,0)\}.$

For $q=2$ a set of type $(3,h)_2$ is either a plane of P$G(3,2)$ or the whole space P$G(3,2)$ or the set of points on three pairwise skew lines (cf [8]). For $q>3$ sets of type $(3,q+3)_2$ have been completely classified.

Theorem C ([8], 2012) *Let K be a subset of* P$G(3,q)$ $(q>2)$ *of type* $(3,q+3)$ *with respect to planes. Then,*

(i) If $q=3$, K is the set of points on three pairwise skew lines, or one of the three sets K_i, $i=1,2,3$.

(ii) If $q=4$, K is either the set of points on three pairwise skew lines or P$G(3,2)$ *embedded in* P$G(3,4)$.

(iii) If $q>4$, K is the set of points on three pairwise skew lines.

The next result states that in P$G(3,q)$, when $m \geq 2$, there is at least one line intersecting a set of type $(m,h)_2$ in at least three points.

Theorem D ([5], 1980) *If $\mathcal{K}$ is a cap of* P$G(3,q)$ *of type* $(m,n)_2$ *then either $\mathcal{K}$ is an ovoid* $(m=1)$ *or $q=2$, $m=0$ and $\mathcal{K}$ is* P$G(3,2)$ *less a plane.*

In view of these results, we will assume that $m \geq 3$ and that at least one s–line with $s \geq 3$ exists.

2 Preliminary results

Let $\mathcal{K}$ denote a k–set of type $(m,h)_2$. By an *m–plane* (*h–plane*) we mean a plane intersecting $\mathcal{K}$ in m–points (h–points).

Let $s \geq 2$ an integer, a line is a *s–secant* (or *s–line*) if it intersects $\mathcal{K}$ in s points. A line is *external* (*tangent*) if it intersects $\mathcal{K}$ in no point (one point).

Proposition 1. *Let $\mathcal{K}$ be a set of type $(m,h)_2$ in* PG$(3,q)$ *with $m \leq q$. Then $(h-m)|q$, and so $h-m \leq q$.*

Proof. Let π be an m–plane, since $m \leq q$, each point of π not belonging to $\mathcal{K}$ is on at least one external line and each point of $\mathcal{K}$ in π lies on at least one tangent line. Hence, there are both an external line and a tangent line. Let ℓ

be an external line and α be the number of m–planes on ℓ, counting k via the planes on ℓ gives:

$$k = (q+1)h - \alpha(h-m).$$

Let ℓ' be a tangent line, and let β denote the number of m–planes on ℓ', counting k via the planes on ℓ' gives

$$k = (q+1)h - q - \beta(h-m).$$

Comparing these two values of k gives $(h-m)|q$ and so $h-m \le q$.

QED

From the results contained in [9] it follows that $1+mq \le k \le hq$ and

$$k^2(q+1) - k[(h+m)(q^2+q+1) - q^2] + mh(q^3+q^2+q+1) = 0. \quad (1)$$

For $k = m(q+1)$ Equation (1) gives $h = m+q$. Vice versa, for $h = m+q$ it has the two solutions

$$k_- = m(q+1) \text{ and } k_+ = (q+m)(q-1) + 2 + \frac{2(m-1)}{q+1}.$$

So, for $h = m+q$ and $m \le q$ either $k = m(q+1)$ or $q = 2m-3$ and $k = 3(2m^2 - 6m + 5)$.

Let ℓ be an s–line, $s \le m$, and let α denote the number of m–planes on ℓ, then

$$m(q+1) = k = s + \alpha(m-s) + (q+1-\alpha)(q+m-s)$$

from which it follows that

$$\alpha = q+1-s.$$

Thus, *the number of h–planes passing through a s–line is s.*

3 Proof of Theorem 1

In this section $\mathcal{K}$ denotes a subset of $m(q+1)$ points of PG$(3,q)$ of type $(m,n)_2$ such that each line intersecting $\mathcal{K}$ in at least three points intersects $\mathcal{K}$ in at least m points.

