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ON WEAK COTYPE AND WEAK TYPE IN BANACH SPACES

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INTRODUCTION: In 1977, T. Figiel, J. Lindenstrauss and V.D. Milman [6] used a refined version of Dvoretzky's theorem to prove that a Banach space X of cotype q $(q \geq 2)$ enjoys the following property:

(P_q) ***For every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that, for every n and every n-dimensional subspace E of X, we can find a subspace F of E such that***

$$\dim F \geq C_\epsilon n^{2/q} \quad \text{and} \quad d(F, \ell_2^{\dim F}) \leq 1 + \epsilon$$

(here d(., .) denotes the usual Banach-Mazur distance).

In [6] some examples were also given to show that this implication may not be reversed.

Later on, in 1986, property (P_2) was thoroughly investigated by V.D. Milman and G. Pisier [32], who proposed to call it weak cotype 2, in view of the fact that the well-known concept of cotype 2 is modified by replacing in a specific manner ℓ_1-convergence by what is known elsewhere as «weak ℓ_1»-convergence. More precisely, one of the results contained in [32] asserts that X has weak cotype 2 if and only if there exists a constant C such that, for all n,

$$(*) \qquad \sigma_{1,\infty}^a(vu) := \sup_k k a_k(vu) \leq C\pi_\gamma(u)\pi_2(v), \qquad \forall u \in L(\ell_2^n, X), \quad v \in L(X, \ell_2^n),$$

where $a_k(\,.)$ denotes the k-th approximation number and π_γ (resp. π_2) is the γ-summing (resp. 2-summing) idea1 norm (see §0 for the definitions). The usual cotype 2 property is obtained by replacing in $(*)$ $\sigma_{1,\infty}^a(vu)$ by the ℓ_1-norm

$$\sigma_1^a(vu) := \sum_k a_k(vu),$$

which is known to define the trace class norm for operators on Hilbert spaces.

Motivated by this, G. Pisier [43] went on only recently to exploit such concepts further and to develop in particular a theory of so-called weak Hilbert spaces. In this work, he also introduces a procedure to define weak properties in general.

Starting from this general point of view, we intend to develop to some extent a theory of weak cotype an weak type. This will be done in §2 and §3, after we have provided the necessary background on weak properties in § 1.

We shall clarify, in the context of local Banach space theory, the relations of weak cotype and weak type to distance to Hilbert spaces, volume ratios, and spaces of vector-valued L_p-functions , and we shall discuss extension properties of certain operators.

It will turn out that severa1 known consequences of cotype and type actually characterize weak cotype and weak type, thus allowing a deeper insight in the local theory of Banach spaces. Generalizations of old results and «weak analogues» of well-known theorems (of Grothendieck's Theorem, for instance) will also be obtained.

Among others, we shall see that for $q > 2$ (resp. $p < 2$) spaces of weak cotype q (resp. weak type p) show a behaviour which is different from what is known for $q = 2$ (resp. $p = 2$). For example, weak cotype q coincides with a well-known propery introduced by L. Tzafriri [47] and called equal-norm cotype q , provided $q > 2$, whereas in case $q = 2$ this latter notion is known to be the same as cotype 2, cf. [12] (of course, an analogue statement holds for weak type p, $p < 2$).

The concluding §4 contains some further results related to Hilbert spaces. We shall prove that being a weak Hilbert space is not a three space property, and we shall generalize some characterizations of weak Hilbert spaces to Banach spaces having weak type p and weak cotype $p/(p-1)$, $1 < p \leq 2$.

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0. NOTATION AND BACKGROUND

(0.1) x, Y, ... E, F will denote (mal) Banach spaces. The letters $E, F, \ldots,$ will be reserved for finite-dimensiona1 spaces. Given X , we will denote by X* its dual and by B_X its closed unit ball, i.e. $\{x \in X: \|x\| \leq 1\}$, where $\|.\|$ is the norm in X. The canonical embedding of a Banach space X in its bidual X** will be denoted by K_X . The family of all finite-dimensional (resp. of all finite-codimensional) subspaces of X will be denoted by Dim (X) (resp. by Cod (X)).

If $\dim(E) = \dim(F) < \infty$, then $d(E, F) := \inf\{\|T\|\|T^{-1}\| :$ T anisomorphism $E \to F\}$ is the so-called ***Banach-Mazur distance*** between ***E an F.***

(0.2) The set of all operators (= continuous linear maps) between X and Y is denoted by $L($ X, Y) and is endowed with the usual operator norm. T^* is the continuous adjoint of an operator T .

(0.3) We shall use the standard Banach spaces

$$\ell_p := \{(\alpha_k) \in \mathbf{R}^{\mathbf{N}} : \Sigma |\alpha_k|^p < \infty\}, \quad 1 \leq p < \infty,$$

$$\ell_\infty := \{(\alpha_k) \in \mathbf{R}^{\mathbf{N}} : \sup |\alpha_k| < \infty\},$$

with the norms

$$||(\alpha_k)||_p := (\Sigma|\alpha_k|^p)^{1/p}, \quad p < \infty,$$

$$||(\alpha_k)||_\infty := \sup|\alpha_k|.$$

Theindex p in $||\cdot||_p$ will often be dropped. If n $\in \mathbb{N}$, $\ell_p^n(1 \leq p \leq \infty)$ is the n-dimensional analogue of ℓ_p. Note that $(\ell_p^n)^* = \ell_{p^*}^n$ isometrically, where $p^* = \frac{p}{(p-1)}$, with the usual conventions if p = 1 or p = ∞.

(0.4) We say that ***X contains the*** ℓ_p^n***'s uniformly*** if there exists a constant C such that, for each $n \in \mathbb{N}$, there is an isomorphic embedding $j_n : \ell_p^n \to X$ such that $||j_n|| \, ||j_n^{-1}|| < C$.

(0.5) X is ***K-convex*** if and only if X contains the ℓ_1^n 's uniformly (see [40]).

(0.6) Let p $\in$ [1, ∞] . A Banach space X is an $\mathcal{L}_p$*-space* if there is an $\epsilon > 0$ such that, for every $E \in$ Dim (X) , we can find an $F \in$ Dim (X) containing E such that d(F, $\ell_p^{\dim F}$) $\leq$ 1+ ϵ. X is an $\mathcal{L}_2$ -space if and only if it is isomorphic to a Hilbert space. Details on $\mathcal{L}_p$-spaces can be found in [23].

(0.7) Let p $\in$ (0 , ∞) . Given $(\alpha_k) \in$ IR" , denote by (α_k^*) the nonincreasing rearrangement of ($|\alpha_k|$) . Then ***we*** can define ***the Lorentz sequence spaces***

$$\ell_{p,1} := \{(\alpha_n) \in \mathbb{R}^{\mathbb{N}} : \Sigma \alpha_n^* n^{-1/p^*} < \infty\}$$

and

$$\ell_{p,\infty} := \{(\alpha_n) \in \mathbb{R}^{\mathbb{N}} : \sup \alpha_n^* n^{1/p} < \infty\},$$

endowed with the quasi-norms

$$\sigma_{p,1}((\alpha_n)) := \Sigma \alpha_n^* n^{-1/p^*} \quad (\text{resp } .\sigma_{p,\infty}((a_n)) := \sup \alpha_n^* n^{1/p}).$$

$\ell_{p,1}$ and $\ell_{p,\infty}$ are thus complete quasi-normed spaces. Equivalent norms can be given if p $\in$ (1, ∞) (see [38] 13.9.5). We shall not explicitly deal with the more general Lorentz sequence spaces $\ell_{p,q}$.

(0.8) An ***Orlicz jiinction*** $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing and convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. Given such ***an*** M, ***we*** define ***the Orlicz sequence space*** ℓ_M by

$$\ell_M := \left\{(\alpha_n) \in \mathbb{R}^{\mathbb{N}} : \Sigma M\left(\frac{|\alpha_n|}{\rho}\right) < \infty \quad \text{for some } \rho > 0\right\}$$

with the norm

$$||(\alpha_n)|| := \inf\left\{\rho > 0 : \Sigma M\left(\frac{|\alpha_n|}{\rho}\right) \leq 1\right\}.$$

ℓ_M is a Banach space. An extensive account of the theory of Orlicz sequence spaces is given in [25].

(0.9) As concerns quasi-normed operator ideals, we adopt more or less the notation of A. Pietsch's book [38]. In particular, all the components of a given quasi-normed ideal are supposed to be quasi-Banach spaces (with respect to the ideal quasi-norm under consideration). If A is a quasi-normed ideal with the quasi-norm α (denoted by $[A, \alpha]$), $[A^d, \alpha^d]$ denotes the *dual ideal*. An operator T is in A^d if an only if T^* is in A, and in this case $\alpha^d(T) := \alpha(T^*)$. Further, $[A^*, \alpha^*]$ denotes the *adjoint ideal*. Recall that if X (or Y) is finite-dimensional, then T is in $A^*(X, Y)$ if and only if

$$\alpha^*(T) := \sup\{tr(TS) : S \in L(Y,X), ||S|| \leq 1\}$$

is finite. Here tr denotes the usual trace of finite rank operators. We shall use the fact that, if $[A, \alpha]$ is a normed ideal, $(A^d)^* = (A^*)^d$ isometrically, i.e. $(\alpha^d)^*$ and $(\alpha^*)^d$ coincide as well (see [38] 9.1.6).

(0.10) Let $[A, \alpha]$ and $[B, \beta]$ be quasi-normed ideals. Using Pietsch's notation (see [38] Ch. 7), an operator $T \in L(X, Y)$ belongs to the «left-hand quotient» $A^{-1} \cdot B$ whenever

$$\alpha^{-1} \cdot \beta(T) := \sup\{\beta(ST) : S \in A(Y,Z), \alpha(S) \leq 1\} < \infty.$$

Here Z ranges over all Banach spaces. $\alpha^{-1} \cdot \beta$ is a quasi-nomi on $A^{-1} \cdot B$ and a norm if β is. The «right-hand quotient» $A \cdot B^{-1}$ and its quasi-norm $\alpha \cdot \beta^{-1}$ are defined analogously. If XisaBanachspace,wewrite $B(.,X) \subset A(.,X)$ (resp. $A(X,.) \subset B(X,.)$) whenever the identiy map id, belongs to $A \cdot B^{-1}$ (resp. to $A^{-1} \cdot B$).

(0.11) For $0 < p \leq q < \infty$, *the* ideal $\Pi_{q,p}$ of all (q, p) *-summing operators* consists of all operators T: X $\rightarrow$ Y for which a constant C exists such that

$$\left(\sum_{i=1}^{n} ||Tx_i||^q\right)^{1/q} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x^*, x_i\rangle|^p\right)^{1/p}$$

for all finite sequences $x_1, \ldots, x_n \in X$. The least such C is denoted by $\pi_{q,p}(T)$. This tums $\Pi_{q,p}$ into a quasi-normed ideal (it is normed if $p \geq 1$). If $p = q$ *we* write $[\Pi_p, \pi_p]$ instead of $[\Pi_{p,p}, \pi_{p,p}]$: this is the idea1 of *p-summing operators*.

We shall in particular use the follwing properties of 2-summing operators:

(0.12) $[\Pi_2, \pi_2] = [\Pi_2^*, \pi_2^*]$ (see [38] 19.2.8 and 19.2.13).

(0.13) A particular case of the Pietsch factorization theorem states that T: X $\rightarrow$ Y is 2-summing if and only if there exist a compact space K, a probability measure μ on K andoperators $A \in L(X, C(K))$, $B \in L(L_2(K, \mu), Y)$ such that $T = BJ_2A$, where

$J_2 : C(K) \to L_2(K, \mu)$ is the canonica1 injection. From the metric extension property of the spaces $L_\infty(\mu)$ one deduces that, given $T \in \Pi_2(X, Y)$ and a Banach space Z containing X as a subspace, there is an extension $T' \in \Pi_2(Z, Y)$ of such that $\pi_2(T') = \pi_2(T)$ (see [38] 17.3.7 and C.3.2).

(0.13') The following statements follow from the fundamental Grothendieck's inequality, and they are usually referred to as «Grothendieck's Theorem» (see for instance [38] 22.4.2 and 22.4.4):

(a) All operators defined on an $\mathcal{L}_\infty$-space and taking values in an $\mathcal{L}_2$ -space are 2-summing.

