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On an initial-value problem for second order partial differential equations with self-reference

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Abstract. In this paper, we study the local existence and uniqueness of the solution to an initial-value problem for a second-order partial differential equation with self-reference.

Keywords: Cauchy problem, second-order partial differential equation, self-reference

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1 Introduction

In [1], Eder obtained the existence, uniqueness, analyticity and analytic dependence of solutions to the following equation of an one-variable unknown function $u : I \subset \mathbb{R} \to \mathbb{R}$:

$$u'(t) = u\left(u(t)\right). \tag{1.1}$$

This is so-called a differential equation with self-reference, since the right-hand side is the composition of the unknown and itself. This equation has attracted much attention. As a more general case than (1.1), Si and Cheng [4] investigated the functional-differential equation

$$u'(t) = u\left(at + bu(t)\right), \tag{1.2}$$

where $a \neq 1$ and $b \neq 0$ are complex numbers; the unknown $u : \mathbb{C} \to \mathbb{C}$ is a complex function. By using the power series method, analytic solutions of this equation are obtained. By generalizing (1.2), in [9] Cheng, Si and Wang considered the equation

$$\alpha t + \beta u'(t) = u\big(at + bu'(t)\big),$$

where α and β are complex numbers. Existence theorems are established for the analytic solutions, and systematic methods for deriving explicit solutions are also given.

In [11], Stanek studied maximal solutions of the functional-differential equation

$$u(t)u'(t) = ku\left(u(t)\right) \tag{1.3}$$

with $0 < |k| < 1$. Here $u : I \subset \mathbb{R} \to \mathbb{R}$ is a real unknown. This author showed that properties of maximal solutions depend on the sign of the parameter k for two separate cases $k \in (-1, 0)$ and $k \in (0, 1)$. For earlier work of Stanek than (1.3), see [16]–[21].

For a more general model than the above, in [6], Miranda and Pascali studied the existence and uniqueness of a local solution to the following initial-valued problem for a partial differential equation with self-reference and heredity

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = u\left(\displaystyle\int_0^t u(x,s)ds, t\right), \ x \in \mathbb{R}, \ \text{a.e. } t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}, \end{cases} \tag{1.4}$$

by assuming that u_0 is a bounded, Lipschitz continuous function. With suitable weaker conditions on u_0, namely u_0 is a non-negative, non-decreasing, bounded, lower semi-continuous real function, in [3], Pascali and Le obtained the existence of a global solution of (1.4).

In [22], T. Nguyen and L. Nguyen, generalizing [7], studied the system of partial differential equations with self-reference and heredity

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = u\left(\alpha v(x,t) + v\left(\displaystyle\int_0^t u(x,s)ds, t\right) t\right), \\ \dfrac{\partial}{\partial t}v(x,t) = v\left(\beta u(x,t) + u\left(\displaystyle\int_0^t v(x,s)ds, t\right) t\right), \end{cases} \tag{1.5}$$

associated with initial conditions

$$u(x,0) = u_0(x), \ v(x,0) = v_0(x), \tag{1.6}$$

where α and β are non-negative coefficients. By the boundedness and Lipschitz continuity of u_0 and v_0, we obtained the existence and uniqueness of a local solution to this system. We also proved that this system has a global solution, provided u_0 and v_0 are non-negative, non-decreasing, bounded and lower semi-continuous functions.

In [5], Pascali and Miranda considered an initial-valued problem for a second-order partial differential equation with self-reference as follows:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(x,t) = k_1 u\left(\dfrac{\partial^2}{\partial t^2}u(x,t) + k_2 u(x,t), t\right), \\ u(x,0) = \alpha(x), \\ \dfrac{\partial}{\partial t}u(x,0) = \beta(x). \end{cases} \tag{1.7}$$

These authors proved that if $\alpha(x)$ and $\beta(x)$ are bounded and Lipschitz continuous functions, k_1 and k_2 are given real numbers, this problem has a unique local solution. It is noted that this result still holds when $k_i \equiv k_i(x,t)$, $i = 1, 2$, are real functions satisfying some technical conditions.

Motivated from problem (1.7) and related questions in [5], in this paper we establish the existence and uniqueness of a local solution to the following Cauchy problem of an partial differential equation with self-reference:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(x,t) = \mu_1 u\left(\dfrac{\partial^2}{\partial t^2}u(x,t) + \mu_2 u\left(\dfrac{\partial^2}{\partial t^2}u(x,t) + \mu_3 u(x,t), t\right), t\right), \\ u(x,0) = p(x) \\ \dfrac{\partial}{\partial t}u(x,0) = q(x), \end{cases} \tag{1.8}$$

where p and q are given functions, μ_i, $i = 1, 2, 3$, given real numbers $x \in \mathbb{R}$ and $t \in [0, T]$ for some $T > 0$. It is clear that this problem is a non-trivial generalization of (1.7). Let us specify some reasons as follows:

- The operator

  $$\frac{\partial^2}{\partial t^2}u(x,t) + \mu_2 u\left(\frac{\partial^2}{\partial t^2}u(x,t) + \mu_3 u(x,t), t\right)$$

  is actually a doubly self-reference form, which is more complicated than that of (1.7);

- If $k_2 = \mu_2 = 0$, problem (1.8) coincides with problem (1.7). This is the only coincidence of these two problems. This means that the problem we study in this paper is not a “natural” generalization of (1.7), not including (1.7) as a special case.