Lemma 1. *If there is an s–secant secant line ℓ with $s \geq 3$ such that $s \geq m+1$, then ℓ is contained in $\mathcal{K}$ and $\mathcal{K}$ is the set of points of the union of m lines of a regulus containing ℓ.*

Proof. Let ℓ be an s–secant line with $s \geq m+1$. Any plane on ℓ is an h–plane, so $k = s + (q+1)(h-s) = (q+1)h - sq$. By assumptions, $k = m(q+1)$, thus

$$m(q+1) = (q+1)h - sq.$$

It follows that $q+1|s$ and so $s = q+1$ and $h = m+q$. Therefore, $\mathcal{K}$ contains at least one line.

Let ℓ be a line contained in $\mathcal{K}$. Being $m \geq 3$ each plane through ℓ contains at least two points of $\mathcal{K}$ outside ℓ. Let π be a plane on ℓ, and p and q two distinct points of $\mathcal{K} \cap (\pi \setminus \{\ell\})$. The line connecting p and q is an m–secant line ℓ' and so $\pi \cap \mathcal{K} = \ell \cup \ell'$.

Let π_i, $i = 1, \ldots, q+1$, denote the $q+1$ planes containing ℓ, and for each $i = 1, \ldots, q+1$ let ℓ_i denote the m–line in π_i.

All the lines ℓ_i, $i = 1, \ldots, q+1$ intersect ℓ and are pairwise skew. Indeed, if there were two lines ℓ_i and ℓ_j intersecting each other in a point, say p, then $p \in \ell$. Let π be the h–plane containing ℓ_i and ℓ_j. Since $h = q+m$, each point of $\pi \cap (\ell_i \setminus \mathcal{K})$ (and of $\pi \cap (\ell_j \setminus \mathcal{K})$ belongs to exactly one m–line and q tangent lines. It follows that there is a point x of $\mathcal{K}$ in π outside ℓ_i and ℓ_j. Connecting x with a point of $(\ell_i \cap \mathcal{K}) \setminus \{p\}$ gives a line ℓ' intersecting ℓ_j in a point $\mathcal{K}$ and so ℓ' is an m–line. Since ℓ' is skew *to* ℓ, all the planes on ℓ' are $(q+m)$–planes, contradicting the fact that the number of h–planes on an m–line is m.

Since the lines ℓ_i are pairwise skew and all of them intersect ℓ, it follows that $\mathcal{K}$ is the set of points on $q+1$ pairwise skew m–lines.

Let $i \neq j$, consider a point $x \in \ell_i \setminus \{\ell \cap \ell_i\}$ and a point $y \in \ell_j \setminus \{\ell \cap \ell_i\}$. Since ℓ_i and ℓ_j are skew and have ℓ as a transversal, xy must be skew to ℓ. If xy contains another point of $\mathcal{K}$ distinct from x and y, the line xy meets $\mathcal{K}$ in $\beta \geq m$ points, by assumption. Then every plane π through xy meets ℓ in a single point, forcing π to be an h–plane, as it contains at least $\beta - 1 > m$ points. Thus, $k = m(q+1) = (h-\beta)(q+1)+\beta = (m+q-\beta)(q+1)+\beta = m(q+1)+q(q+1)-\beta q$. This forces $\beta = q+1$. i.e. xy is contained in $\mathcal{K}$. Therefore the line xy must either be a 2–line or fully contained in $\mathcal{K}$.

