Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **35** (2015) no. 1, 95–107. doi:10.1285/i15900932v35n1p95

Solvability for a coupled system of nonlinear fractional integro-differential equations

Mohamed Amin Abdellaoui

Department of Mathematics and Informatics, Faculty of Exact Sciences and Informatics, UMAB, University of Mostaganem, Algeria
abdellaouiamine13@yahoo.fr

Zoubir Dahmani

Laboratory LPAM, UMAB, University Mostaganem, Algeria
zzdahmani@yahoo.fr

Received: 11.1.2014; accepted: 12.2.2015.

Abstract. In this paper, we prove the existence of solutions for fractional coupled systems of integro-differential equations in Banach spaces. The results are obtained using fractional calculus and fixed point theorems. Some concrete examples are also presented to illustrate the possible application of the established analytical results.

Keywords: Caputo derivative, fixed point theorem, coupled system, existence, uniqueness.

MSC 2000 classification: primary 34A34, secondary 34B10.

1 Introduction

The fractional differential equations theory is increasingly used for many mathematical models in science and engineering, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to ([7, 9, 11]) and the references therein. Recently, there has been a significant progress in the investigation of these equations (see [2, 4, 5]). Moreover, the study of coupled systems of fractional differential equations is also of great importance. Such systems occur in various problems of applied science. For some recent results on the fractional systems, we refer the reader to (see [1, 3, 6, 12]).
In this paper, we study the following coupled system of fractional integro-

differential equations:

$$\begin{cases} D^{\alpha}u(t) = f_1(t, u(t), v(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} g_1(s, u(s), v(s))\, ds, t \in [0,1], \\ D^{\beta}v(t) = f_2(t, u(t), v(t)) + \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} g_2(s, u(s), v(s))\, ds, t \in [0,1], \\ \sum\limits_{k=0}^{n-2} \left(\left| u^{(k)}(0) \right| + \left| v^{(k)}(0) \right| \right) = 0, \\ u^{(n-1)}(0) = \gamma I^p u(\eta), \eta \in]0,1[, \\ v^{(n-1)}(0) = \delta I^q v(\zeta), \zeta \in]0,1[, \end{cases} \tag{1}$$

where D^{α} and D^{β} denote the Caputo fractional derivatives with $n-1 < \alpha < n$, $n-1 < \beta < n$, $n \in \mathbb{N}^*, \delta, \gamma \in \mathbb{R}^* \sigma, \theta, p$ and q are non negative reals numbers, f_1, f_2, g_1 and g_2 are the functions which will be specified later.
To the best of our knowledge, no paper has considered the fractional integro-differential system (1) with two arbitrary orders α, β and using Riemann-Liouville fractional integral conditions. Our aim is to study the existence and uniqueness of solutions of the fractional coupled system (1). However, with fractional integral conditions, it will become more complicated. So, we can take the first $n-1$ classical derivatives equal to 0, for the unknown functions u and v. Then, we shall use some fixed point theorems and other technics to overcome the difficulties.
We organize the rest of this paper as follows: In section 2, we present some definitions, preliminaries and lemmas. In Section 3, we prove our main results for the problem (1). At the last section, some examples are presented to illustrate the effectiveness of the main results.

2 Preliminaries

The following notations, definitions and preliminary facts will be used throughout this paper.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a,b]$ is defined as:

$$I^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, d\tau, \alpha > 0, a < t < b, \tag{2}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ for $f \in C^n([a,b], \mathbb{R})$ is defined as:

$$D^{\alpha}f\left(t\right)=\int_{a}^{t}\frac{\left(t-\tau\right)^{n-\alpha-1}}{\Gamma\left(n-\alpha\right)}f^{(n)}\left(\tau\right)d\tau,n-1<\alpha<n,n\in\mathbb{N}^{*},a<t<b. \quad (3)$$

For more details about fractional calculus, see [10].

The following lemmas give some properties of Riemann-Liouville fractional integrals and Caputo fractional derivatives [7, 9].

Lemma 1. *Let* $r,s>0$, $f\in L^{1}\left([a,b]\right)$. *Then* $I^{r}I^{s}f(t)=I^{r+s}f(t),D^{s}I^{s}f(t)=f(t),t\in[a,b]$.

Lemma 2. *Let* $s>r>0,f\in L^{1}\left([a,b]\right)$. *Then* $D^{r}I^{s}f(t)=I^{s-r}f(t),t\in[a,b]$.

To study the coupled system (1), we need the following two lemmas [8, 10].