Finally we present the problem (1.8) in the case that $p(x) = p_0$ and $q(x) = q_0$, where p_0 and q_0 are two given constants and we remark a particular strange situation.

2 Existence and uniqueness of a local solution

By integrating the partial differential equation in (1.8), we obtain the following integral equation:

$$u(x,t) = u_0(x,t) + \int_0^t \int_0^\tau \mu_1 u\Big(\frac{\partial^2}{\partial s^2}u(x,s) + \mu_2 u\Big(\frac{\partial^2}{\partial s^2}u(x,s) + \mu_3 u(x,s), s\Big), s\Big) ds d\tau, \quad (2.9)$$

where $u_0(x,t) = p(x) + tq(x)$ and $x \in \mathbb{R}$ and $t \in [0,T]$.

The following theorem is so clear that its proof is omitted.

Theorem 2.1. If u is a continuous solution of problem (2.9), then it is also a solution of problem (1.8).

This theorem allows us to consider problem (2.9) only in the rest of this paper. For simplicity, we assume that $|\mu_1| = |\mu_2| = |\mu_3| = 1$. Now we state our main result.

Theorem 2.2. Assume that p and q are bounded and Lipschitz continuous on $\mathbb{R}$. Let σ be the lipschitz constant of p and assume that $\sigma < 1$. Then there exists a positive constant T_0 such that problem (2.9) has a unique solution, denoted by $u_\infty(x,t)$, in $\mathbb{R}\times[0,T_0]$. Moreover, the function $u_\infty(x,t)$ is also bounded and Lipschitz continuous with respect to each of variables $x \in \mathbb{R}$ and $t \in [0,T_0]$.

Proof. To prove this theorem, we use an iterative algorithm. The proof includes some steps as below.

Step 1:An iterate sequence of functions. We define the following sequence of real functions $(u_n)_n$ defined for $x \in \mathbb{R}, t \in [0.T]$ for $T > 0$:

$$\begin{aligned} u_0(x,t) &= p(x) + tq(x), \\ u_1(x,t) &= u_0(x,t) + \int_0^t \int_0^\tau \mu_1 u_0\Big(\mu_2 u_0(\mu_3 u_0(x,s), s), s\Big) ds d\tau, \\ u_{n+1}(x,t) &= u_0(x,t) + \int_0^t \int_0^\tau \mu_1 u_n\Big(\frac{\partial^2}{\partial s^2}u_n(x,s) + \mu_2 u_n\Big(\frac{\partial^2}{\partial s^2}u_n(x,s) \\ &\quad + \mu_3 u_n(x,s), s\Big), s\Big) ds d\tau. \end{aligned} \quad (2.10)$$

Step 2: Proof of the boundedness of (u_n). With simple calculations, taking into account the boundedness of p and q, we get

$$|u_0(x,t)| \le |p(x)| + t|q(x)| \le \|p\|_{L^\infty} + t\|q\|_{L^\infty},$$

$$
\begin{aligned}
|u_1(x,t)| &\le |u_0(x,t)| + \int_0^t \int_0^\tau \Big|\mu_1 u_0(\mu_2 u_0(\mu_3 u_0(x,s),s),s)\Big| ds d\tau \\
&\le \|p\|_{L^\infty} + t\|q\|_{L^\infty} + \int_0^t \int_0^\tau \Big(\|p\|_{L^\infty} + s\|q\|_{L^\infty}\Big) ds d\tau \\
&= \Big(1+\frac{t^2}{2!}\Big)\|p\|_{L^\infty} + \Big(t+\frac{t^3}{3!}\Big)\|q\|_{L^\infty}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|u_2(x,t)| \le |u_0(x,t)| + \int_0^t \int_0^\tau &\Big|\mu_1 u_1\Big(\frac{\partial^2}{\partial s^2} u_1(x,s) + \mu_2 u_1\Big(\frac{\partial^2}{\partial s^2} u_1(x,s) \\
&+ \mu_3 u_1(x,s), s\Big), s\Big)\Big| ds d\tau \\
\le \|p\|_{L^\infty} + t\|q\|_{L^\infty} &+ \int_0^t \int_0^\tau \Big(1+\frac{s^2}{2!}\Big)\|p\|_{L^\infty} + \Big(s+\frac{s^3}{3!}\Big)\|q\|_{L^\infty} ds d\tau \\
= \Big(1+\frac{t^2}{2!}+\frac{t^4}{4!}\Big)&\|p\|_{L^\infty} + \Big(t+\frac{t^3}{3!}+\frac{t^5}{5!}\Big)\|q\|_{L^\infty}.
\end{aligned}
$$

By induction on n we find

$$
|u_n(x,t)| \le e^T\Big(\|p\|_{L^\infty} + \|q\|_{L^\infty}\Big),\ n \in \mathbb{N},\ t \in [0,T]. \tag{2.11}
$$

Step 3: Every u_n is lipschitz with respect to the first variable. From the Lipschitz continuity of p and q

$$
\begin{aligned}
|p(x) - p(y)| &\le \sigma|x-y|,\ \forall\, x,y \in \mathbb{R}, \\
|q(x) - q(y)| &\le \omega|x-y|,\ \forall\, x,y \in \mathbb{R}.
\end{aligned} \tag{2.12}
$$

where $0 < \sigma, \omega$ are real numbers (with $\sigma < 1$ as in the hypotheses).