Consider one of the lines ℓ_i, w.l.o.g. let such a line be ℓ_1. We prove that, the intersection of $\mathcal{K}$ and an h–plane through ℓ_1 consists of the points of ℓ_1 union those of a line contained in $\mathcal{K}$. Namely, this is true for the plane containing ℓ and ℓ_1. Let π an h–plane through ℓ_1 different from the plane containing ℓ and ℓ_1. Such a plane intersects each of the lines ℓ_i, $i \neq 1$, in a point of $\mathcal{K}$. Let x and y be two points of $\pi \cap \mathcal{K} \setminus \{\ell_1\}$. The line xy intersects ℓ_1 in a point p of

$\mathcal{K}$, indeed being π an h–plane it has no external line and so in π every point of ℓ_1 not in $\mathcal{K}$ lies on one m–line (clearly, ℓ_1) and q tangent lines. Thus, from the above argument the line xy is fully contained in $\mathcal{K}$. Since the number of h–planes on ℓ_1 is m , there are exactly m lines intersecting ℓ_1 and contained in $\mathcal{K}$, say L, $L_2, \ldots L_m$. For every $i = 2, \ldots m$, the line L_i is skew with L, otherwise the plane containing L ad L_i should contain ℓ_1 and so its size should be greater then $q+m$. Similarly, since any line L_i intersects every line ℓ_j it follows that the lines L_i and L_j are pairwise skew. So on each point of $\ell_1 \cap \mathcal{K}$ there is exactly one line contained in $\mathcal{K}$. Since $m \geq 3$ there are at least three lines contained in $\mathcal{K}$, thus the lines ℓ_i are the the transversal lines of the regulus generated by these three lines and so $\mathcal{K}$ is the set of points of a hyperbolic quadric with $q+1-m$ lines of one of its reguli deleted. $\boxed{QED}$

Lemma 2. *If any secant line with more than two points in $\mathcal{K}$ intersects $\mathcal{K}$ in exactly m points, then either $m = \sqrt{q}$ and $\mathcal{K}$ is $\mathrm{PG}(3, \sqrt{q})$ or $m = 3$ and $\mathcal{K}$ is the set K_1 described in Example 1.1.*

Proof. By the assumptions, $\mathcal{K}$ contains no line. Since $k = m(q+1)$ a h–plane contains no external line. Let ℓ be an m–line. On ℓ there are exactly $m \geq 3$ planes intersecting $\mathcal{K}$ in h points.

Each h–plane π contains at least one m–line, otherwise $\mathcal{K} \cap \pi$ should be an arc with $q+m \geq q+3$ points.

Since external lines lie only on m–planes, an h–plane π intersects $\mathcal{K}$ in a blocking set, and so if ℓ is an m–line of π, being $h = q+m$, on each point of $\ell \setminus \mathcal{K}$ there are exactly one m–line and q tangent lines. It follows that in an h–plane π if there are two m–lines, they intersect each other in a point of $\mathcal{K} \cap \pi$.

Let π be an h–plane, and let ℓ be an m–line of π. The q points of $\pi \cap \mathcal{K} \setminus \ell$ do not lie on a single line, so they give rise to at least q secant lines. The points of $X := \pi \cap \mathcal{K} \setminus \ell$ together with the secant lines connecting any two of its points may be seen as a linear space on q points. Since all these lines have a constant number m of points, by the de Bruijn–Erdos Theorem [2] it follows that either $q = m = 3$ and X is a triangle or there is a point of ℓ on at least three m–secant lines.

If X is a triangle then $k = 3(q+1) = 12$ and $m = 3$, $h = 6$ and so by the result in [8] $\mathcal{K}$ is the set K_1 described in Example 1.1.

Finally, assume that there is a point p on ℓ on at least three m lines. Let ℓ_1 and ℓ_2 two m–lines sharing with ℓ the point p of $\mathcal{K}$. Connecting a point of $(\ell_1 \setminus \{p\}) \cap \mathcal{K}$ with a point of $\ell \cap \mathcal{K}$ gives an m–line intersecting ℓ_2 in a point of $\mathcal{K}$ otherwise ℓ_2 should contain a point not in $\mathcal{K}$ on a 2–secant line. The same occur connecting a point of ℓ_1 with a point of $\mathcal{K} \cap \pi$ outside $\ell \cup \ell_2$ so on each point of ℓ_1 there are exactly m lines intersecting $\mathcal{K} \cap \pi$ in m points. It follows that $h = q + m = 1 + m(m-1)$ and so $q = (m-1)^2$.