Lemma 3. *For* $\alpha>0$, *the general solution of the equation* $D^{\alpha}x\left(t\right)=0$ *is given by*

$$x\left(t\right)=c_{0}+c_{1}t+c_{2}t^{2}+...+c_{n-1}t^{n-1}, \quad (4)$$

where $c_{i}\in\mathbb{R},i=0,1,2,..,n-1,n=[\alpha]+1$.

Lemma 4. *Let* $\alpha>0$. *Then*

$$I^{\alpha}D^{\alpha}x\left(t\right)=x\left(t\right)+c_{0}+c_{1}t+c_{2}t^{2}+...+c_{n-1}t^{n-1}, \quad (5)$$

for some $c_{i}\in\mathbb{R},i=0,1,2,...,n-1,n=[\alpha]+1$.

We prove the following auxiliary result:

Lemma 5. *Let* $g\in C\left([0,1],\mathbb{R}\right),n-1<\alpha<n,\sigma>0,n\in\mathbb{N}^{*}$. *Then, the solution of the problem*

$$D^{\alpha}x\left(t\right)=f\left(t\right)+\int_{0}^{t}\frac{\left(t-s\right)^{\sigma-1}}{\Gamma\left(\sigma\right)}g\left(s\right)ds, \quad (6)$$

associated with the conditions

$$\left|x\left(0\right)\right|+\left|x'\left(0\right)\right|+...+\left|x^{(n-2)}\left(0\right)\right|=0$$

and

$$x^{(n-1)}\left(0\right)=\gamma I^{p}x\left(\eta\right),\eta\in\left]0,1\right[,p>0,$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + \int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)} g(s)\,ds \\ &+ \frac{\gamma\Gamma(p+n)\,t^{n-1}}{\Gamma(n)\left(\Gamma(p+n)-\gamma\eta^{p+n-1}\right)} \\ &\times \left(\int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f(s)\,ds + \int_0^\eta \frac{(\eta-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} g(s)\,ds \right), \end{aligned} \tag{7}$$

provided that $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$.

Proof: We have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + \int_0^t \frac{(t-\tau)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)} g(\tau)\,d\tau \\ & -c_0 - c_1 t - c_2 t^2 - \ldots - c_{n-1}t^{n-1}. \end{aligned} \tag{8}$$

For $k = 1, 2, ..., n-1$, we obtain

$$x^{(k)}(0) = -k!c_k.$$

Since $x(0) = x'(0) = \ldots = x^{(n-2)}(0) = 0$, then $c_0 = c_1 = \ldots = c_{n-2} = 0$.
Thanks to Lemma 2, we can write

$$\begin{aligned} I^p x(t) &= \int_0^t \frac{(t-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f(s)\,ds \\ &+ \int_0^t \frac{(t-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} g(s)\,ds - c_{n-1}\frac{\Gamma(n)\,t^{p+n-1}}{\Gamma(p+n)}. \end{aligned} \tag{9}$$

Using the last boundary condition of Lemma 5, we get

$$c_{n-1} = -\frac{\gamma\Gamma(p+n)}{\Gamma(n)\left(\Gamma(p+n)-\gamma\eta^{p+n-1}\right)} \left(I^{p+\alpha} f(\eta) + I^{p+\alpha+\sigma} g(\eta)\right). \tag{10}$$

Substituting $(c_i)_{i=0,..,n-1}$ in (8), we obtain the desired quantity (7).

Let us now introduce the spaces:

$$X := \left\{u(t)\,/u \in C^n\left([0,1], \mathbb{R}^+\right)\right\},$$

$$Y := \left\{v(t)\,/v \in C^n\left([0,1], \mathbb{R}^+\right)\right\}.$$

We define the norms

$$\|u\|_X := \sup_{t\in[0,1]} |u(t)|, \|v\|_Y := \sup_{t\in[0,1]} |v(t)|.$$

For the Banach space $X \times Y$, we define the norm

$$\|(u,v)\|_{X\times Y} := \max\Big(\|u\|_X, \|v\|_Y \Big).$$

3 Main Results

We introduce the following quantities:

$$\begin{aligned} \omega_1 &= \frac{|\gamma|\,\Gamma(p+n)}{\Gamma(n)\left(\Gamma(p+n)-\gamma\eta^{p+n-1}\right)}, \\ \omega_2 &= \frac{|\delta|\,\Gamma(q+n)}{\Gamma(n)\left(\Gamma(q+n)-\delta\zeta^{q+n-1}\right)}, \end{aligned}$$

$$\begin{aligned} M_1 &: = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\sigma+1)} + \frac{|\omega_1|\,\eta^{p+\alpha}}{\Gamma(p+\alpha+1)} + \frac{|\omega_1|\,\eta^{p+\sigma+\alpha}}{\Gamma(p+\alpha+\sigma+1)}, \\ M_2 &: = \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+\theta+1)} + \frac{|\omega_2|\,\zeta^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{|\omega_2|\,\zeta^{q+\theta+\beta}}{\Gamma(q+\theta+\beta+1)}. \end{aligned}$$