Using (2.12), we derive

$$
\begin{aligned}
|u_0(x,t) - u_0(y,t)| &\le |p(x) - p(y)| + t|q(x) - q(y)| \\
&\le \Big(\sigma + t\omega\Big)|x-y| := L_0(t)|x-y|,
\end{aligned} \tag{2.13}
$$

where $L_0(t) := \sigma + t\omega$.

In addition,

$$\begin{aligned}|u_1(x,t)-u_1(y,t)| &\le L_0(t)|x-y| + \int_0^t\int_0^\tau \Big|\mu_1 u_0(\mu_2 u_0(\mu_3 u_0(x,s),s),s)\\ &\qquad\qquad - \mu_1 u_0(\mu_2 u_0(\mu_3 u_0(y,s),s),s)\Big| ds d\tau\\ &\le \left(L_0(t) + \int_0^t\int_0^\tau L_0^3(s) ds d\tau\right)|x-y|\\ &:= L_1(t)|x-y|,\end{aligned} \tag{2.14}$$

where $L_1(t) := L_0(t) + \int_0^t\int_0^\tau C_0(s)dsd\tau$, with $C_0(t) := L_0^3(t)$.

Moreover

$$\begin{aligned}\Big|\frac{\partial^2}{\partial t^2}u_1(x,t) - \frac{\partial^2}{\partial t^2}u_1(y,t)\Big| &\le L_0(t)\Big|\mu_2 u_0(\mu_3 u_0(x,t),t) - \mu_2 u_0(\mu_3 u_0(y,t),t)\Big|\\ &\le L_0^2(t)\Big|\mu_3 u_0(x,t) - \mu_3 u_0(y,t)\Big|\\ &\le L_0^3(t)|x-y| := C_0(t)|x-y|.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&|u_2(x,t)-u_2(y,t)|\\ &\le L_0(t)|x-y| + \int_0^t\int_0^\tau L_1(s)\Bigg(\Big|\frac{\partial^2}{\partial s^2}u_1(x,s) - \frac{\partial^2}{\partial s^2}u_1(y,s)\Big|\\ &\qquad + \Big|\mu_2 u_1\Big(\frac{\partial^2}{\partial s^2}u_1(x,s) + \mu_3 u_1(x,s), s\Big)\\ &\qquad\qquad - \mu_2 u_1\Big(\frac{\partial^2}{\partial s^2}u_1(y,s) + \mu_3 u_1(y,s), s\Big)\Big|\Bigg) ds d\tau\\ &\le L_0(t)|x-y| + \int_0^t\int_0^\tau L_1(s)\Big(L_0^3(s)|x-y|\\ &\qquad + L_1(s)\Big(L_0^3(s)|x-y| + L_1(s)|x-y|\Big)\Big) ds d\tau\\ &\le \left(L_0(t) + \int_0^t\int_0^\tau \Big(\Big(L_1(s) + L_1^2(s)\Big)C_0(s) + L_1^3(s)\Big) ds d\tau\right)|x-y|\\ &:= \left(L_0(t) + \int_0^t\int_0^\tau C_1(s) ds d\tau\right)|x-y| := L_2(t)|x-y|,\end{aligned} \tag{2.15}$$

where

$$L_2(t) := L_0(t) + \int_0^t \int_0^\tau C_1(s) ds d\tau,$$

$$C_1(t) := \Big(L_1(t) + L_1^2(t)\Big) C_0(t) + L_1^3(t).$$

Moreover

$$\begin{aligned}
&\Big|\frac{\partial^2}{\partial t^2} u_2(x,t) - \frac{\partial^2}{\partial t^2} u_2(y,t)\Big| \\
&\leq L_1(t)\bigg(\Big|\frac{\partial^2}{\partial t^2} u_1(x,t) - \frac{\partial^2}{\partial t^2} u_1(y,t)\Big| + \Big|\mu_2 u_1\Big(\frac{\partial^2}{\partial t^2} u_1(x,t) \\
&\qquad + \mu_3 u_1(x,t), t\Big) - \mu_2 u_1\Big(\frac{\partial^2}{\partial t^2} u_1(y,t) + \mu_3 u_1(y,t), t\Big)\Big|\bigg) \\
&\leq L_1(t)\bigg(L_0^3(t)|x-y| + L_1(t)\Big(\Big|\frac{\partial^2}{\partial t^2} u_1(x,t) - \frac{\partial^2}{\partial t^2} u_1(y,t)\Big| \\
&\qquad + |\mu_3 u_1(x,t) - \mu_3 u_1(y,t)|\Big)\bigg) \\
&\leq \bigg(\Big(L_1(t) + L_1^2(t)\Big) L_0^3(t) + L_1^3(t)\bigg)|x-y| \\
&= \bigg(\Big(L_1(t) + L_1^2(t)\Big) C_0(t) + L_1^3(t)\bigg)|x-y| := C_1(t)|x-y|.
\end{aligned}$$