Hence, $m = 1 + \sqrt{q}$ and so by the results in [1] the assertion follows. *QED*

4 Proof of Theorem 2

Let $\mathcal{K}$ denote a set of points of P$G(3, q)$ of type $(m, h)_2$. The next lemma gives the proof of Theorem 2.

Lemma 3. *If $m \leq q$, $q \geq 3$, and there are at least $q - \sqrt{q} - 1$ lines contained in $\mathcal{K}$, then $k = m(q+1)$ if and only if $\mathcal{K}$ is either the set of the points of the union of m pairwise skew lines or q is a square, $m = q$, $k = q^2 + q$ and $\mathcal{K}$ is the set of points of* P$G(3, \sqrt{q})$ *union the points of $q - \sqrt{q} - 1$ lines.*

Proof. Since $h \leq m + q \leq 2q$, the lines contained in $\mathcal{K}$ are pairwise skew. Assume $k = m(q+1)$, and so $h = q + m$.

Let $\mathcal{L}$ the set of all the the lines lines contained in $\mathcal{K}$ and let α denote the size of $\mathcal{L}$. Namely, $\alpha \leq m$.

Consider the set $\mathcal{K}' = \mathcal{K} \setminus \bigcup_{\ell \in \mathcal{L}} \ell$. If it is the empty set then $m = \alpha$ and $\mathcal{K}$ is the set of the points of the union of m pairwise skew lines.

Thus, let $\mathcal{K}'$ be different from the empty set. By its definition, $\mathcal{K}'$ contains no line, has size $(m - \alpha)(q+1)$ and it is of type $(m - \alpha, q + m - \alpha)_2$.

An external line lies only on $(m-\alpha)$–planes, so any $(h-\alpha)$–plane is a blocking set, thus

$$q + m - \alpha \geq q + m - \alpha \geq q + \sqrt{q} + 1$$

from which it follows that $\alpha \leq m - \sqrt{q} - 1 \leq q - \sqrt{q} - 1$. Therefore, $\alpha = q - \sqrt{q} - 1$, and so q is a square.

Hence, $m - \alpha = m - q + \sqrt{q} + 1 \leq q - q + \sqrt{q} + 1 = \sqrt{q} + 1$, and by the results in [1] it follows that $\mathcal{K}'$ is P$G(3, \sqrt{q})$. *QED*

5 On sets of type $(3, h)_2$

Let $\mathcal{K}$ denote a subset of points of PG$(3, q)$ of size k and of type $(3, h)_2$. As remarked in the Introduction, the sets of type $(3, h)_2$ in P$G(3, 2)$ are the planes, the set of the points of the union of three pairwise skew lines and the whole space P$G(3, 2)$, so we may assume $q \geq 3$. We are going to prove Theorem 3.

For $m = 3$, Equation (1) becomes:

$$k^2(q+1) - k[(h+3)(q^2+q+1) - q^2] + 3h(q^3+q^2+q+1) = 0. \qquad (2)$$

The following Lemmata give the proof of Theorem 3. Recall, that by Theorem (C) when $h = q+3$ the set $\mathcal{K}$ is determined, so in the following (j) Theorem 3 will follow whenever one has $h = q+3$.