Also, we consider the following hypotheses
$(H1)$: There exist non negative real numbers $m_i, m_i', n_i, n_i', (i=1,2)$, such that for all $t\in[0,1], (u_1,v_1), (u_2,v_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} |f_1(t,u_2,v_2) - f_1(t,u_1,v_1)| &\leq m_1|u_2-u_1| + m_2|v_2-v_1|, \\ |g_1(t,u_2,v_2) - g_1(t,u_1,v_1)| &\leq m_1'|u_2-u_1| + m_2'|v_2-v_1|, \\ |f_2(t,u_2,v_2) - f_2(t,u_1,v_1)| &\leq n_1|u_2-u_1| + n_2|v_2-v_1|, \\ |g_2(t,u_2,v_2) - g_2(t,u_1,v_1)| &\leq n_1'|u_2-u_1| + n_2'|v_2-v_1|. \end{aligned}$$

with $\varpi = \max\{m_i, m_i', n_i, n_i'\}\,(i=1,2)$.

$(H2)$: The functions f_1, f_2, g_1 and $g_2 : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous.
$(H3)$: There exist positive constants L_1, L_2, L_1' and L_2', such that for all $t \in [0,1], (u,v)\in\mathbb{R}^2$, we have

$$|f_i(t,u,v)| \leq L_i, i=1,2, \quad |g_i(t,u,v)| \leq L_i', i=1,2$$

Our first main result is based on Banach contraction principle. We have:

Theorem 1. *Suppose that* $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}, \delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ *and assume that* $(H1)$ *holds. If*

$$\varpi \max\left(M_1, M_2\right) < \frac{1}{2}, \tag{11}$$

then the fractional system (1) *has a unique solution on* $[0,1]$.

Proof: Consider the operator $T : X \times Y \to X \times Y$ defined by

$$T\left(u,v\right)\left(t\right) = \left(\ T_1\left(u,v\right)\left(t\right), T_2\left(u,v\right)\left(t\right)\ \right), \tag{12}$$

where,

$$\begin{array}{c} T_1\left(u,v\right)\left(t\right) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s,u\left(s\right),v\left(s\right)\right)ds \\ + \int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\sigma+\alpha)} g_1\left(s,u\left(s\right),v\left(s\right)\right)ds \\ +\omega_1 t^{n-1} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1\left(s,u\left(s\right),v\left(s\right)\right)ds \\ +\omega_1 t^{n-1} \int_0^\eta \frac{(\eta-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} g_1\left(s,u\left(s\right),v\left(s\right)\right)ds \end{array} \tag{13}$$

and

$$\begin{array}{c} T_2\left(u,v\right)\left(t\right) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s,u\left(s\right),v\left(s\right)\right)ds \\ + \int_0^t \frac{(t-s)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} g_2\left(s,u\left(s\right),v\left(s\right)\right)ds \\ +\omega_2 t^{n-1} \int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2\left(s,u\left(s\right),v\left(s\right)\right)ds \\ +\omega_2 t^{n-1} \int_0^\zeta \frac{(\zeta-s)^{q+\beta+\theta-1}}{\Gamma(q+\beta+\theta)} g_2\left(s,u\left(s\right),v\left(s\right)\right)ds. \end{array} \tag{14}$$

We shall show that T is contractive:

Let $(u_1,v_1), (u_2,v_2) \in X \times Y$. Then, for each $t \in [0,1]$, we have

$$\begin{array}{c} \left|T_1\left(u_2,v_2\right)\left(t\right) - T_1\left(u_1,v_1\right)\left(t\right)\right| \leq \\ \leq \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + |\omega_1| \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} ds\right) \\ \times \sup_{0\leq s\leq 1} \left|f_1\left(s,u_2\left(s\right),v_2\left(s\right)\right) - f_1\left(s,u_1\left(s\right),v_1\left(s\right)\right)\right| \\ + \left(\int_0^t \frac{(t-s)^{\sigma+\alpha-1}}{\Gamma(\sigma+\alpha)} ds + |\omega_1| \int_0^\eta \frac{(t-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} ds\right). \\ \times \sup_{0\leq s\leq 1} \left|g_1\left(s,u_2\left(s\right),v_2\left(s\right)\right) - g_1\left(s,u_1\left(s\right),v_1\left(s\right)\right)\right| \end{array} \tag{15}$$