(b) All operators defined on an $\mathcal{L}_1$ -space and taking values in an $\mathcal{L}_2$ -space are 1-summing.

(0.14) The next result connects the concept of 2-summing operator with the existence of ellipsoids of maximal volume in the unit balls of finite-dimensiona1 spaces (cf. [14]). For a proof see, for instance, [4] (Lemma 2):

If dim E = n, then there exists an isomorphism $u_E \in L(\ell_2^n, E)$ such that $\|u_E\| = 1$ and $\pi_2(u_E^{-1}) = n^{1/2}$. Moreover,

$$\mathcal{E} := u_E(B_{\ell_2^n})$$

is the ellipsoid of maximal volume contained in B_E.

(0.15) Let ***E***, u_E , and $\mathcal{E}$ be as in (0.14). ***The volume ratio*** of ***E*** is defined by

$$vr(E) := \left(\frac{\text{vol } B_E}{\text{vol } \mathcal{E}}\right)^{1/n} = \left(\frac{\text{vol } u_E^{-1}(B_E)}{\text{vol } B_{\ell_2^n}}\right)^{1/n}.$$

The main results about the volume ratio may be found in [33], [36] and [46].

(0.16) If $p \in [1, \infty]$, $[\Gamma_p, \gamma_p]$ is ***the*** ideal of *p-factorable* ***operators.*** Recall that $T \in \Gamma_p(X, Y)$ if there are a space $L_p = L_p(\mu)$ and operators $A : X \to L_p$, $B : L_p \to Y^{**}$ such that $BA = K_Y T$. Theideal norm γ_p on Γ_p is given by $\gamma_p(T) := \inf \|A\| \cdot \|B\|$, where the infimum extends over all factorizations as above.

(0.17) $[\Gamma_\infty^*, \gamma_\infty^*] = [\Pi_1, \pi_1]$ and $[\Gamma_1^*, \gamma_1^*] = [\Pi_1^d, \pi_1^d]$. Moreover, $[\Pi_1^*, \pi_1^*] = [\Gamma_\infty, \gamma_\infty]$ and $[(\Pi_1^d)^*, (\pi_1^d)^*] = [\Gamma_1, \gamma_1]$ (see [38] 19.3.10and9.3.1).

(0.18) Throughtout this work, (g_k) will be used to denote a sequence of independent standard gaussian variables on some probability space. An important property of (g_k) is the following result of J. Hoffman-Jørgensen [11]: if $0 < p < q < \infty$ there is a constant c_{pq} such that, for every finite sequence $x_1, \ldots, x_n$ from a Banach space X ,

$$\left(E\left\|\sum_{k=1}^n g_k(\omega)x_k\right\|^q\right)^{1/q} \leq c_{pq}\left(E\left\|\sum_{k=1}^n g_k(\omega)x_k\right\|^p\right)^{1/p}.$$

Here E is the expectation (integral) sign.

(0.19) $[\Pi_\gamma, \pi_\gamma]$ is ***the*** ideal of γ*-summing* ***operutors,*** which was first defined in [22]. An operator T belongs to Π_γ(X , Y) if there is a constant C such that, for all $x_1, \ldots, x_n \in$ X ,

$$\left(E\left\|\sum_{k=1}^{n} g_k(\omega)Tx_k\right\|^2\right)^{1/2} \leq C \sup_{x^*\in B_{X^*}} \left(E\sum_{k=1}^{n} |\langle x^*, x_k\rangle|^2\right)^{1/2},$$

where (g_k) is as in (0.18). $\pi_\gamma(T)$ is the least constant C satisfying the above inequality. Note that

$$\pi_\gamma(T) = \sup\{\pi_\gamma(Tu) : u \in L(\ell_2^n, X), n \in \mathbb{N}, \|u\| \leq 1).$$

(0.20) Let $u \in L(\ell_2^n$, X) . Then, by rotational invariance of the gaussian measure on $\mathbb{R}^n$,

$$\pi_\gamma(u) = \left(E\left\|\sum_{k=1}^{n} g_k(\omega)u(f_k)\right\|^2\right)^{1/2}$$

for some (in fact, all) orthonormal basis $f_1, \ldots, f_n$ of ℓ_2^n.

(0.21) If $0 < p < \infty$, then Π_p c Π_γ, and there is a constant c_p such that, for all T in Π_p , $\pi_\gamma(T) \leq c_p\pi_p(T)$ ([22] Th. 6).

(0.22) Let T $\in$ ***L(X , Y) .*** The ***n-th approximation*** (resp. ***Weyl, Hilbert, entropy) number*** of T is defined by

$$a_n(T) := \inf\{\|S - T\| : S \in L(X,Y), \mathrm{rank}(S) < n\}$$

(resp. by

s,(T) : = $\sup\{a_n(Tu) : u \in L(\ell_2, X), \|u\| \leq 1\}$

h,(T) : = $\sup\{x_n(vT) : v \in L(X,\ell_2), \|v\| \leq 1\}$

$e_n(T) := \inf\{\epsilon > 0 : \exists y_1, \ldots, y_{2^{n-1}} \in Y$ such that $T(B_X) \subset$

$$\subset \bigcup_{i=1}^{2^{n-1}} (y_i + \epsilon B_Y)\}).$$

The following facts on these numbers are taken from [38] Chs. 11 and 12, and from [39].

(0.23) If $s \in \{a, x, h, e\}$, then $(s_n(T))_{n\in\mathbb{N}}$ isanonincreasingsequenceand s,(T) = $\|T\|$. Moreover, a,(T) $\geq$ s,(T) $\geq$ h,(T) for all n $\in$ N and 0 = a,(T) = $x_n(T)$ = h,(T) if rank $(T) <$ n.

(0.24) If X is a Hilbert space, then $a_n(T) = x_n(T)$ for all $n \in \mathbb{N}$, and

$$a_n(T) = \sup_{E \in \dim(X)} a_n(T|_E).$$

(0.25) If T is compact, then $a_n(T) = a_n(T^*)$ for all $n \in \mathbb{N}$.
(0.26) $h_n(T) = h_n(T^*)$ for all T and all $n \in \mathbb{N}$.
(0.27) Let $s \in \{a, x, h, e\}$. We define the quasi-normed operator ideals

$$S^s_{p,q} := \{T : (s_n(T))_{n\in\mathbb{N}} \in \ell_{p,q}\}, \quad 0 < p < \infty, \quad q \in \{1, \infty\}$$

and

$$S^s_p := \{T : (s_n(T))_{n\in\mathbb{N}} \in \ell_p\}, \quad 1 \le p \le \infty,$$

the quasi-norm being given by

$$\sigma^s_{p,q}(T) := \sigma_{p,q}((s_n(T))),$$

(cf. (0.7)) resp. by

$$\sigma^s_p(T) := \|(s_n(T))\|_p.$$

(0.28) Let $r, p, q, u, v, w \in [1, \infty]$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad \frac{1}{u} + \frac{1}{v} = \frac{1}{w},$$

and let $s \in \{a, x, e\}$. Then, if $T \in S^s_{q,v}(X, Y)$, $S \in S^s_{p,u}(Y, Z)$, we have $ST \in S^s_{r,w}(X, Z)$ and

$$\sigma^s_{r,w}(ST) \le 2^{1/r} \sigma^s_{p,u}(S) \sigma^s_{q,v}(T).$$

(0.29) Let $q \in [2, \infty)$. Then $S^x_{q,1} \subset \Pi_{q,2} \subset S^x_{q,\infty}$ and there are constants c_q, c'_q such that

$$\sigma^x_{q,\infty}(T) \le c_q \pi_{q,2}(T) \le c'_q \sigma^x_{q,1}(T)$$

for all operators T belonging to the appropriate ideals.
(0.30) If $q \in (2, \infty)$, then $S^x_q \subset \Pi_{q,2}$ and there is a constant c" such that

$$\pi_{q,2}(T) \le c''_q \sigma^x_q(T)$$

for all T. Further, if X is a Hilbert space, then $S^x_q(X, .) = \Pi_{q,2}(X, .)$.

(0.31) Let $q \in (2, \infty)$ and $r \in [2, q)$. Then there is a constant c_{rq} such thtat

$$\pi_{r,2}(T) \leq c_{rq} n^{1/r-1/q} \sigma^{x}_{q,\infty}(T)$$

for all rank n operators T.

(0.32) The next lemma, due to G. Pisier [43], will be often useful to us:
Let α be any idea1 quasi-norm on $L(\ell_2^n, X)$. Suppose there is a constant C such that, for all $u \in L(\ell_2^n, X)$ and for all $n \in \mathbb{N}$,

$$a_{[n/2]}(u) \leq Cn^{-1/q}\alpha(u)$$

(here $[x]$ denotes the greatest integer less or equal to x). Then there is a constant C', depending only on C, such that

$$\sigma^{a}_{q,\infty}(u) \leq C'\alpha(u)$$

for all $u \in L(\ell_2^n, X)$ and for all $n \in \mathbb{N}$.

Pisier's proof actually shows that $C' \leq (3/2)^{1/q} C$.

(0.33) To conclude, we introduce the notions of type, cotype and related concepts. We restrict to the Gaussian case. For details about the relation between Gaussian and Rademacher type or cotype see, for instance, [33].

A Banach space X is said to have *cotype q* ($q \in [2, \infty)$) if there is a constant C such that, for all $x_1, \ldots, x_n \in X$,

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leq C \left(E\left\|\sum_{i=1}^{n} g_i(\omega) x_i\right\|^2\right)^{1/2},$$

where (g_i) is as in (0.18). For fixed n, let $C_q(X, n)$ be the least such C, and put $C_q(X) = \sup_{n\in\mathbb{N}} C_q(X, n)$, so that X has cotype q if and only if $C_q(X) < \infty$. $C_q(X)$ is the so-called *cotype constant* of X.

X has *equal-norm cotype q* if the inequality above is only supposed to hold for vectors x_i of equal norm (e.g. such that $\|x_i\| = 1$ for all i).

Similarly, X is said to have *type p* ($p \in (1, 2]$) if there is a constant C such that, for all $x_1, \ldots, x_n \in X$,

$$\left(E\left\|\sum_{i=1}^{n} g_i(\omega) x_i\right\|^2\right)^{1/2} \leq C \left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q}.$$

For fixed n, let $T_p(X, n)$ be the least such C . Put $T_p(X) := \sup_{n\in\mathbb{N}} T_p(X, n)$, so that X has type p and only *if* $T_p(X) < \infty$. $T_p(X)$ is the *type constant* of X .

X has *equal-norm type* p if the inequality in the definition of type is only supposed to hold for vectors x_i of equa1 norm.

(0.33') A Banach space X has type p if and only if it is K-convex and X* has cotype p^* (cf. (0.5)). This fact is fundamental for the so-called «duality» between type and cotype. Of course, a similar statement holds for equal-norm type and equal-norm cotype as well.

Let $p \in [1, \infty)$. Then the $\mathcal{L}_p$-spaces (cf. (0.6)) have type min(p, 2) and cotype max(p, 2) . $\mathcal{L}_\infty$-spaces have neither type nor cotype, as it follows from (0.36). A result of Kwapień [19] states that X is an $\mathcal{L}_2$-space (i.e. is isomorphic to a Hilbert space) if and only if X has type 2 and cotype 2.

(0.34) Given a Banach space X let p(X) := $\sup\{p$: X has type p$\}$ and g(X) := inf $\{q$: X has cotype $q\}$. Then the Maurey-Pisier Theorem asserts that X contains the $\ell^n_{p(X)}$'s and the $\ell^n_{q(X)}$'s uniformly (see [30]). This has the following corollaries:

(0.35) X does not contain the ℓ^n_1 's uniformly (i.e. X is K-convex) if and only if X has type p for some $p > 1$.

(0.36) X does not contain the ℓ^n_∞ 's uniformly if and only if X has cotype q for some $q < \infty$.

(0.37) Since a *K-convex* space does not contain the ℓ^n_1 's uniformly, it does not contain the ℓ^n_∞ 's uniformly as well, so that (0.36) implies that K-convex spaces also have cotype q for some finite q.