Repeating the previous calculation for u_3 we get

$$\begin{aligned}
&|u_3(x,t) - u_3(y,t)| \\
&\leq L_0(t)|x-y| + \int_0^t \int_0^\tau L_2(s)\bigg(\Big|\frac{\partial^2}{\partial s^2} u_2(x,s) \frac{\partial^2}{\partial s^2} u_2(y,s)\Big| \\
&\quad + \Big|\mu_2 u_2\Big(\frac{\partial^2}{\partial s^2} u_2(x,s) + \mu_3 u_2(x,s), s\Big) \\
&\qquad\qquad - \mu_2 u_2\Big(\frac{\partial^2}{\partial s^2} u_2(y,s) + \mu_3 u_2(y,s), s\Big)\Big|\bigg) ds d\tau \qquad (2.16) \\
&\leq \bigg(L_0(t) + \int_0^t \int_0^\tau \bigg(\Big((L_2(s) + L_2^2(s)\Big) C_1(s) + L_2^3(s) ds d\tau\bigg)\bigg)|x-y| \\
&:= \bigg(L_0(t) + \int_0^t \int_0^\tau C_2(s) ds d\tau\bigg)|x-y| := L_3(t)|x-y|,
\end{aligned}$$

where

$$L_3(t) := L_0(t) + \int_0^t \int_0^\tau C_2(s) ds d\tau,$$

and

$$C_2(t) := \Big(L_2(t) + L_2^2(t)\Big) C_1(t) + L_2^3(t).$$

We have also

$$\Big|\frac{\partial^2}{\partial t^2} u_3(x,t) - \frac{\partial^2}{\partial t^2} u_3(y,t)\Big| \le C_2(t)|x-y|.$$

Next, we proceed by induction. Let $L_0(t) := \sigma + t\omega$ and $C_0(t) := L_0^3(t)$,

$$\begin{aligned} C_n(t) &:= \Big(L_n(t) + L_n^2(t)\Big) C_{n-1}(t) + L_n^3(t) \\ L_n(t) &:= L_0(t) + \int_0^t \int_0^\tau C_{n-1}(s) ds d\tau, \ n \ge 1. \end{aligned} \tag{2.17}$$

From (2.13) – (2.16), by induction on n, we obtain

$$|u_{n+1}(x,t) - u_{n+1}(y,t)| \le L_{n+1}(t)|x-y|, \tag{2.18}$$

$$\Big|\frac{\partial^2}{\partial t^2} u_{n+1}(x,t) - \frac{\partial^2}{\partial t^2} u_{n+1}(y,t)\Big| \le C_n(t)|x-y|. \tag{2.19}$$

We introduce a definition. We call (v_n) a stationary sequence in x if

$$|v_{n+1}(x,t) - v_n(x,t)| \le f_n(t),$$

where (f_n) is a non-negative sequence of real function defined on $[0,T]$. If $f_n = f$ for all n, we say that (v_n) is uniformly stationary sequence in x.

Step 4: (u_n) and $(\frac{\partial^2}{\partial t^2} u_n)$ are stationary sequence in x. Direct calculations show that

$$\begin{aligned} |u_1(x,t) - u_0(x,t)| &= \int_0^t \int_0^\tau \Big|\mu_1 u_0\Big(\mu_2 u_0(\mu_3 u_0(x,s),s),s\Big)\Big| ds d\tau \\ &\le \int_0^t \int_0^\tau \Big(\|p\|_{L^\infty} + t\|q\|_{L^\infty}\Big) ds d\tau \\ &= \frac{t^2}{2}\|p\|_{L^\infty} + \frac{t^3}{6}\|q\|_{L^\infty} := A_1(t). \end{aligned} \tag{2.20}$$

$$\begin{aligned} \Big|\frac{\partial^2}{\partial t^2} u_1(x,t) - \frac{\partial^2}{\partial t^2} u_0(x,t)\Big| &= \Big|\mu_1 u_0\Big(\mu_2 u_0(\mu_3 u_0(x,s),s),s\Big)\Big| \\ &\le \|p\|_{L^\infty} + t\|q\|_{L^\infty} := B_1(t). \end{aligned} \tag{2.21}$$

From (2.20) and (2.21), we deduce

$$A_1(t) := \int_0^t \int_0^\tau B_1(s) ds d\tau. \tag{2.22}$$

$$\begin{aligned}
&|u_2(x,t) - u_1(x,t)| \\
&\le \int_0^t \int_0^\tau \Big(A_1(s) + L_0(s) \Big(\Big| \frac{\partial^2}{\partial s^2} u_1(x,s) - \frac{\partial^2}{\partial s^2} u_0(x,s) \Big| \\
&\quad + \Big| \mu_2 u_0 \Big(\frac{\partial^2}{\partial s^2} u_1(x,s) + \mu_3 u_1(x,s), s \Big) \\
&\qquad - \mu_2 u_0 \Big(\frac{\partial^2}{\partial s^2} u_0(x,s) + \mu_3 u_0(x,s), s \Big) \Big| \Big) \Big) ds d\tau \\
&\le \int_0^t \int_0^\tau \Big(\Big(1 + L_0(s) + L_0^2(s) \Big) A_1(s) + \Big(L_0(s) + L_0^2(s) \Big) B_1(s) \Big) ds d\tau \\
&:= A_2(t).
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&\Big| \frac{\partial^2}{\partial t^2} u_2(x,t) - \frac{\partial^2}{\partial t^2} u_1(x,t) \Big| \\
&\le A_1(t) + L_0(t) \Big(\Big| \frac{\partial^2}{\partial t^2} u_1(x,t) - \frac{\partial^2}{\partial t^2} u_0(x,t) \Big| + \Big| \mu_2 u_0 \Big(\frac{\partial^2}{\partial t^2} u_1(x,t) \\
&\qquad + \mu_3 u_1(x,t), t \Big) - \mu_2 u_0 \Big(\frac{\partial^2}{\partial t^2} u_0(x,t) + \mu_3 u_0(x,t), t \Big) \Big| \Big) \\
&\le A_1(t) \Big(1 + L_0(t) + L_0^2(t) \Big) + \Big(L_0(t) + L_0^2(t) \Big) B_1(t) \\
&:= B_2(t).
\end{aligned} \tag{2.24}$$