Lemma 4. *Either $h = q+3$ or $\mathcal{K}$ contains no line.*

Proof. Assume that $\mathcal{K}$ contains a line. Then $h \ge q+2$. Since $(h-3)|q$ and $q \ge 3$ it follows that $h = q+3$. QED

Lemma 5. *If $h < q+3$ then $s \le 3$ for every s–line.*

Proof. Let ℓ be a line intersecting $\mathcal{K}$ in s points, with $4 \le s \le q$. Thus, $k = (q+1)h - sq$, and Equation (1.1) becomes

$$(q+1)^3h^2 + s^2q^2(q+1) - 2shq(q+1)^2 - [h(q^2+q+1) + 2q^2 + 3q + 3](q+1)h +$$
$$[h(q^2+q+1) + 2q^2 + 3q + 3]sq + 3h(q^3+q^2+q+1) = 0$$

and so

$$h^2(q+1) - h[(s-1)q^2 + (3s+2)q + (s+3)] + s(s+2)q^2 + s(s+3)q + 3s = 0.$$

The discriminant of such equation is

$$\Delta = (s-1)^2q^4 + (2s^2-10s-4)q^3 + (3s^2-4s-2)q^2 + (2s^2-2s+12)q + (s-3)^2.$$

Since $\Delta > [(s-1)q^2 - (s+2)q]^2$ and $q \ge 3$ it follows that

$$h_+ = \frac{(s-1)q^2 + (3s+2)q + (s+3) + \sqrt{\Delta}}{2q+2} >$$
$$\frac{(s-1)q^2 + (3s+2)q + (s+3) + (s-1)q^2 - (s+2)q}{2q+2} =$$
$$= \frac{2(s-1)q^2 + 2sq + (s+3)}{2q+2} = (s-1)q + 1 + \frac{s+1}{2q+2}$$

which cannot occur since $h \le q+3$.

Now, we prove that $h_- = \dfrac{(s-1)q^2+(3s+2)q+(s+3)-\sqrt{\Delta}}{2q+2} \leq s+3$.

Assume to the contrary that $h_- = \dfrac{(s-1)q^2+(3s+2)q+(s+3)-\sqrt{\Delta}}{2q+2} \geq s+4$.

Thus, $(s-1)q^2+(s-6)q-(s+5) \geq \sqrt{\Delta}$, which gives

$$(s-4)q^3+(s^2+4s-12)q^2+(s^2-s-12)q-(s+4) \leq 0$$

which is not possible since $4 \leq s \leq q$.

From $3q+3 \leq k = (q+1)h - sq$, it follows that $h \geq s+2$.

If $h = s+2$, then $3q+3 \leq (q+1)(s+2) - sq = 2q+s+2$ and so $s = q+1$. A contradiction to Lemma 4.

If $h = s+3$ then: $k = 3(q+1)+s = 3q+h$.

For $k = 3q+h$ Equation (2) gives:

$$(3q+h)^2(q+1)-(3q+h)[hq^2+hq+h+2q^2+3q+3]+3h(q^3+q^2+q+1) = 0$$

$$(3q+h)(3q^2+3q+hq+h-hq^2-hq-h-2q^2-3q-3)+3q^3h+3q^2h+3qh+3h = 0$$

$$(3q+h)(q^2-hq^2-3)+3q^3h+3q^2h+3qh+3h = 0$$

$$3q^3-3hq^3-9q+hq^2-h^2q^2-3h+3q^3h+3q^2h+3qh+3h = 0$$

$$3q^3-9q+hq^2-h^2q^2+3q^2h+3qh = 0$$

$$3q^2-9+4hq-h^2q+3h = 0$$

$$3h-9 = h^2q-4hq-3q^2$$

and so

$$3 = \frac{q}{h-3}(h^2-4h-3q). \tag{3}$$

Since $h-3|q$, the ratio $\dfrac{q}{h-3}$ is an integer and so either $\dfrac{q}{h-3} = 1$ or $\dfrac{q}{h-3} = 3$.

In the former case, it follows that $h = q+3$ against the assumptions of the Lemma.

In the latter case, being $q = 3h-9$ Equation 3 gives $1 = h^2-4h-3q$ and so $h^2-13h+26 = 0$ whose roots are not integers. $\boxed{QED}$

By the results contained in [5] and the previous lemma it there exists at least one line intersecting $\mathcal{K}$ in exactly 3 points. Thus, counting the size of $\mathcal{K}$ via the planes on a 3–line gives $k \leq 3 + q(h-3)$.