1. WEAK PROPERTIES

Let X be a Banach space and $[A, \alpha]$, $[B, \beta]$ be quasi-normed ideals. Following G. Pisier [43] we say that X *has the property* $P(\alpha, \beta)$ if there is a constant C such that

$$\alpha(u) \leq C\beta(u), \ \forall u \in L(\ell^n_2, X) \ \forall n \in N .$$

Clearly, if $id_X \in A \cdot B^{-1}$ then X has $P(\alpha, \beta)$. One may show that the converse does not hold in general.

Similarly, we say that X *has the property* $Q(\alpha, \beta)$ if there is a constant C such that

$$\alpha(v) \leq C\beta(v), \ \forall v \in L(X, \ell^n_2), \ \forall n \in N.$$

As above, we can easily see that if $id_X \in B^{-1} \cdot A$ then X has $Q(\alpha, \beta)$, the converse being again false in general.

The two concepts are essentially dual to each other, as it is seen by the following straight-forward lemma:

Lemma 1.1. ***(a) If X has $P(\alpha,\beta)$ then X has $Q(\beta^*,\alpha^*)$***

(b) If X has $Q(\alpha,\beta)$ then X has $P(\beta^*,\alpha^*)$.

(c) If α^{**} is equivalent to α and β^{**} is equivalent to β, then $P(\alpha,\beta)$ is equivalent to $Q(\beta^*,\alpha^*)$.

We illustrate these concepts by

Proposition 1.2. ***(a) X has cotype q, $q\in[2,\infty)$, iff X has $P(\pi_{q,2},\pi_\gamma)$.***

(b) X has type p, $p\in(1,2]$, iff X has $P(\pi_\gamma,(\pi_{p^*,2})^{*d})$, or else, iff X has $Q((\pi_{p^*,2})^d,\pi_\gamma^*)$.

(c) X does not contain the ℓ_1^n's uniformly (i.e. X is K-convex) iff X has $P(\pi_\gamma,(\pi_\gamma)^{*d})$, or else, iff X has $Q(\pi_\gamma^d,\pi_\gamma^*)$.

(d) X does not contain the ℓ_∞^n's uniformly iff X has $P(\pi_\gamma,\gamma_\infty)$.

Proof . (a) and (b) follow from [44], whereas (c) follows form a characterization of K-convexity by T. Figiel and N. Tomczak-Jaegermann [8] and from (0.5).

We only prove (d). By the Maurey-Pisier Theorem (0.36), X does not contain the ℓ_∞^n's uniformly iff X has cotype q for some $q<\infty$. The latter implies that all operators from an $\mathcal{L}_\infty$-space into X are $(q+\epsilon)$-summing for all $\epsilon>0$ (use, for instance, [42] Cor. 2.7 and [38] 22.6.4). By (0.21), all Γ_∞-operators into X must be γ-summing. In particular, X has $P(\pi_\gamma,\gamma_\infty)$.

Assume now that X has $P(\pi_\gamma, 7,)$. We get immediately $\Gamma_\infty(\cdot,X)\subset\Pi_\gamma(\cdot,X)$ and a constant C_1 not depending on n such that $\pi_\gamma(s)\le C_1\|s\|$ for all $s\in L(\ell_\infty^n,X)$. We shall reach a contradiction from assuming that X contains the ℓ_∞^n's uniformly: let C_2 be a constant such that, for some isomorphic embeddings $j_n:\ell_\infty^n\to X$,

$$\sup_{n\in\mathbb{N}}\|j_n\|\,\|j_n^{-1}\|\le C_2.$$

This implies

$$\pi_\gamma(id_{\ell_\infty^n})\le\|j_n^{-1}\|\pi_\gamma(j_n)\le\|j_n^{-1}\|C_1\|j_n\|\le C_1C_2.$$

Now, if (g_k) is as in (0.18) we have, by the definition of γ-summing operators,

$$\int_{\mathbb{R}}\sup_{1\le k\le n}|g_k(\omega)|\,d\omega\le\pi_\gamma(id_{\ell_\infty^n}).$$

The integra1 on the left is known to be of the order of magnitude of $(\log n)^{1/2}$ (see [1] Cor. VIII.4.4), hence we have reached the desired contradiction. □

Let $[A, \alpha]$ bea quasi-normed ideal and X a Banach space. On $L(\ell_2^n, X)$ and $L(X, \ell_2^n)$ we define the quasi-norm $w\alpha$ by

$$w\alpha(u) := \sup\{\sigma_{1,\infty}^a(vu) : v \in L(X,\ell_2^n),\ \alpha^*(v) \leq 1\},\ \forall u \in L(\ell_2^n, X),$$

and

$$w\alpha(v) := \sup\{\sigma_{1,\infty}^a(vu) : u \in L(\ell_2^n, X),\ \alpha^*(u) \leq 1\},\ \forall v \in L(X,\ell_2^n),$$

respectively. One may easily show, for example, that $w\alpha = w(\alpha^{**})$ is always true and that $w\alpha$ is maximal on $L(\ell_2^n, X)$, resp. $L(X, \ell_2^n)$, if α^* is surjective, resp. injective.

Let $[B, \beta]$ be another quasi-normed ideal. Following Pisier ([43], §3), we say that X *has the property weak-$P(\alpha, \beta)$* if X has $P(w\alpha, \beta)$. Thus X has weak-(α, β) if and only if there is a constant C such that

$$\sigma_{1,\infty}^a(vu) \leq C\alpha^*(v)\beta(u),\ \forall n,\ \forall u \in L(\ell_2^n, X),\ \forall v \in L(X, \ell_2^n).$$

Similarly, X ***has the** property weak-$Q(\alpha, \beta)$* if X has $Q(w\alpha, \beta)$ i.e. if and only if there is a constant C such that

$$\sigma_{1,\infty}^a(vu) \leq C\alpha^*(u)\beta(v),\ \forall n,\ \forall u \in L(\ell_2^n, X),\ \forall v \in L(X, \ell_2^n).$$

Lemma 1.3. *(a) $w\alpha \leq \alpha$ **both on** $L(\ell_2^n, X)$ **and** $L(X, \ell_2^n)$.*

(b) $P(\alpha, \beta) \Rightarrow$ weak-$P(\alpha, \beta)$.

(c) $Q(\alpha, \beta) \Rightarrow$ weak-$Q(\alpha, \beta)$.

***(d)** If β **is** equivalent **to** β^{**} **on** $L(X, \ell_2^n)$**, then weak-Q(**α, β) is **the** same **as** weak-$P(\beta^*, \alpha^*)$.*

(e) $w(\alpha^d) = (w\alpha)^d$ on $L(\ell_2^n, X)$, and $w(\alpha^d) \leq (w\alpha)^d$ on $L(X, \ell_2^n)$.

Proof . (a) Let $u \in L(\ell_2^n, X)$, $v \in L(X, \ell_2^n)$. Since the nuclear norm (denoted by ν_1) of operators between Hilbert spaces coincides with σ_1^a (see [38] 15.5.3), we have

$$\sigma_{1,\infty}^a(vu) \leq \sigma_1^a(vu) = \nu_1(vu) \leq \alpha^*(v)\alpha(u),$$

which proves (a).

(b), (c) and (d) follow easily from the definitions and part (a).

(e) follows from the definitions and from the identity $\alpha^{*d} = \alpha^{d*}$. □

There are important properties which coincide with their weakened form, notably the properties of not containing the ℓ_∞^n's (resp. the ℓ_1^n's) uniformly.

Proposition 1.4. ***The weakproperty associated to having finite cotype is finite cotype.***

Proof . By (0.36), X has finite cotype if and only if X does not contain the ℓ_∞^n 's uniformly. Hence, by 1.2 (d) and 1.3 (b), it suffices to prove that $P(w\pi_\gamma, 7,)$ implies $P(\pi_\gamma, \gamma_\infty)$. Suppose that X contains the ℓ_∞^n's uniformly and let $j_n : \ell_\infty^n \to X$, $n \in \mathbb{N}$, be isomorphic embeddings such that sup $||j_n|| \, ||j_n^{-1}|| = C < \infty$. By $i_{p,q}$ we denote the identity of $\mathbb{R}^n$ regarded as a map $\ell_p^n \to \ell_q^n$. Then

$$\gamma_\infty(j_n i_{2,\infty}) \le ||j_n|| \, ||i_{2,\infty}|| = ||j_n||.$$

Let $v_n \in L(X, \ell_\infty^n)$ be an extension of j_n^{-1} such that $||v_n|| = ||j_n^{-1}||$ (there exists such an extension by the metric extension property of ℓ_∞^n). We have, by duality,

$$\pi_\gamma^*(i_{\infty,2} v_n) \le ||j_n^{-1}|| \pi_\gamma^*(i_{\infty,2}) \le ||j_n^{-1}|| C_2(\ell_\infty^n) \pi_2(i_{\infty,2})$$

Now, by Grothendieck's Theorem (0.13') ,

$$\pi_2(i_{\infty,2}) \le \kappa_1 ||i_{\infty,2}|| = \kappa_1 n^{1/2}$$

for some constant κ_1. Further it is known that there is a constant κ_2 such that

$$C_2(\ell_\infty^n) \le \kappa_2 \, . \, [n/\log(n+1)]^{1/2}$$

(see [46] Ch. 1.4). Hence, if X is supposed to have $P(w\pi_\gamma, 7,)$, there must be a constant κ such that, for all n,

$$n \le \sigma_{1,\infty}^a(i_{\infty,2} v_n j_n i_{2,\infty}) \le \kappa \pi_\gamma^*(i_{\infty,2} v_n) \gamma_\infty(j_n i_{2,\infty}) \le$$
$$\le \kappa C \kappa_1 \kappa_2 n [\log(n+1)]^{-1/2},$$

a contradiction. □

Proposition 1.5. ***Weak K-convexity is equivalent to K-convexity.***

This is due to Pisier [43]. We provide a proof for completeness. We start by an easy lemma.

Lemma 1.6. *There is a constant κ such that*

$$\pi_\gamma^*(t) \le \kappa ||t||$$

for all t $\in L(\ell_1, \ell_2^n)$ ***andfor all n.***

Proof . Let $t \in L(\ell_1, \ell_2^n)$, $s \in L(\ell_2^n, \ell_1)$. Since $\Pi_2^* = \Pi_2$ isometrically,

$$|tr(st)| \leq \pi_2(s)\pi_2(t).$$

By Grothendieck's Theorem (0.13') , there is a constant C such that $\pi_2(t) \leq C||t||$. Furhter, since ℓ_1 has cotype 2, $\pi_2(s) \leq C_2(\ell_1)$. n,,(s) . Hence

$$|tr(st)| \leq \kappa\pi_\gamma(s)||t||,$$

where κ : = CC, (ℓ_1) . By definition, this means that $\pi_\gamma^*(t) \leq \kappa||t||$. □

Next we quote the following simple observation form [33] 15.5:

Lemma 1.7. ***Let α be an injective norm defined on L(E, F) for some finite dimensiona1 normed space E and all finite dimensiona1 normed spaces F . Then, for any Banach space X $\supset$ F, every v $\in$ L(F , E) admits an extension V $\in$ L(X , E) with $\alpha^*(V) = \alpha^*(v)$.***

Proof of Proposition 1.5. By proposition 1.2 (c) an lemma 1.3 (b) it suffices to show that $P(w\pi_\gamma, (\pi_\gamma)^{*d})$ implies $P(\pi_\gamma, (\pi_\gamma)^{*d})$. Suppose X is not K-convex, i.e. let X contain the ℓ_1^n 's uniformly (cf. (0.5)). Let $j_n : \ell_1^n \to X$ be isomorphisms such that sup $||j_n|| \, ||j_n^{-1}|| = C < \infty$, and let $i_{p,q}$ be the identity of $\mathbf{R}^n$ regarded as a map $\ell_p^n \to \ell_q^n$. Since π_γ is injective, we may apply lemma 1.7 to obtain an extension $v_n \in$ ***L(X*** , ℓ_2^n) of $i_{1,2}j_n^{-1}$ such that

$$\pi_\gamma^*(v_n) = \pi_\gamma^*(i_{1,2}j_n^{-1}) \leq ||j_n^{-1}||\pi_\gamma^*(i_{1,2}) \leq \kappa||j_n^{-1}||,$$

where κ is the (universal) constant appearing in lemma 1.6. Just as in the proof of proposition 1.4, there is a constant κ' such that

$$\pi_\gamma^*((i_{2,1})^*) = \pi_\gamma^*(i_{\infty,2}) \leq C_2(\ell_\infty^n)\pi_2(i_{\infty,2}) \leq \kappa' n[\log(n+1)]^{-1/2}.$$

If we assume that X has $P(w\pi_\gamma, (\pi_\gamma)^{*d})$, there must be a constant κ_0 such that, for each $n \in \mathbf{N}$,

$$n = \sigma_{1,\infty}^a(v_n j_n i_{2,1}) \leq \kappa_0 \pi_\gamma^*((j_n i_{2,1})^*)\pi_\gamma^*(v_n) \leq$$
$$\leq \kappa_0 C\kappa\kappa' n[\log(n+1)]^{-1/2},$$

which is impossible.