For all $t \in [0,1]$, and using $(H1)$, we can write:

$$\begin{array}{c} \left|T_1\left(u_2,v_2\right)\left(t\right) - T_1\left(u_1,v_1\right)\left(t\right)\right| \\ \leq \left(\frac{1}{\Gamma(\alpha+1)} + \frac{|\omega_1|\eta^{p+\alpha}}{\Gamma(p+\alpha+1)}\right) \\ \times\varpi\left(\sup_{0\leq t\leq 1}\left|u_2\left(t\right) - u_1\left(t\right)\right| + \sup_{0\leq t\leq 1}\left|v_2\left(t\right) - v_1\left(t\right)\right|\right) \\ + \left(\frac{1}{\Gamma(\alpha+\sigma+1)} + \frac{|\omega_1|\eta^{p+\sigma+\alpha}}{\Gamma(p+\alpha+\sigma+1)}\right) \\ \times\varpi\left(\sup_{0\leq t\leq 1}\left|u_2\left(t\right) - u_1\left(t\right)\right| + \sup_{0\leq t\leq 1}\left|v_2\left(t\right) - v_1\left(t\right)\right|\right). \end{array} \tag{16}$$

Therefore,

$$\begin{aligned} &\left|T_1\left(u_2, v_2\right)(t)-T_1\left(u_1, v_1\right)(t)\right| \\ &\leq\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\left|\omega_1\right| \eta^{p+\alpha}}{\Gamma(p+\alpha+1)}\right) \\ &\times 2 \varpi \max \left\{\left\|u_2-u_1\right\|_X,\left\|v_2-v_1\right\|_Y\right\} \\ &+\left(\frac{1}{\Gamma(\alpha+\sigma+1)}+\frac{\left|\omega_1\right| \eta^{p+\sigma}}{\Gamma(p+\alpha+\sigma+1)}\right) \\ &\times 2 \varpi \max \left\{\left\|u_2-u_1\right\|_X,\left\|v_2-v_1\right\|_Y\right\}. \end{aligned} \tag{17}$$

Consequently,

$$\left\|T_1\left(u_2, v_2\right)-T_1\left(u_1, v_1\right)\right\|_X \leq 2 M_1 \varpi\left\|\left(u_2-u_1, v_2-v_1\right)\right\|_{X \times Y}. \tag{18}$$

Similarly,

$$\left\|T_2\left(u_2, v_2\right)-T_2\left(u_1, v_1\right)\right\|_Y \leq 2 M_2 \varpi\left\|\left(u_2-u_1, v_2-v_1\right)\right\|_{X \times Y}. \tag{19}$$

Using (18) and (19), we deduce that

$$\begin{gathered} \left\|T\left(u_2, v_2\right)-T\left(u_1, v_1\right)\right\|_{X \times Y} \leq \\ 2 \varpi \max \left(M_1, M_2\right)\left\|\left(u_2-u_1, v_2-v_1\right)\right\|_{X \times Y}. \end{gathered} \tag{20}$$

Thanks to (11), we conclude that T is a contraction mapping. Hence by Banach fixed point theorem, there exists a unique fixed point which is a solution of (1).

The second main result is given by the following theorem:

Theorem 2. *Suppose that* $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$, $\delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ *and assume that* $(H2)$ *and* $(H3)$ *are satisfied. Then the fractional system* (1) *has at least one solution on* $[0,1]$.

Proof: We show that the operator T is completely continuous. (Note that T is continuous on $X \times Y$ in view of the continuity of f_1, f_2, g_1 and g_2). We proceed on the following steps:

Step 1: Let us take $\overline{L}=\max \left\{L_i, L_i^{\prime}, i=1,2\right\}$ and $B_r:=\{(u, v) \in X \times Y ; \|(u, v)\|_{X \times Y} \leq r, r>0\}$. For $(u, v) \in B_r$, we have

$$\begin{aligned} \left\|T_1(u, v)\right\|_X \leq & \frac{L_1}{\Gamma(\alpha+1)}+\frac{L_1^{\prime}}{\Gamma(\alpha+\sigma+1)} \\ & +\frac{\left|\omega_1\right| L_1 \eta^{p+\alpha}}{\Gamma(p+\alpha+1)}+\frac{\left|\omega_1\right| L_1^{\prime} \eta^{p+\alpha+\sigma}}{\Gamma(p+\alpha+\sigma+1)}, \end{aligned} \tag{21}$$