Combining (2.23) and (2.24) gives

$$A_2(t) := \int_0^t \int_0^\tau B_2(s) ds d\tau. \tag{2.25}$$

From (2.20) and (2.23), by inducting on n, we derive

$$|u_{n+1}(x,t) - u_n(x,t)| \le A_{n+1}(t) \tag{2.26}$$

and

$$\Big| \frac{\partial^2}{\partial t^2} u_{n+1}(x,t) - \frac{\partial^2}{\partial t^2} u_n(x,t) \Big| \le B_{n+1}(t), \tag{2.27}$$

where

$$\begin{aligned} A_{n+1}(t) &:= \int_0^t \int_0^\tau B_{n+1}(s) ds d\tau, \\ B_{n+1}(t) &:= \Big(1 + L_{n-1}(t) + L_{n-1}^2(t)\Big) A_n(t) \\ &\quad + \Big(L_{n-1}(t) + L_{n-1}^2(t)\Big) B_n(t),\ n \geq 1. \end{aligned} \tag{2.28}$$

In the following step, we select T_0 for which we prove also that (u_n) and $(\frac{\partial^2}{\partial t^2} u_n)$ are uniformly stationary sequences.

Step 5: Existence of a local solution. Because $\sigma < 1$, we can find $T_0 > 0,\ 0 < M < 1,\ 0 < h < 1$ such that for $t \in [0, T_0]$, we have

$$\sigma + t\omega + M\frac{t^2}{2} \leq M < 2M < h; M + 2M^2 \leq 1;\ 2M + (1 + 2M)\frac{t^2}{2} < h. \tag{2.29}$$

From (2.29) we obtain

$$\begin{aligned} L_0(t) &= \sigma + t\omega \leq M, \\ L_1(t) &\leq \sigma + t\omega + \int_0^t \int_0^\tau M^3 ds d\tau = \sigma + t\omega + M^3\frac{t^2}{2} \leq M, \\ L_2(t) &\leq \sigma + t\omega + \int_0^t \int_0^\tau \Big(M^3 + M^4 + M^5\Big) ds d\tau \leq \sigma + t\omega + M\frac{t^2}{2} \leq M, \\ C_0(t) &= L_0^3(t) \leq M^3 \leq M, \\ C_1(t) &\leq \Big(M + M^2\Big)M + M^3 = M(M + 2M^2) \leq M, \\ C_2(t) &\leq \Big(M + M^2\Big)M + M^3 = M(M + 2M^2) \leq M. \end{aligned} \tag{2.30}$$

Now, by induction on n, we conclude that

$$\begin{aligned} C_n(t) &\leq M, \\ L_{n+1}(t) &\leq \sigma + t\omega + M\frac{t^2}{2} \leq M. \end{aligned} \tag{2.31}$$

Hence we derive

$$
\begin{aligned}
B_2(t) &\le A_1(t)(1+M+M^2) + B_1(t)(M+M^2) \\
&\le (1+M+M^2)\int_0^t \int_0^\tau B_1(s)\,ds\,d\tau + B_1(t)(M+M^2) \\
&\le \|B_1\|_{L^\infty}\frac{t^2}{2}(1+M+M^2) + \|B_1\|_{L^\infty}(M+M^2) \\
&\le \|B_1\|_{L^\infty}\left(\frac{t^2}{2}\Big(1+2M\Big)+2M\right) \\
&\le \|B_1\|_{L^\infty}h.
\end{aligned}
\tag{2.32}
$$

From (2.32) we obtain

$$\|B_2\|_{L^\infty} \le \|B_1\|_{L^\infty}h. \tag{2.33}$$

By a similar argument, we get

$$
\begin{aligned}
B_3(t) &\le \|B_2\|_{L^\infty}\frac{t^2}{2}\Big(1+M+M^2\Big) + \|B_2\|_{L^\infty}(M+M^2) \\
&\le \|B_2\|_{L^\infty}\left(\frac{t^2}{2}\Big(1+2M\Big)+2M\right) \le \|B_2\|_{L^\infty}h.
\end{aligned}
\tag{2.34}
$$

So

$$\|B_3\|_{L^\infty} \le \|B_2\|_{L^\infty}h. \tag{2.35}$$

From (2.33) and (2.35), by induction on n, we conclude that

$$\|B_{n+1}\|_{L^\infty} \le \|B_n\|_{L^\infty}h. \tag{2.36}$$

In addition, from (2.28) we deduce

$$\|A_{n+1}\|_{L^\infty} \le \|B_{n+1}\|_{L^\infty}\frac{T_0^2}{2}. \tag{2.37}$$

Due to (2.36), we see the series $\sum B_{n+1}(t)$ converges absolutely and uniformly, hence by (2.27) there exists ϕ_∞ such that

$$\frac{\partial^2}{\partial t^2}u_n \to \phi_\infty \tag{2.38}$$

uniformly in $\mathbb{R}\times[0,T_0]$.

Similarly, from (2.26)) and (2.37), we conclude that $\sum A_{n+1}(t)$ converges absolutely and uniformly and there exists u_∞ such that

$$u_n \to u_\infty \tag{2.39}$$

uniformly in $\mathbb{R} \times [0, T_*]$.