Of course, weak-$P(\alpha, \beta)$ is nothing but $P(\alpha, \beta)$ whenever $w\alpha$ is equivalent to α on $L(\ell_2^n, X)$. In the light of 1.4 and 1.5 it is tempting to conjecture that $w\pi_\gamma$ is equivalent to π_γ on $L(\ell_2^n, X)$. This, however, is false, since for example weak type p is always strictly weaker than type p, as we will see in $3. On the other hand, the next proposition will enable us to show that weak (weak cotype q) is again weak cotype q, whenever $q > 2$ (see Corollary 2.2). As we will sec, an analogous result holds for weak type p, p < 2.

Proposition 1.8. ***Consider the quasi-norms $\pi_{q,2}$, $\sigma^a_{q,\infty}$ ($q \in [2, \infty)$) on $L(\ell_2^n, X)$ for some Banach space X. Then:***

(a) $w\pi_{q,2}$ is equivalent $\sigma^a_{q,\infty}$, $q \in [2, \infty)$.

(b) $w\sigma^a_{q,\infty}$ isequivalent $\sigma^a_{q,\infty}$, $q \in (2, \infty)$.

In case $q = 2$, (a) was already stated in [43], §3. We shall need the following lemma:

Lemma 1.9. ***Let $q \in (1, \infty)$. There is a constant C such that***

$$\sigma^x_{q^*,\infty}(v) \leq (\sigma^x_{q,1})^*(v) \leq C\sigma^x_{q^*,\infty}(v)$$

for all $v \in L(X, \ell_2^n)$.

Proof. Let $v \in L(X, \ell_2^n)$, $u \in L(\ell_2^n, X)$. By [38] 13.9.6 and 154.6,

$$\sigma^x_{q^*,\infty}(vu) \leq (\sigma^x_{q,1})^*(vu) \leq (\sigma^x_{q,1})^*(v)\|u\|.$$

By the definiton of the Weyl numbers, this implies

$$\sigma^x_{q^*,\infty}(v) \leq (\sigma^x_{q,1})^*(v).$$

By (0.28), if $u \in L(\ell_2^n, X)$ and $v \in L(X, \ell_2^n)$,

$$|tr(vu)| = \sigma^x_1(vu) \leq 2\sigma^x_{q^*,\infty}(v)\sigma^x_{q,1}(u),$$

which means that

$$(\sigma^x_{q,1})^*(v) \leq C\sigma^x_{q^*,\infty}(v),$$

by the definition of the adjoint norms. □

Proof of Proposition 1.8 We consider only the case $2 < q < \infty$. Because of the identity $\Pi_2^* = \Pi$, (0.12), the case $q = 2$ is even easier to deal with.

Let $u \in L(\ell_2^n, X)$ and let E be an n-dimensional subspace of X which contains $u(\ell_2^n)$. Further, let j_E: ***E*** $\to$ X be the natural embedding. By (0.14) there is an isomorphism

$v \in L(E, \ell_2^n)$ suchthat $\pi_2(v) = n^{1/2}$ and $\|v^{-1}\| = 1$. By (0.13), there exists an extension $V \in L(X, \ell_2^n)$ of v such that $\pi_2(V) = \pi_2(v) = n^{1/2}$. For all $k \leq n$ we get

$$\begin{aligned} ka_k(u) &= ka_k(j_E v^{-1} V u) \leq ka_k(Vu) = \\ &= ka_k(vu) \leq n^{1/2} \sigma^a_{1,\infty}\left(\frac{v}{n^{1/2}} u\right) \leq \\ &\leq n^{1/2} \sup\{\sigma^a_{1,\infty}(tu) : t \in L(X, \ell_2^n), \pi_2(t) \leq 1\} \leq \\ &\leq n^{1/2} w\pi_2(u). \end{aligned}$$

By (0.31) and since $\alpha \leq C\beta$ implies $w\alpha \leq Cw\beta$ (by definition), there is a constant c_q depending only on q such that

$$w\pi_2(u) \leq c_q n^{1/2-1/q} w\sigma^a_{q,\infty}(u).$$

Therefore, letting $k := [n/2]$ we obtain, for some constant κ,

$$a_{[n/2]}(u) \leq \kappa n^{-1/q} w\sigma^a_{q,\infty}(u).$$

By (0.32), there is a constant κ' depending only on κ such that

$$\sigma^a_{q,\infty}(u) \leq \kappa' w\sigma^a_{q,\infty}(u),$$

which proves (b) by 1.3 (a).
Further, by (0.29) and 1.3 (b), there is a constant κ_1 such that

$$w\sigma^a_{q,\infty}(u) \leq \kappa_1 w\pi_{q,2}(u)$$

for all $u \in L(\ell_2^n, X)$. To complete the proof, it remains to show that there is a constant κ_2 such that, for any $u \in L(\ell_2^n, X)$,

$$w\pi_{q,2}(u) \leq \kappa_2 \sigma^a_{q,\infty}(u).$$

To see this, note that, by (0.28),

$$\begin{aligned} w\pi_{q,2}(u) &= \sup\{\sigma^a_{1,\infty}(vu) : v \in L(X, \ell_2^n), n \in \mathbb{N}, (\pi_{q,2})^*(v) \leq 1\} \leq \\ &\leq 2\sigma^a_{q,\infty}(u) \sup\{\sigma^a_{q^*,\infty}(v) : v \in L(X, \ell_2^n), n \in \mathbb{N}, (\pi_{q,2})^*(v) \leq 1\}. \end{aligned}$$

By trace duality and (0.29), there is a constant κ_3 such that

$$(\sigma^a_{q,1})^*(v) \leq \kappa_3 (\pi_{q,2})^*(v)$$

for all $v \in L(X, \ell_2^n)$. By lemma 1.9,

$$w\pi_{q,2}(u) \leq 2\kappa_3 \sigma^a_{q,\infty}(u),$$

and thus the proof is complete (let $\kappa_2 := 2\kappa_3$).

Lemma 1.10. ***Let*** $v \in L(X, \ell_2^n)$, $g \in [2, \infty)$. ***Then***

(a) $w\alpha(v) = \sup_{E \in \mathrm{Dim}(X)} w\alpha(v|_E)$, ***for all quasi-norms*** α ***such that*** α^* ***is injective.***

(a') $w\alpha(v) = \sup_{F \in \mathrm{Cod}(X)} w\alpha(Q_F v)$, ***for all quasi-norms*** α ***such that*** α^* ***is surjective.***

(b) $\sigma^a_{q,\infty}(v) = \sup_{E \in \mathrm{Dim}(X)} \sigma^a_{q,\infty}(v|_E)$.

***Proof*.** We have only to show «$\leq$».

(a)Let $u \in L(\ell_2^n, X)$, $\alpha^*(u) \leq 1$. Put $E_0 := u(\ell_2^n)$ and let $u_0 \in L(\ell_2^n, E_0)$ be such that $u = j u_0$, where j is the natural embcdding of E_0 in X . Since α^* is injective, $\alpha^*(u_0) = \alpha^*(u)$ and

$$\sigma^a_{1,\infty}(vu) = \sigma^a_{1,\infty}(v|_{E_0} u_0) \leq w\alpha(v|_{E_0}) \leq \sup_{E \in \mathrm{Dim}(X)} w\alpha(v|_E)$$

Since u was arbitrary, (a) is proved.

The proof of (a') is similar,

(b) By (0.24) and (0.29,

$$\sigma^a_{q,\infty}(v) = \sigma^a_{q,\infty}(v^*) = \sup \sigma^a_{q,\infty}(Qv^*) ,$$

where the supremum extends over all quotient maps Q defined on X^* with finite dimensiona1 range. For all such Q we have, by (0.25),

$$\sigma^a_{q,\infty}(Qv^*) = \sigma^a_{q,\infty}(v^{**}Q^*) \leq \sup_{E \in \mathrm{Dim}(X^{**})} \sigma^a_{q,\infty}(v^{**}|_E) =$$

$$= \sup_{E \in \mathrm{Dim}(X)} \sigma^a_{q,\infty}(v|_E) ,$$

the last inequality following from local reflexivity (see, e.g., [13] 17.57). □

We are now ready to prove a companion result to proposition 1.8. We point out that proposition 1.11 (b) will be used to prove that weak (weak type p) is nothing but weak type p' when $1 < p < 2$ (Corollary 3.2).

Proposition 1.11. ***Consider the quasi-norms*** $(\pi_{q,2})^d, \sigma^a_{q,\infty}$ ***(***$g \in [2, \infty)$***) on*** $L(X, \ell_2^n)$ ***for some Banach space X . Then:***

(a) $w(\pi_{q,2})^d$ ***isequivalent to*** $\sigma^a_{q,\infty}$, $g \in [2, \infty)$.

(b) $w\sigma^a_{q,\infty}$ ***is equivalent to*** $\sigma^a_{q,\infty}$, $g \in (2, CO)$.

Proof . (a) Let v $\in L(X, \ell_2^n)$. Since $(\pi_{q,2})^{dd} = \pi_{q,2}$, proposition 1.8 (a) yields two absolute constants κ_1, κ_2 such that

$$w\pi_{q,2}(v^*) \leq \kappa_1 \sigma^a_{q,\infty}(v^*) \leq \kappa_2 w\pi_{q,2}(v^*) = \kappa_2 w((\pi_{q,2})^{dd})(v^*).$$

By (0.25) and 1.3 (e) we get

$$w((\pi_{q,2})^d)(v^{**}) \le \kappa_1 \sigma^a_{q,\infty}(v) \le \kappa_2 w((\pi_{q,2})^d)(v^{**}).$$

Application of 1.10 (b) completes the proof of (a).
(b) is proved analogously, using 1.8 (b) and (0.25). □

2. WEAK COTYPE

Let $q \in [2, \infty)$. By 1.2 (a) and 1.3 (d), a Banach space X has *weak cotype q* if X has $P(w\pi_{q,2}, \pi_\gamma)$, i.e. if there is a constant C such that

$$\sigma^a_{1,\infty}(vu) \le C(\pi_{q,2})^*(v)\pi_\gamma(u)$$

for all $u \in L(\ell_2^n, X)$, $v \in L(X, \ell_2^n)$ and $n \in \mathbb{N}$.

Given a Banach space X and $q \in [2, \infty)$, we define $wC_q(X)$ to be the smallest C such that

$$\sigma^a_{q,\infty}(u) \le C\pi_\gamma(u), \quad \forall u \in L(\ell_2^n, X), \quad \forall n \in \mathbb{N},$$

with the usual agreement inf $\emptyset = \infty$. We call $wC_q(X)$ *the weak cotype q constant* of X.

Proposition 2.1. ***Let $q \in [2, \infty)$, and X be a Banach space. The following conditions are equivalent:***

(a) X has weak cotype q.