which implies that

$$\left\|T_1(u, v)\right\|_X \leq \overline{L} M_1<+\infty. \tag{22}$$

Similarly, it can be shown that

$$\|T_2(u,v)\|_Y \leq \overline{L}M_2 < +\infty. \tag{23}$$

By (22) and (23), we obtain

$$\|T(u,v)\|_{X\times Y} \leq \overline{L}\max\{M_1,M_2\} < +\infty. \tag{24}$$

Step 2: The equi-continuity of T :
Let $t_1,t_2 \in [0,1], t_1 < t_2$ and $(u,v) \in B_r$. We have

$$\begin{gathered}\|T_1(u,v)(t_2) - T_1(u,v)(t_1)\|_X \leq \\ \frac{L_1\left(t_2^{\alpha}-t_1^{\alpha}\right)}{\Gamma(\alpha+1)} + \frac{L_1'\left(t_2^{\alpha+\sigma}-t_1^{\alpha+\sigma}\right)}{\Gamma(\alpha+\sigma+1)} \\ +\left(\frac{|\omega_1|L_1\eta^{p+\alpha}}{\Gamma(p+\alpha+1)} + \frac{|\omega_1|L_1'\eta^{p+\alpha+\sigma}}{\Gamma(p+\alpha+\sigma+1)}\right)\left(t_2^{n-1}-t_1^{n-1}\right).\end{gathered} \tag{25}$$

With the same arguments, we can write

$$\begin{gathered}\|T_2(u,v)(t_2) - T_2(u,v)(t_1)\|_Y \leq \\ \frac{L_2\left(t_2^{\beta}-t_1^{\beta}\right)}{\Gamma(\beta+1)} + \frac{L_2'\left(t_2^{\theta+\beta}-t_1^{\theta+\beta}\right)}{\Gamma(\theta+\beta+1)} \\ +\left(\frac{|\omega_2|L_2\eta^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{|\omega_2|L_2'\eta^{q+\beta+\theta}}{\Gamma(q+\beta+\theta+1)}\right)\left(t_2^{n-1}-t_1^{n-1}\right).\end{gathered} \tag{26}$$

As $t_2 \to t_1$, the right-hand sides of the inequalities (25) and (26) tend to zero. Then, as a consequence of steps $1,2$, and by Arzela-Ascoli theorem, we conclude that T is completely continuous.

Next, we show that the set

$$\Omega := \{(u,v) \in X\times Y, (u,v) = \lambda T(u,v), 0<\lambda<1\} \tag{27}$$

is bounded.

Let $(u,v) \in \Omega$, then $(u,v) = \lambda T(u,v)$, for some $0<\lambda<1$. Hence, for $t\in[0,1]$, we have:

$$u(t) = \lambda T_1(u,v)(t), v(t) = \lambda T_2(u,v)(t). \tag{28}$$

Thanks to $(H3)$ and using the same arguments as before, we obtain

$$\begin{aligned}\|u\|_X &\leq \lambda\|T_1(u,v)\|_X \leq \lambda\overline{L}M_1 < +\infty, \\ \|v\|_Y &\leq \lambda\|T_2(u,v)\|_Y \leq \overline{L}M_2 < +\infty.\end{aligned} \tag{29}$$

Hence, we can state that

$$\|(u,v)\|_{X\times Y} \quad \leq \quad \lambda\overline{L}\max\{M_1,M_2\}<+\infty. \tag{30}$$

This shows that Ω is bounded.
As a conclusion of Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of (1).

The third main result is based on Krasnoselskii theorem [9]. We prove the following result:

Theorem 3. *Let* $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$ *and* $\delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$. *Suppose that* $(H1)$,$(H2)$ *and* $(H3)$ *are satisfied. If*

$$\varpi\max\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\sigma+\alpha+1)},\frac{1}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\theta+\beta+1)}\right)<\frac{1}{2}, \tag{31}$$

then, the system (1) *has at least one solution in* $X\times Y$.