We remark that $|u_\infty(x,t) - u_\infty(y,t)| \leq M|x-y|$.

Now we are proving that $u_\infty(x,t)$ is a solution of (2.9). It is clear that

$$\begin{aligned}
&\Big|\mu_1 u_n\Big(\frac{\partial^2}{\partial t^2}u_n(x,t) + \mu_2 u_n\Big(\frac{\partial^2}{\partial t^2}u_n(x,t) + \mu_3 u_n(x,t), t\Big), t\Big) \\
&\qquad - \mu_1 u_\infty\Big(\phi_\infty(x,t) + \mu_2 u_\infty\Big(\phi_\infty(x,t) + \mu_3 u_\infty(x,t), t\Big), t\Big)\Big| \\
&\leq \|u_n - u_\infty\|_{L^\infty} + M\bigg(\Big|\frac{\partial^2}{\partial t^2}u_n(x,t) - \phi_\infty(x,t)\Big| \\
&\qquad + \Big|\mu_2 u_n\Big(\frac{\partial^2}{\partial t^2}u_n(x,t) + \mu_3 u_n(x,t), t\Big) - \mu_2 u_\infty\Big(\phi_\infty(x,t) \\
&\qquad + \mu_3 u_\infty(x,t), t\Big)\Big|\bigg) \\
&\leq \|u_n - u_\infty\|_{L^\infty}\Big(1 + M + M^2\Big) \\
&\qquad + \Big\|\frac{\partial^2}{\partial t^2}u_n - \phi_\infty\Big\|_{L^\infty}\Big(M + M^2\Big) \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{2.40}$$

From 2.40, we deduce that

$$\begin{aligned}
u_\infty(x,t) = u_0(x,t) + \int_0^t \int_0^\tau \mu_1 u_\infty\Big(\phi_\infty(x,s) + \mu_2 u_\infty\Big(\phi_\infty(x,s) \\
+ \mu_3 u_\infty(x,s), s\Big), s\Big) ds d\tau.
\end{aligned} \tag{2.41}$$

Moreover, we have

$$\begin{aligned}
&\Big|\phi_\infty(x,t) - \frac{\partial^2}{\partial t^2}u_\infty(x,t)\Big| \le \Big\|\phi_\infty - \frac{\partial^2}{\partial t^2}u_n\Big\|_{L^\infty} + \Big|\frac{\partial^2}{\partial t^2}u_n(x,t) - \frac{\partial^2}{\partial t^2}u_\infty(x,t)\Big| \\
&\le \Big\|\phi_\infty - \frac{\partial^2}{\partial t^2}u_n\Big\|_{L^\infty} + \|u_{n-1} - u_\infty\|_{L^\infty} + M\Big(\Big|\frac{\partial^2}{\partial t^2}u_{n-1}(x,t) \\
&\quad - \frac{\partial^2}{\partial t^2}u_\infty(x,t)\Big| + \Big|\mu_2 u_{n-1}\Big(\frac{\partial^2}{\partial t^2}u_{n-1}(x,t) + \mu_3 u_{n-1}(x,t), t\Big) \\
&\quad - \mu_2 u_\infty\Big(\frac{\partial^2}{\partial t^2}u_\infty(x,t) + \mu_3 u_\infty(x,t), t\Big)\Big|\Big) \\
&\le \Big(\Big\|\phi_\infty - \frac{\partial^2}{\partial t^2}u_n\Big\|_{L^\infty} + \|u_{n-1} - u_\infty\|_{L^\infty}\Big)\Big(1 + M + M^2\Big) \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{2.42}$$

Hence,

$$\begin{aligned}
u_\infty(x,t) = u_0(x,t) + \int_0^t \int_0^\tau &\mu_1 u_\infty\Big(\frac{\partial^2}{\partial s^2}u_\infty(x,s) \\
&+ \mu_2 u_\infty\Big(\frac{\partial^2}{\partial s^2}u_\infty(x,s) + \mu_3 u_\infty(x,s), s\Big), s\Big) ds d\tau,
\end{aligned} \tag{2.43}$$

for all $x \in \mathbb{R},\ t \in [0, T_0]$. Then u_∞ is a solution of (2.9) in $\mathbb{R} \times [0, T_0]$.

Step 6: Uniqueness of the local solution u_∞. We assume that there exists another lipschitz solution $u_\star(x,t)$ of (2.9). Then