(b) There is a constant C such that, for all $u \in L(\ell_2^n, X)$ and all $n \in N$,

$$\sigma^a_{q,\infty}(u) \le C\pi_\gamma(u).$$

(c) There is a constant C such that, for every finite-dimensional subspace E of X,

$$wC_q(E) \le C.$$

(d) $id_X \in S^x_{q,\infty} \cdot \Pi_\gamma^{-1}$, i.e. all γ-summing operators t with values in X satisfy

$$(x_n(t))_{n\in\mathbb{N}} \in \ell_{q,\infty}.$$

Proof. (a) $\Leftrightarrow$ (b). By the considerations above and by 1.8 (a), X has weak cotype q iff X has $P(\sigma^a_{q,\infty}, \pi_\gamma)$.

(b) ⇒ (c). Let $E \in \mathrm{Dim}(X)$ and let $u \in L(\ell_2^n, X)$. If j_E is the natural embedding of E into X we have

$$\sigma_{q,\infty}^a(u) = \sigma_{q,\infty}^a(j_E u) \leq w\mathrm{C}_q(X)\pi_\gamma(j_E u) \leq w\mathrm{C}_q(X)\pi_\gamma(u).$$

It follows that $w\mathrm{C}_q(E) \leq w\mathrm{C}_q(X)$.

(c) ⇒ (b). Let $u \in L(\ell_2^n, X)$ and $E := u(\ell_2^n)$. Then, if j_E is the natural embedding of E into X and $u_0 : \ell_2^n \to E$ is such that $u = j_E u_0$, we have

$$\sigma_{q,\infty}^a(u) = \sigma_{q,\infty}^a(u_0) \leq w\mathrm{C}_q(E)\pi_\gamma(u_0) = w\mathrm{C}_q(E)\pi_\gamma(u).$$

Consequently, $w\mathrm{C}_q(X) \leq \sup_{E \in \mathrm{Dim}(X)} w\mathrm{C}_q(E)$ and (b) follows.

(b) ⇒ (d). By (0.24) we have

$$\sigma_{q,\infty}^a(u) \leq w\mathrm{C}_q(X)\pi_\gamma(u)$$

for all $u \in L(\ell_2^n, X)$. Let now $T \in \Pi_\gamma(Z, X)$, Z being an arbitrary Banach space, and $u \in L(\ell_2^n, Z)$, $\|u\| \leq 1$. We get

$$\sigma_{q,\infty}^a(Tu) \leq w\mathrm{C}_q(X)\pi_\gamma(Tu) \leq w\mathrm{C}_q(X)\pi_\gamma(T).$$

By the definition of the Weyl numbers, (d) follows.

(d) ⇒ (b). Is trivial. □

Remark: We have actually proved that $w\mathrm{C}_q(X) = \sup_{E \in \mathrm{Dim}(X)} w\mathrm{C}_q(E)$ for all Banach spaces X.

As it was already announced, from 2.1 and 1.8 (b) we deduce the next

Corollary 2.2. *If $2 < q < \infty$, weak (weak cotype q) is equivalent to weak cotype q.*

Problem 2.2.*. *Does the same holdfor weak cotype 2?*

Equal-norm cotype q is a natura1 weakening of cotype q (see (0.33) for the definitions). However, Pisier [12] has proved that the two notions are the same in the case $q = 2$. In particular, it will be clear from the examples given after theorem 2.10 that weak cotype 2 is strictly weaker than equal-norm cotype 2. In this light it is surprising to discover that, if $q > 2$, weak cotype q and equal-norm cotype q coincide. This, together with several other characterizations, is the content of the next theorem:

Theorem ***2.3. Let $q \in (2, \infty)$, and X be a Banach space. The following conditions are equivalent:***

(a) X has weak cotype q .

(b) For each $r \in [2, q)$ there is a constant κ_r such that

$$C_r(X, n) \le \kappa_r n^{1/r-1/q}, \quad \forall n \in N.$$

(d) X has equal-norm cotype q.

(e) $L_r(\mu, X)$ has weak cotype q for all $r \in [1, q)$ and all measure spaces (Ω, μ).

(f) $L_r(\mu, X)$ has weak cotype q for some $r \in [1, q)$ and some (nontrivial) measure space (Ω, μ).

Proof . (a) $\Rightarrow$ (b). Let $r \in [2, q)$. It is easy to deduce from the definitions (0.33) that $C_r(X, n)$ is the least κ such that, for all $u \in L(\ell_2^n, X)$,

$$\pi_{r,2}(u) \le \kappa \cdot \pi_r(u).$$

It follows then from (0.3 1) that there is a constant C such that

$$\pi_{r,2}(u) \le Cn^{1/r-1/q}\sigma^a_{q,\infty}(u) \le wC_q(X)Cn^{1/r-1/q}\pi_r(u)$$

for all $u \in L(\ell_2^n, X)$, which proves (b).

(b) $\Rightarrow$ (c), is trivial.

(c) $\Rightarrow$ (d). Let $r \in [2, q)$ and κ be such that

$$C_r(X, n) \le \kappa n^{1/r-1/q}, \quad \forall n \in N,$$

and let $x_1, x_2, \dots, x_n \in X$ be norm-one vectors. We have

$$n^{1/r} = \left(\sum_{i=1}^n \|x_i\|^r\right)^{1/r} \le \kappa n^{1/r-1/q}\left(E\left\|\sum_{i=1}^n g_i x_i\right\|^2\right)^{1/2},$$

i.e.

$$n^{1/q} \le \left(\kappa E\left\|\sum_{i=1}^n g_i x_i\right\|^2\right)^{1/2}$$

Consequently, X has equal-norm cotype q .

(d) $\Rightarrow$ (a). Let X have equal-norm cotype q. We first show that, for every $m \in \mathbb{N}$, every $w \in L(\ell_2^m, X)$, and every orthonormal basis $f_1, f_2, \ldots, f_m$ of ℓ_2^m,

$$m^{1/q} \min_{k \leq m} \|w(f_k)\| \leq C\pi_\gamma(w),$$

C being a constant which depends only on X. Define

$$h_i := \frac{\min_{k\leq m} \|w(f_k)\|}{\|w(f_i)\|} f_i, \quad i = 0, \ldots, m,$$

(0/0: = 0). Then, since $\|w(h_i)\| = \min_k \|wf_k\|$ for all i, we have

$$m^{1/q} \min_{k\leq m} \|wf_k\| = \left(\sum_{i=1}^m \|w(h_i)\|^q\right)^{1/q} \leq$$

$$\leq C\left(E\left\|\sum_{i=1}^m g_i w(h_i)\right\|^2\right)^{1/2},$$

by the equal-norm cotype q property of X.
Since

$$\left(\sum_{i=1}^m |\langle h, h_i\rangle|^2\right)^{1/2} \leq \left(\sum_{i=1}^m |\langle h, f_i\rangle|^2\right)^{1/2}$$

for all $h \in \ell_2^m$, by proposition 3.7 of [41] we have

$$\left(E\left\|\sum_{i=1}^m g_i w(h_i)\right\|^2\right)^{1/2} \leq \left(E\left\|\sum_{i=1}^m g_i w(f_i)\right\|^2\right)^{1/2} = \pi_\gamma(w),$$

and the claim is proved.

Let now $u \in L(\ell_2^n, X)$ be given. Using a well-known lemma (see e.g. [37] lemma 7), we may construct an orthonormal basis $f_1, f_2, \ldots, f_n$ of ℓ_2^n such that $a_k(u) \leq \|u(f_k)\|$, $k = 1, \ldots, n$. Then, by what was shown above,

$$a_k(u) \leq \min_{k\leq n} \|u(f_k)\| \leq Ck^{-1/q}\pi_\gamma(u|_{\mathrm{span}\{f_1,\ldots,f_k\}}) \leq Ck^{-1/q}\pi_\gamma(u),$$

i.e. X has weak cotype q (by proposition 2.1).

(a) $\Rightarrow$ (e). Let first $r \in [1, 2]$. Let (Ω, μ) be a measure space and $x_1, x_2, \ldots, x_n \in L_r(\mu, X)$. By (a) $\Leftrightarrow$ (b) and (0.18), there is a constant κ such that

$$\left(\sum_{i=1}^{n} \|x_i(\omega)\|^r\right)^{1/r} \leq n^{1/r-1/2} \left(\sum_{i=1}^{n} \|x_i(\omega)\|^2\right)^{1/2} \leq$$
$$\leq \kappa n^{1/r-1/2} n^{1/2-1/q} \left(E\left\|\sum_{i=1}^{n} g_i x_i(\omega)\right\|^r\right)^{1/r}$$

and hence

$$\sum_{i=1}^{n} \|x_i(\omega)\|^r \leq (\kappa n^{1/r-1/q})^r E\left\|\sum_{i=1}^{n} g_i x_i(\omega)\right\|^r,$$

for all w $\in \Omega$. Integrating with respect to w we get, by Fubini's Theorem,

$$\left(\sum_{i=1}^{n} \|x_i\|^r_{L_r(X)}\right)^{1/r} \leq \kappa n^{1/r-1/q} \left(E\left\|\sum_{i=1}^{n} g_i x_i\right\|^r_{L_r(X)}\right)^{1/r}$$

If we suppose $\|x_1\| = \|x_2\| = \ldots = \|x_n\| = 1$, this becomes

$$n^{1/q} \leq \kappa \left(E\left\|\sum_{i=1}^{n} g_i x_i\right\|^r_{L_r(X)}\right)^{1/r} \leq \kappa \left(E\left\|\sum_{i=1}^{n} g_i x_i\right\|^2_{L_r(X)}\right)^{1/2},$$

which means that $L_r(\mu, X)$ has equal-norm cotype q, i.e. weak cotype q, by (a) $\Leftrightarrow$ (d).

Next, we consider the case $r \in (2, q)$. By (a) $\Leftrightarrow$ (b), if $x_1, x_2, \ldots, x_n \in L_r(\mu, X)$ we have

$$\left(\sum_{i=1}^{n} \|x_i(\omega)\|^r\right)^{1/r} \leq \kappa_r n^{1/r-1/q} \left(E\left\|\sum_{i=1}^{n} g_i x_i(\omega)\right\|^r\right)^{1/r}$$

i.e.

$$\sum_{i=1}^{n} \|x_i(\omega)\|^r \leq (\kappa_r n^{1/r-1/q})^r E\left\|\sum_{i=1}^{n} g_i x_i(\omega)\right\|^r,$$

for all w $\in \Omega$. Integration against μ yields, again by Fubini's Theorem,

$$C_r(L_r(\mu, X), n) \leq \kappa_r n^{1/r-1/q},$$

which shows that $L_r(\mu, X)$ has weak cotype q, by (a) $\Leftrightarrow$ (c).

(e) $\Rightarrow$ (f) $\Rightarrow$ (a) are trivial.

Remarks: (A) It is not clear whether $r = q$ can be included in (e) and (f) or not. If $q = 2$, $L_2(X)$ has weak cotype 2 iff X has cotype 2, as it is proved in [32].

(B) A first proof of the equivalence of(a) and (d) was obtained in collaboration with U. Matter: the one given above is somewhat different from the original one (compare [28]).

Corollary 2.4. *Let $q \in (2, \infty)$, and X be a Banach space. Then X has weak cotype q iff there is a constant κ such that, for any n-dimensional subspace E of X,*

$$C_2(E) \leq \kappa n^{1/2-1/q}$$

Proof . By theorem 2.3, X has weak cotype q iff there is a constant κ_2 such that

$$C_2(X, n) \leq \kappa n^{1/2-1/q}, \quad n \in \mathbb{N}.$$

By [44] th. 2, for any n-dimensional subspace E of X we have

$$C_2(E) \leq 2C_2(E, n).$$

Since clearly $C_2(E, n) \leq C_2(X, n)$, we get

$$C_2(E) \leq 2\kappa n^{1/2-1/q}.$$

To see the converse, let $x_1, x_2, \ldots, x_n \in X$ be arbitrarily given. Since

$$C_2(\text{span}\{x_1, \ldots, x_n\}) \leq \kappa n^{1/2-1/q},$$

we get from the definitions

$$\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \leq \kappa n^{1/2-1/q} \left(E\left\|\sum_{i=1}^{n} g_i x_i\right\|^2\right)^2.$$

Hence,

$$C_2(X, n) \leq \kappa n^{1/2-1/q}, \quad \forall n \in \mathbb{N},$$

so that X has weak cotype q by theorem 2.3. □

The concept of weak cotype is closely related to the existence of almost euclidean finite dimensional subspaces. To provide some further information, we will need a couple of lemmas. The proof of the first one can be found in [32].