Proof: Let us fix $\theta \geq \max(L_1M_1, L_2M_2)$ and consider $B_\theta := \{(u,v)\in X\times Y, ||(u,v)||_{X\times Y}\leq\theta\}$. On B_θ, we define the operators R and S as follows:

$$\begin{array}{c} R(u,v)(t)=(R_1(u,v)(t),R_2(u,v)(t)),\\ S(u,v)(t)=(S_1(u,v)(t),S_2(u,v)(t)), \end{array} \tag{32}$$

where,

$$\begin{array}{c} R_1(u,v)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,u(s),v(s))\,ds\\ +\int_0^t\frac{(t-s)^{\sigma+\alpha-1}}{\Gamma(\sigma+\alpha)}g_1(s,u(s),v(s))\,ds, \end{array} \tag{33}$$

$$\begin{array}{c} R_2(u,v)(t)=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f_2(s,u(s),v(s))\,ds\\ +\int_0^t\frac{(t-s)^{\theta+\beta-1}}{\Gamma(\theta+\beta)}g_2(s,u(s),v(s))\,ds, \end{array} \tag{34}$$

$$\begin{array}{c} S_1(u,v)(t)=\omega_1t^{n-1}\int_0^\eta\frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)}f_1(s,u(s),v(s))\,ds\\ +\omega_1t^{n-1}\int_0^\eta\frac{(\eta-s)^{p+\sigma-1}}{\Gamma(p+\sigma)}g_1(s,u(s),v(s))\,ds \end{array} \tag{35}$$

and

$$\begin{array}{c} S_2(u,v)(t)=\omega_2t^{n-1}\int_0^\zeta\frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)}f_2(s,u(s),v(s))\,ds\\ +\omega_2t^{n-1}\int_0^\zeta\frac{(\zeta-s)^{q+\theta-1}}{\Gamma(q+\theta)}g_2(s,u(s),v(s))\,ds. \end{array} \tag{36}$$

Let $(u_1,v_1),(u_2,v_2)\in B_\theta$. Using $(H3)$, for all $t\in[0,1]$, we can write

$$|R_1(u_1,v_1)(t)+S_1(u_2,v_2)(t)|$$

$$\begin{aligned}
&\leq \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds + |\omega_1| \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)}ds\right) \\
&\times \sup_{0\leq s\leq 1} \left|f_1\left(s,u_1\left(s\right),v_1\left(s\right)\right)\right| \\
&+\left(\int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\sigma+\alpha)}ds + |\omega_1| \int_0^t \frac{(t-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)}ds\right) \\
&\times \sup_{0\leq s\leq 1} \left|g_1\left(s,u_2\left(s\right),v_2\left(s\right)\right)\right|.
\end{aligned} \tag{37}$$

Therefore,

$$\left\|R_1\left(u_1,v_1\right)+S_1\left(u_2,v_2\right)\right\|_X \leq L_1M_1. \tag{38}$$

Similarly, we obtain

$$\left\|R_2\left(u_1,v_1\right)+S_2\left(u_2,v_2\right)\right\|_Y \leq L_2M_2. \tag{39}$$

The inequalities (38) and (39) allow us to obtain

$$\left\|R\left(u_1,v_1\right)+S\left(u_2,v_2\right)\right\|_{X\times Y} \leq \max\left(L_1M_1,L_2M_2\right) \leq \theta. \tag{40}$$

Hence,

$$R\left(u_1,v_1\right)+S\left(u_2,v_2\right) \in B_\theta. \tag{41}$$

Now, we prove that R is contractive:

Using $(H1)$, we can write

$$\begin{aligned}
&\left|R_1\left(u_2,v_2\right)(t)-R_1\left(u_1,v_1\right)(t)\right| \leq \\
&\frac{\varpi}{\Gamma(\alpha+1)}\left(\sup_{0\leq s\leq 1}\left|u_2(s)-u_1(s)\right|+\sup_{0\leq s\leq 1}\left|v_2(s)-v_1(s)\right|\right) \\
&+\frac{\varpi}{\Gamma(\sigma+\alpha+1)}\left(\sup_{0\leq s\leq 1}\left|u_2(s)-u_1(s)\right|+\sup_{0\leq s\leq 1}\left|v_2(s)-v_1(s)\right|\right).
\end{aligned} \tag{42}$$

Thus,

$$\begin{aligned}
&\left\|R_1\left(u_2,v_2\right)-R_1\left(u_1,v_1\right)\right\|_X \leq \\
&\left(\frac{2\varpi}{\Gamma\left(\alpha+1\right)}+\frac{2\varpi}{\Gamma\left(\sigma+\alpha+1\right)}\right)\left\|\left(u_2-u_1,v_2-v_1\right)\right\|_{X\times Y}.
\end{aligned} \tag{43}$$

Similarly,

$$\begin{aligned}
&\left\|R_2\left(u_2,v_2\right)-R_2\left(u_1,v_1\right)\right\|_Y \leq \\
&\left(\frac{2\varpi}{\Gamma\left(\beta+1\right)}+\frac{2\varpi}{\Gamma\left(\theta+\beta+1\right)}\right)\left\|\left(u_2-u_1,v_2-v_1\right)\right\|_{X\times Y}.
\end{aligned} \tag{44}$$