$$
\begin{aligned}
&|u_\star(x,t) - u_\infty(x,t)| \\
&\leq \int_0^t \int_0^\tau \Bigg(\bigg| u_\star\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star(x,s), s\bigg), s\bigg) \\
&- u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star(x,s), s\bigg), s\bigg) \bigg| \\
&+ \bigg| u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star(x,s), s\bigg), s\bigg) \\
&- u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\infty(x,s) + u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\infty(x,s) + u_\infty(x,s), s\bigg), s\bigg) \bigg| \Bigg) ds d\tau \\
&\leq \int_0^t \int_0^\tau \Bigg(\|u_\star - u_\infty\|_{L^\infty} + M\bigg(\bigg| \frac{\partial^2}{\partial s^2} u_\star(x,s) - \frac{\partial^2}{\partial s^2} u_\infty(x,s) \bigg| \\
&+ \bigg| u_\star\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star(x,s), s\bigg) - u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\infty(x,s) + u_\infty(x,s), s\bigg) \bigg| \bigg) \Bigg) ds d\tau \\
&\leq \int_0^t \int_0^\tau \Bigg(\|u_\star - u_\infty\|_{L^\infty} + M\bigg(\bigg| \frac{\partial^2}{\partial s^2} u_\star(x,s) - \frac{\partial^2}{\partial s^2} u_\infty(x,s) \bigg| \\
&+ \bigg| u_\star\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star(x,s), s\bigg) - u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star(x,s), s\bigg) \bigg| \\
&+ \bigg| u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\star(x,s) + u_\star(x,s), s\bigg) - u_\infty\bigg(\frac{\partial^2}{\partial t^2} u_\infty(x,s) + u_\infty(x,s), s\bigg) \bigg| \bigg) \Bigg) ds d\tau \\
&\leq \int_0^t \int_0^\tau \Bigg(\|u_\star - u_\infty\|_{L^\infty} + M\bigg(\bigg\| \frac{\partial^2}{\partial s^2} u_\star - \frac{\partial^2}{\partial s^2} u_\infty \bigg\|_{L^\infty} + \|u_\star - u_\infty\|_{L^\infty} \\
&+ M\bigg(\bigg\| \frac{\partial^2}{\partial s^2} u_\star - \frac{\partial^2}{\partial s^2} u_\infty \bigg\|_{L^\infty} + \|u_\star - u_\infty\|_{L^\infty} \bigg) \bigg) \Bigg) ds d\tau \\
&\leq \int_0^t \int_0^\tau \bigg(\Big(1 + M + M^2 \Big) \|u_\star - u_\infty\|_{L^\infty} \\
&+ \Big(M + M^2 \Big) \bigg\| \frac{\partial^2}{\partial s^2} u_\star - \frac{\partial^2}{\partial s^2} u_\infty \bigg\|_{L^\infty} \bigg) ds d\tau.
\end{aligned}
\tag{2.44}
$$

Additionally,

$$
\begin{aligned}
&\Big|\frac{\partial^2}{\partial t^2}u_\star(x,t)-\frac{\partial^2}{\partial t^2}u_\infty(x,t)\Big|\\
&=\Bigg|u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg),t\bigg)\\
&-u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\infty(x,t)+u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\infty(x,t)+u_\infty(x,t),t\bigg),t\bigg)\Bigg|\\
&\le\Bigg|u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg),t\bigg)\\
&-u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg),t\bigg)\Bigg|\\
&+\Bigg|u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg),t\bigg)\\
&-u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\infty(x,t)+u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\infty(x,t)+u_\infty(x,t),t\bigg),t\bigg)\Bigg|\\
&\le\|u_\star-u_\infty\|_{L^\infty}+M\Bigg(\Big|\frac{\partial^2}{\partial t^2}u_\star(x,t)-\frac{\partial^2}{\partial t^2}u_\infty(x,t)\Big|\\
&+\Big|u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg)-u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\infty(x,t)+u_\infty(x,t),t\bigg)\Big|\Bigg)\\
&\le\|u_\star-u_\infty\|_{L^\infty}+M\Bigg(\Big\|\frac{\partial^2}{\partial t^2}u_\star-\frac{\partial^2}{\partial t^2}u_\infty\Big\|_{L^\infty}\\
&+\Big|u_\star\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg)-u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg)\Big|\\
&+\Big|u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\star(x,t)+u_\star(x,t),t\bigg)-u_\infty\bigg(\frac{\partial^2}{\partial t^2}u_\infty(x,t)+u_\infty(x,t),t\bigg)\Big|\Bigg)\\
&\le\|u_\star-u_\infty\|_{L^\infty}+M\Bigg(\Big\|\frac{\partial^2}{\partial t^2}u_\star-\frac{\partial^2}{\partial t^2}u_\infty\Big\|_{L^\infty}+\|u_\star-u_\infty\|_{L^\infty}\\
&+M\bigg(\Big|\frac{\partial^2}{\partial t^2}u_\star(x,t)-\frac{\partial^2}{\partial t^2}u_\infty(x,t)\Big|+|u_\star(x,t)-u_\infty(x,t)|\bigg)\Bigg)\\
&\le\|u_\star-u_\infty\|_{L^\infty}\Big(1+M+M^2\Big)\\
&\qquad+\Big\|\frac{\partial^2}{\partial t^2}u_\star-\frac{\partial^2}{\partial t^2}u_\infty\Big\|_{L^\infty}\Big(M+M^2\Big)\\
&\le\Big(1+2M\Big)\|u_\star-u_\infty\|_{L^\infty}+2M\Big\|\frac{\partial^2}{\partial t^2}u_\star-\frac{\partial^2}{\partial t^2}u_\infty\Big\|_{L^\infty}.
\end{aligned}
\tag{2.45}
$$

From (2.45), we deduce

$$\left\| \frac{\partial^2}{\partial t^2} u_\star - \frac{\partial^2}{\partial t^2} u_\infty \right\|_{L^\infty} \leq \frac{1+2M}{1-2M} \| u_\star - u_\infty \|_{L^\infty}. \tag{2.46}$$

From (2.11), (2.44) and (2.46), we deduce

$$|u_\star(x,t) - u_\infty(x,t)| \leq \left(\frac{1+2M}{1-2M} \right) \frac{T_0^2}{2} \| u_\star - u_\infty \|_{L^\infty}. \tag{2.47}$$

This shows that $u_\infty \equiv u_*$ and the proof is complete. $\boxed{QED}$

Remark 2.1. It is clear that a trivial example for problem (1.8) is that $u(x,t) = 0$ is a solution for $p(x) = q(x) = 0$.