Lemma 2.5. *Let F be a Banach space and $u \in L(\ell_2^n, F)$. Then for every $k \leq n$ there is a subspace G of ℓ_2^n with codim $G < k$ such that*

$$\|u|_G\| \leq k^{-1/2} d(F, \ell_2^{\dim F}) \pi_\gamma(u).$$

Lemma 2.6. *There is a constant c such that, for any n-dimensional space E and any isomorphism $u \in L(E, \ell_\infty^n)$, there exists a volumepreserving operator $v \in L(\ell_\infty^n)$ with*

$$e_{[cn]}(vu) \leq c\, vr(E) n^{-1/2} \|u\|.$$

Proof . By homogeneity, we may assume $\|u\| \leq 1$. Let $\mathcal{E}$ be the maximal volume ellipsoid contained in B_E. By Lemma 10 in [27], there are a volume preserving operator in $L(\ell_\infty^n)$ and an absolute constant c such that

$$e_{[cn]}(vu) \leq \left(\frac{\text{vol } uB_E}{\text{vol } B_{\ell_\infty^n}}\right)^{1/n} = vr(E) \left(\frac{\text{vol } u\mathcal{E}}{\text{vol } B_{\ell_\infty^n}}\right)^{1/n} \leq \frac{vr(E)}{vr(\ell_\infty^n)},$$

since $u\mathcal{E} \subseteq uB_E \subseteq B_{\ell_\infty^n}$. Now, it is known that $vr(\ell_\infty^n) \geq \hat{c} n^{1/2}$ for some absolute constant $\hat{c}$, so the lemma follows. ∎

Theorem 2.7. *Let $q \in [2, \infty)$, and X be a Banach space. Then*

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e):$$

(a) There exists $\delta \in (0, 1)$ and a constant C such that every n-dimensional subspace E of X contains a subspace F with

$$\dim F \geq \delta n \quad \text{and} \quad d(F, \ell_2^{\dim F}) \leq C n^{1/2-1/q}.$$

(b) There is a constant C such that, for all $n \in N$ and every n-dimensional subspace E of X,

$$vr(E) \leq C n^{1/2-1/q}.$$

(c) X has weak cotype q.

(d) There is a constant C such that, for every n-dimensional subspace E of X,

$$C n^{1/q} \leq \pi_\gamma(\text{id}_E).$$

(e) For each $\epsilon > 0$, there is a constant C_ϵ such that every n-dimensional subspace E of X contains a subspace F with

$$\dim F \geq C_\epsilon n^{2/q} \quad \text{and} \quad d(F, \ell_2^{\dim F}) \leq 1+\epsilon.$$

Remark: It follows that if $q = 2$ the five conditons above are equivalent. We get thus some of the characterizations of weak cotype 2 obtained by V.D. Milman and G. Pisier (see [32], Cor. 5).

Proof . (a) $\Rightarrow$ (b). Just take the Milman-Pisier proof for the case $q = 2$, with minor changes [32].

(b) $\Rightarrow$ (c). If (b) holds, by Lemma 2.6 there is a constant C = C(X) such that, for any n-dimensional subspace E of X and any isomorphism $u \in L(E, \ell_\infty^n)$, we can find a volume preserving operator $v \in L(\ell_\infty^n)$ such that $e_{[cn]}(vu) \leq Cn^{-1/q}||u||$. Reasoning as in Pajor's proof of Theorem 2 in [34], we see that this implies that X has weak cotype q.

(c) $\Rightarrow$ (d). By proposition 2.1, if X has weak cotype q then $wC_q(E) \leq wC_q(X)$ for all subspaces E of X. Let $E \in$ Dim (X) be n-dimensional, and let $u \in L(\ell_2^n, E)$. We have

$$\sigma_{q,\infty}^a(u) \leq wC_q(X)\pi_\gamma(u) \leq wC_q(X)\pi_\gamma(id_E)||u||,$$

and, by the definition of the Weyl numbers,

$$\sigma_{q,\infty}^x(id_E) \leq wC_q(X)\pi_\gamma(id_E).$$

The left hand side is greater than $Cn^{1/q}$ for some universal constant C ([37] Th. 12), hence (d) follows.

(d) $\Rightarrow$ (e). Let X satisfy (d) and E be an n-dimensional subspace of X. Since

$$\pi_\gamma(id_E) = \sup\{\pi_\gamma(u) : u \in L(\ell_2^n, E), ||u|| = 1\},$$

a compactness argument yields an $u \in L(\ell_2^n, E)$ such that $\pi_\gamma(u)\pi_\gamma(\text{id},)$ and $||u|| = 1$. Further, we may assume that u is one-to-one. It follows from [33] 15.1.1 and 5.1, that there is a universal constant C' such that

$$\pi_\gamma(u) = n^{1/2}\left(\int_{S^{n-1}} ||u(\xi)||^2 d\mu(\xi)\right)^{1/2} \leq C'n^{1/2}M_r,$$

where $S^{n-1} := \{\xi \in \ell_2^n : ||\xi|| = 1\}$, μ is the normalized Haar measure on S^{n-1} and M_r is the median of the function $r(\xi) := ||u(\xi)||$ on S^{n-1} with respect to μ. So,

$$\frac{C}{C'}n^{1/q-1/2} \leq M_r.$$

Now, by the Figiel-Lindenstrauss-Milman version of Dvoretzky's Theorem ([6], Th. 2.6), given $\epsilon > 0$, there area constant C_ϵ and a subspace F of E with

$$\dim F \geq C_\epsilon n M_r^2 ||u||^{-2} \geq C_\epsilon n M_r^2 \geq C'_\epsilon n^{2/q}$$

and

$$d(F, \ell_2^{\dim F}) \leq 1 + \epsilon.$$ 0

Remark: just as in [32] Th. 1, it is possible to prove that if (a) of theorem 2.7 holds for a Banach space X and for one $\delta \in (0, 1)$, then it holds for all $\delta \in (0, 1)$. Of course, in this case C will depend on δ.

Problem 2.7*. *It would be interesting to know which of the implications appearing in theorem 2.7 may be reversedfor $q > 2$, as it is known to be the case if $q = 2$.*

In the presence of K-convexity, however, we are able to prove the following

Theorem 2.8. *Let $q \in (2, \infty)$ and let X be K-convex. Then X has weak cotype q if and only if there exist $\delta \in (0, 1)$ and a constant C such that, for every $n \in IN$, every n-dimensiona1 subspace E of X contains a subspace F with*

$$\dim F \geq \delta n \text{ and } d(F, \ell_2^{\dim F}) \leq C n^{1/2-1/q}.$$

Proof. Let X be K-convex. If X has weak cotype q, by 2.4 there is a constant κ such that, for any n-dimensional subspace E of X we have

$$T_2(E^*) \leq C_2(E) K(E) \leq \kappa n^{1/2-1/q} K(X),$$

where $K(X) < \infty$ is the (gaussian) K-convexity constant of X. Now, by a result of V.D. Milman ([31], Th. 5.1), given $\delta \in (0,1)$ there is a constant C such that, for every n-dimensional subspace E of X, there is a subspace F of E with

$$\dim F \geq \delta n \text{ and } d(F, \ell_2^{\dim F}) \leq C T_2(E^*)$$

Combining both inequalities for $T_2(E^*)$, we see that the desired property holds. The opposite implication is 2.7 (a) $\Rightarrow$ (c). 0

If we require type 2 instead of K-convexity in theorem 2.8, the situation is more pleasant, since we are now able to avoid the machinery of «proportional subspaces». We prepare our statement again by a lemma:

Lemma 2.9. ***Let $q \in [2,\infty)$ and let X have type 2 and weak cotype q. Then***

$$\Gamma_1(\cdot,X) \subset S^x_{q,\infty}(\cdot,X).$$

Proof . By Grothendieck's Theorem (0.13') , $\Gamma_1 \subset \Pi_2^d \cdot \Gamma_2^{-1}$ and this yields

$$T_2 \cdot \Gamma_1 \subset (T_2 \cdot \Pi_2^d) \cdot \Gamma_2^{-1} \subset \Pi_\gamma \cdot \Gamma_2^{-1} = \Pi_\gamma,$$

by [38] 21.3.5. Hence, if X has weak cotype ***q*** we have, by proposition 2.1,

$$T_2 \cdot \Gamma_1(\cdot,X) \subset S^x_{q,\infty}(\cdot,X).$$

In particular, if X has type 2 (i.e., if $id_X \in T_2$) we get the lemma. □

The converse is also true whenever X has type 2; we shall prove this together with other equivalent statements in 4.5. But 2.9 suffices ah-eady to yield the follwing improvement of 2.7 for spaces of type 2:

Theorem 2.10. ***Let $q \in (2,\infty)$ and let X have type 2. Then X has weak cotype q if and only if there is a constant C such that, for any n-dimensiona1 subspace E of X,***

$$d(E,\ell_2^n) \le C n^{1/2-1/q}.$$

Further, there exists a projection P of X into E with

$$\|P\| \le C T_2(X) n^{1/2-1/q}.$$

Proof . We use an argument from [36] cor. 22.1. Let ***E*** be an n-dimensiona1 subspace of X and let $s: \ell_1 \to E$ be a quotient map. By lemma 2.9, there is a constant C such that

$$\sigma^x_{q,\infty}(s) \le C\|s\| = C.$$

By (0.31), there is a constant C_q depending only on ***q*** such that

$$\pi_2(s) \le C_q n^{1/2-1/q} \sigma^x_{q,\infty}(s) \le C_q C n^{1/2-1/q}.$$

Now, by the surjectivity of γ_2 ;

$$d(E,\ell_2^n) = \gamma_2(id_E) = \gamma_2(s) \le \pi_2(s),$$

and thus the desired estimate follows. The existence of a projection ***P*** of X onto ***E as*** above is an immediate consequence of Maurey's extension theorem for type 2 spaces [29]. □

Remark: Theorem 2.10 is not true for $q = 2$ since there are spaces of type 2 and weak cotype 2 failing to be isomorphic to a Hilbert space. In fact, if $q \in [2, \infty)$, the space $\hat{X}(q, \eta)$ defined in [6] Ex. 5.3 (using a construction by W.B. Johnson) has an unconditional basis, type 2 and weak cotype q (by 2.7 (a) $\Rightarrow$ (c)), but not cotype q. This also shows that the weak cotype q property is strictly weaker than cotype q, for all $q \in [2, \infty)$.

Related spaces have been constucted by L. Tzafriri [47]: they also have an unconditional basis, type 2 and equal-norm cotype q but not cotype q if $q > 2$. Since weak cotype q and equal-norm cotype q are equivalent for $q > 2$ (theorem 2.3), Tzafriri's spaces turn out to be exactly as useful (for our purposes) as Johnson's spaces cited above.

Let us now prove a couple of properties of weak cotype g spaces, which will enable us to provide some counterexarnples to further questions related to our subject.

Proposition 2.11. ***Let $q \in [2, \infty)$ and let X have weak cotype q. Then***

$$\Gamma_\infty(\cdot, X) \subset S^x_{q,\infty}(\cdot, X).$$

Proof . By proposition 2.1 and (0.29), X has cotype $q + \epsilon$ for all $\epsilon > 0$ and thus, by (0.36), X does not contain the ℓ^n_∞ 's uniformly. Hence, by Proposition 1.2 (d) and again by proposition 2.1 the conclusion follows. □

Corollary 2.12. ***Let $q \in [2, \infty)$ and let X have weak cotype q. Then there is a constant C such that, for any n-dimensiona1 subspace E of X, we have***

$$n^{1/q} \leq C\lambda(E),$$

where $X(E) := \gamma_\infty(id_E)$ is the projection constant of E.

Proof . This follows from the fact that $\sigma^a_{q,\infty}(id_E) \geq \kappa n^{1/q}$ for a universal constant κ ([37], ***th. 12)*** and from 2.11. □

As it is clear from the proof, corollary 2.12 holds under the weaker assumption

$$\Gamma_\infty(\cdot, X) \subset S^x_{q,\infty}(\cdot, X).$$

Stated in this general form and for $q > 2$, corollary 2.12 was first proved by U. Matter (personal communication).