Now, we prove that there is at least one 2–line.

As usual, let b_i denote the number of lines intersecting $\mathcal{K}$ in exactly i points (i.e. the number of i– secant lines), and let c_3 denote the number of 3–planes. Let x_3 and y_3 denote the number of T–planes and L–planes, respectively. Thus, $x_3 + y_3 = c_3$.

Being $m = 3 \leq q$, it follows that $b_0, b_1 \ > 0$.

Assume $b_2 = 0$. Using the usual counting argument

$$b_0 + b_1 + b_3 = (q^2+1)(q^2+q+1)$$

$$b_1 + 3b_3 = k(q^2+q+1)$$

$$3b_3 = k(k-1)(q+1).$$

From the last two equations it follows that $b_1 = k(q^2+q+1-k(q+1))$, which gives $k(q+1) \leq q^2+q+1$ being $b_1 \geq 0$. So, from $k \geq 3q+3$ a contradiction follows.

Hence $b_i > 0$ for every $i = 0, 1, 2, 3$.

Since there are 2–secant lines, there are two types of 3–planes: the *T–planes* i.e. those intersecting $\mathcal{K}$ in three non–collinear points and the *L–planes* that is those intersecting $\mathcal{K}$ in three collinear points.

The usual counting argument (cf e.g. [9]) on the types gives:

$$c_3 = \frac{(q^2+1)[h(q+1)-k]-kq}{h-3}.$$

Double counting give:

$$b_2 \frac{(q+1)h-k-2q}{h-3} = 3x_3 \tag{4}$$

$$b_3 \frac{(q+1)h-k-3q}{h-3} = y_3. \tag{5}$$

If $x_3 = 0$, then $k = (q+1)h - 2q$. Counting k via the planes on a 3–line, gives $k = 3 + x(h-3)$, where x is the number of h–planes on the 3–line.

Thus,

$$q(h-3) + q + h - 3 = (q+1)h - 2q - 3 = k - 3 = x(h-3)$$

and so $x > q+1$, which is not possible.

If $y_3 = 0$, then $k = (q+1)h - 3q$. Moreover, all the planes containing a 3–line are h–planes.

Lemma 6. *If $k = (q+1)h - 3q$ then either $h = q+3$, $q = 3$, $k = 15$ and $\mathcal{K}$ is one of the two sets Ω_i, $i = 1, 2$ or $q = 8$, $h = 7$ and $k = 39$.*

Proof. Arguing as in the proof of the previous lemma with $s = 3$ gives

$$h = \frac{2q^2 + 11q + 6 \pm \sqrt{4q^4 - 16q^3 + 13q^2 + 24q}}{2q + 2}.$$

Since the discriminant $\Delta = 4q^4 - 16q^3 + 13q^2 + 24q > (2q^2 - 5q)^2$, it follows that $h_+ = \dfrac{2q^2 + 11q + 6 + \sqrt{4q^4 - 16q^3 + 13q^2 + 24q}}{2q + 2} > 2q+1+2/(q+1) > q+3$ and so $h = h_- = \dfrac{2q^2 + 11q + 6 - \sqrt{4q^4 - 16q^3 + 13q^2 + 24q}}{2q + 2}$.

From $2q + 1 + \dfrac{2}{q+1} + h_- < h_+ + h_- = 2q + 9 - \dfrac{3}{q+1}$ it follows that $h = h_- \le 7$.

If $h \le 6$, then

$$h_- = \frac{2q^2 + 11q + 6 - \sqrt{4q^4 - 16q^3 + 13q^2 + 24q}}{2q + 2} \le 6$$

$$2q^2 + 11q + 6 - \sqrt{4q^4 - 16q^3 + 13q^2 + 24q} \le 12q + 12$$

$$2q^2 - q - 6 \le \sqrt{4q^4 - 16q^3 + 13q^2 + 24q}$$

$$4q^4 + q^2 + 36 - 4q^3 - 24q^2 + 12q \le 4q^4 - 16q^3 + 13q^2 + 24q$$

$$12q^3 - 36q^2 - 12q + 36 \le 0$$

$$q^2(q - 3) - (q - 3) \le 0$$

$$(q^2 - 1)(q - 3) \le 0$$

thus $q = 2, 3$.