Remark 2.2. In this paper, we only consider the existence and uniqueness of a local solution to problem (1.8). It is of course interesting to investigate the behavior of this solution for some special cases of the initial more regular data p and q. We do not think that such problems are trivial.

Remark 2.3. A numerical algorithm for problem (1.8) is still open. We believe that various specific differential equations with self-reference of the general form

$$Au(x,t) = u\left(Bu(x,t), t\right),$$

where $A : X \to \mathbb{R}$ and $B : X \to \mathbb{R}$ are two functionals, X is a function space, $u = u(x,t)$, $(x,t) \in \mathbb{R} \times [0, +\infty)$ is an unknown function, can be solved numerically.

3 A remark for particular initial data

Now we present a particular situation that show as the initial value are very important in the study the iterative procedure considered in previous section, in particular if we assume $p(x) = p_0$, $q(x) = q_0$; p_0 and q_0 are two given real constants.

Now, suppose $p(x) = p_0$ and $q(x) = q_0$, where p_0 and q_0 are two given real constants. We consider, as in previous section

$$u_0(x,t) = p_0 + tq_0, \tag{3.48}$$

and remark that

$$\begin{aligned} u_1(x,t) &= u_0(x,t) + \int_0^t \int_0^\tau u_0(u_0(u_0(x,s),s),s)dsd\tau \\ &= u_0(x,t) + \int_0^t \int_0^\tau u_0(u_0(p_0+sq_0,s),s)dsd\tau \\ &= p_0 + tq_0 + \int_0^t \int_0^\tau (p_0+sq_0)dsd\tau \\ &= p_0 + tq_0 + p_0\frac{t^2}{2} + q_0\frac{t^3}{6} \\ &= p_0\Big(1+\frac{t^2}{2!}\Big) + q_0\Big(t+\frac{t^3}{3!}\Big). \end{aligned} \tag{3.49}$$

Therefore

$$\frac{\partial^2}{\partial t^2}u_1(x,t) = p_0 + tq_0 = u_0(x,t). \tag{3.50}$$

In addition, we get

$$\begin{aligned} u_2(x,t) &= u_0(x,t) + \int_0^t \int_0^\tau u_1\Big(\frac{\partial^2}{\partial s^2}u_1(x,s) + u_1\Big(\frac{\partial^2}{\partial s^2}u_1(x,s) \\ &\qquad + u_1(x,s),s\Big),s\Big)dsd\tau \\ &= u_0(x,t) + \int_0^t \int_0^\tau \Big(p_0\Big(1+\frac{s^2}{2!}\Big) + q_0\Big(s+\frac{s^3}{3!}\Big)\Big)dsd\tau \\ &= p_0\Big(1+\frac{t^2}{2!}+\frac{t^4}{4!}\Big) + q_0\Big(t+\frac{t^3}{3!}+\frac{t^5}{5!}\Big). \end{aligned} \tag{3.51}$$

Now, by induction on k we obtain

$$u_k(x,t) = p_0\sum_{i=0}^{k}\frac{t^{2i}}{(2i)!} + q_0\sum_{i=0}^{k}\frac{t^{2i+1}}{(2i+1)!}.$$

We deduce

$$\begin{aligned} u_{k+1}(x,t) &= u_0(x,t) + \int_0^t \int_0^\tau \Big(p_0\sum_{i=0}^{k}\frac{s^{2i}}{(2i)!} + q_0\sum_{i=0}^{k}\frac{t^{2i+1}}{(2i+1)!}\Big)dsd\tau \\ &= p_0\Big(1+\sum_{i=0}^{k}\frac{t^{2i+2}}{(2i+2)!}\Big) + q_0\Big(t+\sum_{i=0}^{k}\frac{t^{2i+3}}{(2i+3)!}\Big). \end{aligned} \tag{3.52}$$

From (3.48) – (3.52) we obtain

$$u_n(x,t) = p_0\sum_{i=0}^{n}\frac{t^{2i}}{(2i)!} + q_0\sum_{i=0}^{n}\frac{t^{2i+1}}{(2i+1)!}, \tag{3.53}$$

$$\frac{\partial^2}{\partial t^2} u_{n+1}(x,t) = u_{n-1}(x,t). \tag{3.54}$$

Letting n go to infinity, for all $t \in [0,T], T > 0$, we get

$$u_\star(x,t) = \begin{cases} Ce^t, \ p_0 = q_0 = C \\ p_0 \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} + q_0 \sum_{n=0}^\infty \frac{t^{2n+1}}{(2n+1)!} = p_0 \cosh t + q_0 \sinh t, \ p_0 \neq q_0. \end{cases} \tag{3.55}$$

But it is easy to prove that u_* are solution of (1.8). The functions u_* are solution of the ordinary differential equation $\ddot{u}(t) = u(t)$.

Hence we have the following situation. The problem (2.9) generated an integral equation; starting from non-constant initial condition (so that almost one of p, q depend explicitely on x) the iteration procedure give a local solution of problem (2.9). But starting from constant initial condition (p and q togheter constant) the same iteration procedure give a solution of a different problem. This seem to be a very interesting situation.

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