Consequently,

$$\begin{aligned}
&\left\|R\left(u_2,v_2\right)-R\left(u_1,v_1\right)\right\|_{X\times Y} \leq \\
&2\varpi\max\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\sigma+\alpha+1)},\frac{1}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\theta+\beta+1)}\right) \\
&\times\left\|\left(u_2-u_1,v_2-v_1\right)\right\|_{X\times Y}.
\end{aligned} \tag{45}$$

Thanks to (31), we conclude that R is a contraction mapping.

The continuity of f_1, f_2, g_1 and g_2 given in $(H2)$ implies that the operator S is continuous.
Now, we prove the compactness of the operator S :

Let $t_1, t_2 \in [0,1], t_1 < t_2$ and $(u,v) \in B_\theta$. By $(H3)$, we have

$$\|S_1(u,v)(t_2) - S_1(u,v)(t_1)\|_X \leq$$

$$|\omega_1| \left(\frac{L_1 \eta^{p+\alpha}}{\Gamma(p+\alpha+1)} + \frac{L_1' \eta^{p+\alpha+\sigma}}{\Gamma(p+\alpha+\sigma+1)} \right) \left(t_2^{n-1} - t_1^{n-1} \right). \tag{46}$$

With same arguments as before, we have

$$\|S_2(u,v)(t_2) - S_2(u,v)(t_1)\|_Y \leq$$

$$|\omega_2| \left(\frac{L_2 \xi^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{L_2' \xi^{q+\beta+\theta}}{\Gamma(q+\beta+\theta+1)} \right) \left(t_2^{n-1} - t_1^{n-1} \right). \tag{47}$$

The right hand sides of (46) and (47) are independent of (u,v) and tend to zero as $t_1 \to t_2$, so S is relatively compact on B_θ. Then by Ascolli-Arzella theorem, the operator S is compact. Finally, by Krasnoselskii theorem, we conclude that there exists a solution to (1). Theorem 3 is thus proved.

4 Examples

In this section, we present some examples to illustrate the main results.

Example 1. Consider the following fractional differential system:

$$\begin{cases} D^{\frac{5}{2}} u(t) = \frac{\sin(u(t)+v(t))}{32(t^2+1)} + \int_0^t \frac{(t-s)^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \frac{\sin u(s) + \sin v(s)}{16(t+1)} ds + 2t + 1, \\ D^{\frac{5}{2}} v(t) = \frac{|u(t)|+|v(t)|}{(t^2+t+20)(1+|u(t)|+|v(t)|)} + \exp(2t) \\ + \int_0^t \frac{(t-s)^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{|u(s)|}{(15+t)(1+|u(s)|)} + \frac{|v(s)|}{(t+10)(1+|v(s)|)} \right) ds, \\ |u(0)| + |v(0)| + |u'(0)| + |v'(0)| = 0, \\ u''(0) = \gamma I^p u(\eta), \eta \in]0,1[, \\ v''(0) = \delta I^q v(\zeta), \zeta \in]0,1[, \end{cases} \tag{48}$$

where, $\alpha = \beta = \frac{5}{2}, \sigma = \theta = \frac{1}{2}, \eta = \zeta = \frac{1}{2}, p = q = \frac{3}{2}, \gamma = -4, \delta = -8$, and for all $(u, v) \in \mathbb{R}^2, t \in [0, 1]$, we have

$$
\begin{aligned}
f_1(t, u, v) &= \frac{\sin(u+v)}{32(t^2+1)} + 2t + 1, \\
g_1(t, u, v) &= \frac{\sin u + \sin v}{16(t+1)}, \\
f_2(t, u, v) &= \frac{|u| + |v|}{(t^2+t+20)(1+|u|+|v|)} + \exp(2t), \\
g_2(t, u, v) &= \frac{|u|}{(15+t)(1+|u|)} + \frac{|v|}{(t+10)(1+|v|)}.
\end{aligned}
$$

We have also

$$\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}} = 105\sqrt{2\pi}, \delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}} = 105\sqrt{2\pi}.$$