Being $q \ge 3$, it follows that $q = 3$ and so $h = 6$ (since $h - 3|q$), that is $h = q + 3$ against the assumptions.

If $h = 7$,

$$\frac{2q^2+11q+6-\sqrt{4q^4-16q^3+13q^2+24q}}{2q+2}=7$$

$$2q^2+11q+6-\sqrt{4q^4-16q^3+13q^2+24q}=14q+14$$

$$2q^2-3q-8=\sqrt{4q^4-16q^3+13q^2+24q}$$

$$4q^4+9q^2+64-12q^3-32q^2+48q=4q^4-16q^3+13q^2+24q$$

$$4q^3-36q^2+24q+64=0$$

$$(q-2)(q^2-7q-8)=0$$

from which it follows that $q=2,8$. Hence, $q=8, h=7$ e $k=39$. QED

Thus, we may assume that both x_3 and y_3 are different from 0. Hence, equations (4) and (5) give

$$b_2[(q+1)h-k-2q]+3b_3[(q+1)h-k-3q]=3c_3(h-3)$$

From which it follows

$$[(q+1)h-k-2q]b_2+3[(q+1)h-k-3q]b_3=3(q^2+1)[h(q+1)-k]-3kq.$$

Being

$$2b_2+6b_3=k(k-1)$$

we have a system of two linear equations in the unknowns b_2 and b_3 with determinant $6q$, and so:

$$b_2=\frac{[(q+1)h-k][6(q^2+1)-k(k-1)]+3kq(k-3)}{2q}$$

$$b_3=\frac{[(q+1)h-k][k(k-1)-6(q^2+1)]-2kq(k-4)}{6q}.$$

Being $b_2>0$ it follows that

$$3kq(k-3)>[(q+1)h-k][k(k-1)-6(q^2+1)]. \tag{6}$$

From $3q+3\leq k\leq 3+q(h-3)$ it follows that $h-3\geq 3$.

Lemma 7. $h > 13 \Rightarrow k \geq (q+1)h - \frac{7}{2}q$.

Proof. If $[(q+1)h - k] > 9q$ then (6) gives:

$$2k^2 - 18(q^2+1) < 0$$

which cannot occur, since $k \geq 3(q+1)$. Thus, $k \geq (q+1)h - 9q$.

Assume $k \leq 5q$, then $(q+1)h \leq 14q$ and so $14 \leq h \leq 14 - \frac{14}{q+1}$ which is a contradiction. Hence $k > 5q$.

If $[(q+1)h - k] \geq 4q$, from (6) it follows that

$$k^2 + 5k - 24(q^2+1) < 0$$

and so $k < 5q$, a contradiction.

Therefore, $[(q+1)h - k] < 4q$, that is $k > (q+1)h - 4q$.

Write $k = (q+1)h - 4q + x$, $0 < x < q$ and x is divisible for $h-3$.

When $x < q/2$ it follows that $k < (q+1)h - \frac{7}{2}q$ and so $(q+1)h - k > \frac{7}{2}q$ and from Equation (6) one gets

$$6k(k-3) > 7[k(k-1) - 6(q^2+1)]$$

and so

$$k^2 + 11k - 42(q^2+1) < 0$$

which gives $k < 7q$. But $7q > k > (q+1)h - 4q$ gives $h < 11$, contracting our assumptions.

Thus, $x \geq q/2$ and $k \geq (q+1)h - 4q + q/2 = (q+1)h - \frac{7}{2}q$. QED

Thus, Theorem 3 is proved.

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