Proposition 2.13. ***The Lorentz sequence space $\ell_{2,1}$ satisfies a lower 2-estimate (i.e. there is a constant C such that***

$$\left(\sum_{i=1}^n ||x_i||^2\right)^{1/2} \leq C \left\|\sum_{i=1}^n x_i\right\|$$

*for all $x_1, x_2, \ldots, x_n \in \ell_{2,1}$ **with disjoint** support), but **does not** have **weak cotype 2.***

Proof . That $\ell_{2,1}$ satisfies a lower 2-estimate was observed in [3] prop. 3.2. On the other hand, by th. 3.2 of [20], there is a constant C' such that

$$\lambda(\ell_{2,1}^n) \leq C' \left(\frac{n}{\log(\log n)}\right)^{1/2}, \quad \forall n \in \mathbb{N}.$$

By corollary 2.12 above, $\ell_{2,1}$ cannot have weak cotype 2. □

We conclude this section with a result on Orlicz sequence spaces. We will need another lemma which is esentially known. We provide a proof for completeness:

Lemma ***2.14. Let M be an Orliczjùnction and let $X_n \subset \ell_M$ be the*** *subspace* ***spanned*** *by the first* ***n*** *coordinates.* ***Then***

$$d(X_n, \ell_\infty^n) \leq \frac{M^{-1}(1)}{M^{-1}(1/n)}.$$

Proof . Consider the identities $i: X_n \to \ell_\infty^n$ and $j: \ell_\infty^n \to X_n$.
Let $(\alpha_k) \in X_n$, $1 \leq k \leq n$, and $r \in \{1, \ldots, n\}$ be such that

$$|\alpha_r| = \sup_{1\leq k\leq n} |\alpha_k|.$$

If we define $\rho_0 := |\alpha_r|/M^{-1}(1)$ and assume $|\alpha_r| > 0$ we have

$$\sum_{k=1}^{n} M\left(\frac{|\alpha_k|}{\rho_0}\right) \leq M\left(\frac{|\alpha_r|}{\rho_0}\right) = 1$$

and so, by the definition of the norm in ℓ_M,

$$||i((\alpha_k))||_\infty = \sup_{1\leq k\leq n} |\alpha_k| = M^{-1}(1)\rho_0 \leq$$
$$\leq M^{-1}(1)||(\alpha_k)||_{\ell_M} = M^{-1}(1)||(\alpha_k)||_{X_n}$$

Since the latter inequality trivially holds if $|\alpha_r| = 0$, we have $||i|| \leq M^{-1}(1)$.
To estimate $||j||$, notice that there is a vector $(\alpha_k) \in \ell_\infty^n$ such that $\sup_{k\leq n} |\alpha_k| = 1$ and

$$||(\alpha_k)||_{X_n} = ||(\alpha_k)||_{\ell_M} = ||j||.$$

Since it is readily checked that

$$\|(\alpha_k)\|_{\ell_M} = \min\left\{\rho : \sum_{k=1}^{n} M\left(\frac{|\alpha_k|}{\rho}\right) = 1\right\},$$

we have

$$1 = \sum_{k=1}^{n} M\left(\frac{|\alpha_k|}{\|j\|}\right) \leq nM\left(\frac{1}{\|j\|}\right)$$

(M is nondecreasing) and thus $\|j\| \leq 1/M^{-1}(1/n)$.
Combining the estimates for $\|i\|$ and $\|j\|$ we get finally

$$d(X_n, \ell_\infty^n) \leq \|i\|\,\|j\| \leq \frac{M^{-1}(1)}{M^{-1}(1/n)}.$$

Proposition 2.15. *Let M be an Orlicz function and let $q \in [2, \infty)$. If ℓ_M has weak cotype q then there is a constant C such that $M(\epsilon) \geq C\epsilon^q$ for all $\epsilon > 0$ sufficiently small. Zn particular, if $M(\epsilon) = \epsilon^q|\log \epsilon|^{-\alpha}$ ($\alpha > 0$), then ℓ_M (which is known to have type 2 and cotype q' for all $q' > q$) does not have weak cotype q.*

Proof. If ℓ_M has weak cotype q then, by corollary 2.12, there is a constant C' such that, for all $n \in \mathbb{N}$, $n^{1/q} \leq C'\lambda(X_n)$ where $X_n \subset \ell_M$ is the subspace spanned by the first n coordinates. Hence, by lemma 2.14,

$$n^{1/q} \leq C'\lambda(X_n) \leq C' d(X_n, \ell_\infty^n) \leq C' \frac{M^{-1}(1)}{M^{-1}(1/n)}$$

for all $n \in \mathbb{N}$, and so

$$\sup_{n\in\mathbb{N}} n^{1/q} M^{-1}(1/n) \leq C^{1/q},$$

where $C := [C'M^{-1}(1)]^q$. Since M is nondecreasing this is easily seen to imply $M(\epsilon) \geq C\epsilon^q$ for all $\epsilon > 0$ sufficiently small.

Let us now consider the special case $M(\epsilon) := \epsilon^q|\log \epsilon|^{-\alpha}$ ($\alpha > 0$) for all ϵ close to 0. Let $\delta_x(\epsilon)$ (resp. $\rho_x(\tau)$) be the modulus of convexity (resp. smoothness) of the Orlicz space ℓ_M (see [25] 1.e for the definitions). It follows then from th. 1 of [26] that, for every $q' > q$,

$$\delta_X(\epsilon) \geq c_1\epsilon^{q'} \quad \text{and} \quad \rho_X(\tau) \leq c_2\tau^2$$

for all ϵ and τ close to 0 and for some constants c_1, c_2. These inequalities together with the main result of [7] imply that ℓ_M has cotype q', $q' > q$, and type 2. Finally, by what was proved above, ℓ_M does not have weak cotype q.

3. WEAK TYPE

Let $p \in (1,2]$. By 1.2 (b) and 1.3 (d), a Banach space has ***weak type*** p ***if X*** has $Q(w(\pi_{p^*,2})^d, \pi_\gamma^*)$ (or, equivalently, if X has $P(w\pi_\gamma, (\pi_{p^*,2})^{d*}))$, i.e. if and only if there is a constant C such that

$$\sigma_{1,\infty}^a(vu) \leq C(\pi_{p^*,2})^*(u^*)\pi_\gamma^*(v),$$

for all $u \in L(\ell_2^n, X)$, $v \in L(X, \ell_2^n)$ and all $n \in N$. Given a Banach space X and $p \in (1,2]$, we define $w\mathrm{T}_p(X)$ to be the least constant C such that

$$\sigma_{1,\infty}^a(v) \leq C\pi_\gamma^*(v),$$

for all $v \in L(X, \ell_2^n)$ and all $n \in N$ (let $w\mathrm{T}_p(X) := \infty$ if no such constant exists). $w\mathrm{T}_p(X)$ is called the ***weak type p constant*** of X.

Proposition 3.1. ***Let*** $p \in (1,2]$ ***and X be a Banach space. The following conditions are equivalent:***

(a) X has weak type p.

(b) There is a constant C such that, for all n andfor all $v \in L(X, \ell_2^n)$,

$$\sigma_{p^*,\infty}^a(v) \leq C\pi_\gamma^*(v).$$

(c) There is a constant C such that, for every finite-dimensiona1 subspace E of X,

$$w\mathrm{T}_p(E) \leq C.$$

(d) id, $\in (\Pi_\gamma^*)^{-1} \cdot (S_{p^*,\infty}^x)^d$, ***i.e. all*** Π_γ^****-operators t defined on X satisfy***

$$(x_n(t))_{n\in\mathbb{N}} \in \ell_{p^*,\infty}.$$

Proof . (a) $\Leftrightarrow$ (b). By the considerations above and by propositon 1.11, has weak type p if and only if X has $Q(\sigma_{p^*,\infty}^a, \pi_\gamma^*)$.

(b) $\Rightarrow$ (c). Let $E \in \dim(X)$ and let $v \in L(E, \ell_2^n)$. By lemma 1.7, v admits an extension $w \in L(X, \ell_2^n)$ such that $\pi_\gamma^*(w) = \pi_\gamma^*(v)$. Let $j_E: E \to X$ be the natura1 embedding. We have

$$\sigma_{p^*,\infty}^a(v) = \sigma_{p^*,\infty}^a(wj_E) \leq \sigma_{p^*,\infty}^a(w) \leq$$
$$\leq w\mathrm{T}_p(X)\pi_\gamma^*(w) = w\mathrm{T}_p(X)\pi_\gamma^*(v),$$

hence $wT_p(E) \leq wT_p(X)$.

(c) $\Rightarrow$ (b). Let $v \in L(E, \ell_2^n)$. By lemma 1.10 (b), we get

$$\sigma^a_{p^*,\infty}(v) = \sup_{E \in \mathrm{Dim}(X)} \sigma^a_{p^*,\infty}(v|_E) \leq$$

$$\leq \sup_{E \in \mathrm{Dim}(X)} wT_p(E)\pi^*_\gamma(v|_E) \leq C\pi^*_\gamma(v),$$

and thus (b) holds.

(b) $\Leftrightarrow$ (d) is proved in the same manner as the corresponding statement in proposition 2.1.

0

Remark: We have actually proved that $wT_p(X) = \sup_{E \in \mathrm{Dim}(X)} wT_p(E)$ for all Banach spaces X.

It is straightforward to deduce from propositions 3.1 and 1.11 (b) the next

Corollary *3.2. If p $\in (1, 2)$, weak (weak type p) is equivalent to weak type p.*

Problem 3.2*. ***Does the same holdfor weak type 2?***

The analysis of the weak type property is considerably simplified by the following duality theorem:

Theorem 3.3. ***Let p*** $\in (1, 2]$. X ***has weak type p if und only if X is K-convex and X* has weak cotype*** p^*.

Proof. Let X have weak type p and let $u \in L(\ell_2^n, X^*)$. Put v: $= u^*|_X$, so that $u = v^*$ and, by (0.25),

$$\sigma^a_{p^*,\infty}(u) = \sigma^a_{p^*,\infty}(v) \leq wT_p(X)\pi^*_\gamma(v) =$$

$$= wT_p(X)\pi^*_\gamma(u^*) \leq wT_p(X)\pi_\gamma(u),$$

i.e. X* has weak cotype p^* (as for the last of the preceding inequalities, see [46] th. 11.57). To see that X must be K-convex, it is enough to show that the sequence $(wT_p(\ell_1^n))_{n \in \mathbb{N}}$ is unbounded. As always, given p, $q \in [1, \infty]$, let $i_{p,q}$ be the identity of $\mathbb{R}^n$ regarded as a map $\ell_p^n \to \ell_q^n$. Then clearly $\sigma^a_{1,\infty}(i_{1,2}i_{2,1}) = n$ and $\pi^*_\gamma(i_{1,2}) \leq \kappa$, for some universal constant κ (Lemma 1.3). Further, by [38] 9.1.8,

$$(\pi_{p^*,2})^*(i^*_{2,1}) = (\pi_{p^*,2})^*(i_{\infty,2}) = \frac{n}{\pi_{p^*,2}(i_{2,\infty})} \leq \frac{n}{n^{1/p^*}} = n^{1/p}.$$

Assume that $(wT_p(\ell_1^n))_{n\in\mathbf{N}}$ is bounded. Then, by definition, there is a constant C such that, for all n,

$$n = \sigma_{1,\infty}^a(i_{1,2}\cdot i_{2,1}) \leq C(\pi_{p^*,2})^*(i_{2,1}^*)\pi_\gamma^*(i_{1,2}) \leq C\kappa n^{1/p},$$

which is impossible.

Suppose now that X is K-convex and that X* has weak cotype p^*. Then, if $v \in L(X,\ell_2^n)$,

$$\sigma_{p^*,\infty}^a(v) = \sigma_{p^*,\infty}^a(v^*) \leq wC_{p^*}(X^*)\pi_\gamma(v^*) \leq$$
$$\leq wC_{p^*}(X^*)K(X)\pi_\gamma^*(v),$$

i.e.

$$wT_p(X) \leq wC_{p^*}(X^*)K(X) < \infty.$$

In analogy with theorem 2.3 and corollary 2.4 we are now able to prove the next

Theorem 3.4. ***Let p $\in(1,2)$ and X be a Banach space. The following conditions are equivalent:***

(a) X has weak type p.