For $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2, t \in [0, 1]$, we have

$$
\begin{aligned}
|f_1(t, u_2, v_2) - f_1(t, u_1, v_1)| &\leq \frac{1}{32}(|u_2 - u_1| + |v_2 - v_1|), \\
|g_1(t, u_2, v_2) - g_1(t, u_1, v_1)| &\leq \frac{1}{16}(|u_2 - u_1| + |v_2 - v_1|), \\
|f_2(t, u_2, v_2) - f_2(t, u_1, v_1)| &\leq \frac{1}{20}(|u_2 - u_1| + |v_2 - v_1|), \\
|g_2(t, u_2, v_2) - g_2(t, u_1, v_1)| &\leq \frac{1}{15}(|u_2 - u_1| + |v_2 - v_1|).
\end{aligned}
$$

It yields then that

$$M_1 = 0.774, M_2 = 0.788, \varpi = \frac{1}{10}, 2\varpi \max(M_1, M_2) = 0.157 < 1.$$

The conditions of Theorem 1 hold. Therefore, the problem (48) has a unique solution on $[0, 1]$.

We give also the following example:

Example 2. We take:

$$
\begin{cases}
D^{\frac{13}{4}} u(t) = \exp\left(-t^2\right)(\sin(u(t)) + \cos(v(t))) \\
\qquad + \int_0^t \frac{(t-s)^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} \sin(t + u(s)v(s))\,ds, t \in [0, 1], \\
D^{\frac{22}{7}} v(t) = \exp\left(t^2\right)(\sin(u(t)) + v(t)) \\
\qquad + \int_0^t \frac{(t-s)^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} \exp(-t|u(s)| - |v(s)|), t \in [0, 1], \\
u(0) = u'(0) = v(0) = v'(0) = 0.
\end{cases} \tag{49}
$$

For this problem, we remark that $\alpha = \frac{13}{4}, \beta = \frac{22}{7}$, $p = 3$, $q = 2$, $\gamma = \delta = \sqrt{2}$, $\eta = \frac{4}{5}$, $\xi = \frac{1}{5}$, and for all $(u, v, z) \in \mathbb{R}^3$, we have

$$\begin{aligned}
f_1(t,u,v) &= \exp\left(-t^2\right)(\sin u + \cos v),\\
g_1(t,u,v) &= (\sin(t+uv)),\\
f_2(t,u,v) &= \exp\left(t^2\right)(\sin u + v),\\
g_2(t,u,v) &= \exp(-t\,|u| - |v|).
\end{aligned}$$

It's clear that the conditions of Theorem 2 are satisfied. Then the problem (49) has at least one solution on $[0, 1]$.

References

[1] B. Ahmad and J. J. Nieto: *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions,* Computers Mathematics with Applications, **58**, n. 9, 1838-1843, 2009.

[2] A. Anber, S. Belarbi, Z. Dahmani: *New existence and uniqueness results for fractional differential equations,* An. St. Univ. Ovidius Constanta, **21**, n. 3, 3341, 2013.

[3] C.Z. Bai, J.X. Fang: *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations,* Applied Mathematics and Computation, **150**, n. 3, 611-621, 2004.

[4] M.A. Bengrine, Z. Dahmani: *Boundary value problems for fractional differential equations,* Int. J. Open problems compt. Math., **5**, n. 4, 1-10, 2012.

[5] Z. Cui. P. Yu, Z. Mao: *Existence of solutions for nonlocal boundary value problems of nonlinear fractional differential equations,* Advances in Dynamical Systems and Applications, **7** n. 1, 31-40, 2012.

[6] M. Gaber, M.G. Brikaa: *Existence results for a couple system of nonlinear fractional differential equation with three point boundary conditions,* Journal of Fractional Calculus and Applications, **3**, n. 21, 1-10, 2012.

[7] V. Gafiychuk, B. Datsko, and V. Meleshko: *Mathematical modeling of time fractional reaction-diffusion systems,* Journal of Computational and Applied Mathematics, **220**, n. 1-2, 215-225, 2008.

[8] A.A. Kilbas, S.A. Marzan: *Nonlinear differential equation with the Caputo fraction derivative in the space of continuously differentiable functions,* Differ. Equ., **41**, n. 1, 84-89, 2005.

[9] M.A. Krasnoselskii: Positive solutions of operator equations, Nordhoff Groningen Netherland, 1964.

[10] V. Lakshmikantham, A.S. Vatsala: *Basic theory of fractional differential equations,* Nonlinear Anal., **69**, n. 8, 2677-2682, 2008.

[11] F. Mainardi: Fractional calculus: some basic problem in continuum and statistical mechanics. Fractals and fractional calculus in continuum mechanics, Springer, Vienna, 1997.

[12] X. Su: *Boundary value problem for a coupled system of nonlinear fractional differential equations,* Applied Mathematics Letters, **22**, n. 1, 64-69, 2009.