(b) For each $\tau\in(p,2]$ there is a constant κ_τ such that

$$T_\tau(X,n) \leq \kappa_\tau n^{1/p-1/\tau}, \quad \forall n\in\mathbf{N}$$

(c) There are an $\tau\in(p,2]$ and a constant κ such that

$$T_\tau(X,n) \leq \kappa n^{1/p-1/\tau}, \quad \forall n\in\mathbf{N}.$$

(d) X has equal-norm type p.

(e) $L_\tau(\mu,X)$ has weak type p for all $\tau\in(p,\infty)$ and all measure spaces (Ω,μ).

(f) $L_\tau(\mu,X)$ has weak type p for some $\tau\in(p,\infty)$ and some (nontrivial) measure space (Ω,μ).

(g) There is a constant κ such that, for every n-dimensional subspace E of X,

$$T_2(E) \leq \kappa n^{1/p-1/2}$$

Proof. (a) $\Rightarrow$ (b). By [33], 9.9, $T_\tau(X,n) \leq K(X)C_{\tau^*}(X^*,n)$ and thus (b) follows from theorem 3.3 and theorem 2.3 (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d). Suppose (c) holds and let $x_1, \ldots, x_n \in X$ be norm-one vectors. Then

$$\left(E\left\|\sum_{i=1}^{n} g_i x_i\right\|^2\right)^{1/2} \leq \kappa n^{1/p-1/r}\left(\sum_{i=1}^{n}\|x_i\|^r\right)^{1/r} = \kappa n^{1/p},$$

i.e., X has equal-norm type p.

(d) $\Rightarrow$ (a). Since X has equal-norm type p if and only if it is K-convex and X* has equal-norm cotype p^* (cf. (0.33)), (d) $\Rightarrow$ (a) follows from theorem 3.3 and from the implication (d) $\Rightarrow$ (a) in theorem 2.3.

(a) $\Rightarrow$ (e). Let $r \in (p, \infty)$ and let (Ω, μ) be a measure space. By theorem 3.3 and theorem 2.3 (a) $\Rightarrow$ (e), $L_{r^*}(\mu, X^*)$ is K-convex and has weak cotype p^*. It follows then again form theorem 3.3 that $L_{r^*}(\mu, X^*)^*$ has weak type p and, since $L_r(\mu, X)$ is isometric to a subspace of $L_{r^*}(\mu, X^*)^*$, $L_r(\mu, X)$ has weak type p (by proposition 3.1).

(e) $\Rightarrow$ (f) $\Rightarrow$ (a) are trivial.

Finally, the proof of (b) $\Rightarrow$ (g) $\Rightarrow$ (c) carries over without difficulty from the proof of corollary 2.4. □

With the aid of theorem 2.3 and thorem 3.4 it is now possible to obtain a generalization of a result contained in [16] (th. 3). In view of the equivalence between equal-norm cotype q (resp. type p) and weak cotype q (resp. weak type p) for $q \in (2, \infty)$ (resp. $p \in (1,2)$), we can give a concise statement. Accordingly, ***we define*** the ***equal-norm cotype*** q (resp. ***equal-norm type p) constant*** by

$$eC_q := \begin{cases} C_2, & q = 2 \\ wC_q, & q > 2 \end{cases} \quad \left(\text{resp. } eT_p := \begin{cases} T_2, & p = 2 \\ wT_p, & p < 2 \end{cases}\right).$$

Theorem ***3.5. Let*** $p \in (1, 2]$ ***and*** $q \in [2, \infty)$. ***Let X be a Banach space of equal-norm type*** p, $Z \subset X$ ***a subspace, F an n-dimensional normed space and*** $v \in L(Z, F)$. ***Then there is an extension*** $w \in L(X, F)$ ***with***

$$\gamma_2(w) \leq c_{pq} \min\{eC_q(Z), eC_q(F)\} eT_p(X) n^{1/p-1/q}\|v\|,$$

where c_{pq} ***is a constant which depends only on p and*** q.

Sketch of *Proof*. Using (0.31) if $p < 2$ or $q > 2$, it is not difficult to find a constant c_{pq} such that

$$|tr(sab)| \leq c\|s\|\pi_2^d(b)\pi_2(a),$$

for all $s \in L(Z, F)$, $a \in L(\ell_2^n, Z)$, $b \in L(F, \ell_2^n)$, where $c := c_{pq} \min\{eC_q(Z), eC_q(F)\}$ $eT_p(X) n^{1/p-1/q}$.

Since $\gamma_2^* = \pi_2^d \cdot \pi_2$ by a result of Kwapień ([38] 17.4.3), this means that, if $t \in L(F, Z)$,

$$|tr(st)| \leq c||s||\gamma_s^*(t)$$

which in turn is equivalent to

$$\nu_1(t) \leq c\gamma_2^*(t),$$

where ν_1 denotes the 1-nuclear norm ([13], 17.5.2). Now, the last inequality proves also that the operator

$$\phi : N_1(F, Z) \to \Gamma_2^*(F, X)$$

defined by $\phi(z) = iz$, where $i: Z \to X$ is the inclusion, is an isomorphic embedding. Hence, by duality, the adjoint operator

$$\phi^* : \Gamma_2(X, F) \to L(Z, F)$$

is a surjection (with norm c), which proves the theorem. □

Generalizing a result of B. Maurey, V.D. Milman and G. Pisier [32] have proved that X *has* ***weak type 2*** *if and* ***only iffor*** *all* $\delta \in (0, 1)$ *there* ***is a constant*** C_δ ***such that, for every*** *subspace* Z ***of X and every*** *operator* $v \in L(Z, \ell_2^n)$***,*** *there* ***exist*** *an* ***orthogonal projection p:*** $\ell_2^n \to \ell_2^n$ ***with*** $\text{rank}(p) \geq \delta n$ *and* ***anextension*** $w \in L(X, \ell_2^n)$ ***of pv*** *such* ***that*** $||w|| \leq C_\delta ||v||$. Maurey [29] had originally shown that if X has type 2, then there is a constant C such that, if Z and v are as before, there is an extension w $\in L(X, \ell_2^n)$ of v such that $||w|| \leq C||v||$. It is not clear whether the converse holds. Of course, because of the Milman-Pisier result cited above, the last property implies that X has weak type 2.

If p $\in (1, 2)$ then the situation for weak type p is closer to the situation in Maurey's Theorem: in fact, there is no need to work with a projection p.

Theorem 3.6. ***Let p*** $\in (1, 2)$ ***and X be a Banach space. Then X has weak type p*** *if and* ***only if*** *there* ***exists a constant C such that, for every*** *subspace* Z ***of X and evety*** *operator* $v \in L(Z, \ell_2^n)$, *there* ***is*** *an* ***extension*** $w \in L(X, \ell_2^n)$ ***of v*** *such* ***that***

$$||w|| \leq Cn^{1/p-1/2}||v||.$$

Proof . Necessity follows form theorem 3.5 above. As for sufficiency, let the condition hold. Thus, if E is a k-dimensional subspace of X, there exists a projection p: X $\to$ ***E*** with

$$||p|| \leq Cd(E, \ell_2^k) k^{1/p-1/2}.$$

In particular, if X were not K-convex we would be able to construct a projection p from ℓ_1^{2n} onto a (uniformly) Hilbertian n-dimensional subspace F such that $\|p\| \le C' n^{1/p-1/2}$. Since $1/p - 1/2 < 1/2$, this would be a contradiction, since $\gamma_1(\ell_2^n)$ is of order $n^{1/2}$. So, X is K-convex.

To prove that X* has weak cotype p^*, we use the argument of [32] th. 10 (iii) ⇒ (i). Let $u \in L(\ell_2^n, X^*)$. By [32] prop. 7, there exists a subspace Z of X with codim $Z < [n/2]$ such that, for some constant κ,

$$\|u^*|_Z\| \le \kappa \pi_\gamma(u) n^{-1/2}.$$

By our hypothesis, there is an extension $v \in L(X, \ell_2^n)$ of $u^*|_Z$ such that

$$\|v\| \le C n^{1/p-1/2} \|u^*|_Z\|.$$

Since $(u^*|_X - v)|_Z = 0$, we have rank$(u^*|_X - v) \le$ codim $Z < [n/2]$. So, since $(u^*|_X - v)^* = u - v^*$ and

$$\|u - (u - v^*)\| = \|v\| \le \kappa C n^{1/p-1/2} \pi_\gamma(u) n^{-1/2} = \kappa C \pi_\gamma(u) n^{-1/p^*},$$

we get

$$a_{[n/2]}(u) \le \kappa C \pi_\gamma(u) n^{-1/p^*}.$$

By (0.32), $\sigma^a_{p^*,\infty}(u) \le C' \pi_\gamma(u)$ follows with a suitable constant C', so X* has weak cotype p^*. □

Theorem 2.8 and theorem 3.3 lead to the following characterization of weak type p:

Theorem *3.7. Let $p \in (1, 2]$. A Banach space X has weak type p if and only if there are constants C and $\delta \in [0, 1)$ such that, for all n and every n-dimensional subspace E of a quotient of X^*, there exists a subspace F of E with*

$$\dim F > \delta n \quad \text{and} \quad d(F, \ell_2^{\dim F}) \le C n^{1/p-1/2}.$$

Proof. Let E be an n-dimensional subspace of a quotient Z of X*, and note that Z^* is isometric to a subspace of X**, which also has weak type p, by 3.1. It follows easily that the weak type p constant of E^* is bounded by the weak type p constant of X, so that (reasoning as in the proof of 2.8) the verification of necessity is complete.

Assume now that the condition holds. The assertion about subspaces already implies weak cotype p^* for X^* by theorem 2.7 (a) ⇒ (c). K-convexity of X is obtained as follows: if X contains the ℓ_1^n's uniformly, X^* has quotients almost isometric to ℓ_∞^n, so that our hypothesis contradicts the result of Szarek about «large» subspaces of ℓ_∞^n which was used in the proof of theorem 3.6 (cf. [36] th.8.1). □

We conclude this section with an analogue of theorem 2.7 (b) $\Rightarrow$ (c) for weak type, thereby generalizing a recent result of A. Pajor [34]:

Theorem 3.8. *Let $p \in (1, 2]$. A Banach space X has weak type p if and only if there is a constant C such that, for every n-dimensional quotient E of X^*,*

$$vr(E) \leq Cn^{1/p-1/2}.$$

Proof . The case p = 2 has been proved by A. Pajor [34]. Further, if $p < 2$, sufficiency is also seen as in Pajor's paper with only minor modifications. To see that the condition is necessary, we argue as follows: let E be an n-dimensional quotient of X^*, and let $u_E \in L(\ell_2^n, E)$ be an isomophism such that the image by u_E of the unit ball of ℓ_2^n is the ellipsoid of maximal volume inscribed in B_E (0.14). By [35], there is a universal constant κ such that

$$n^{1/2} e_n(u_E^{-1}) \leq \kappa \pi_\gamma((u_E^{-1})^*),$$

where $e_n(\cdot)$ denotes the n-th entropy number (0.22). Since X has weak type p, it is K-convex and thus, by (0.14),

$$\pi_\gamma((u_E^{-1})^*) \leq K(X) \pi_\gamma^*(u_E^{-1}) \leq$$
$$\leq K(X) C_2(E) \pi_2(u_E^{-1}) \leq n^{1/2} K(X) T_2(E^*).$$

Now, E^* is isometric to a subspace of X**, which has weak type p. Since $p < 2$, by theorem 3.4 there is a constant κ' such that $T_2(E^*) \leq \kappa' n^{1/p-1/2}$, so that we get

$$e_n(u_E^{-1}) \leq \kappa\kappa' K(X) n^{1/p-1/2}.$$

This proves the necessity since, by the definition of e_n, we have

$$vr(E) \leq 2 e_n(u_E^{-1}),$$

as it is easy to verify. □

4. APPLICATIONS TO WEAK HILBERT SPACES

By S. Kwapień [193, X is isomorphic to a Hilbert space if and only if X has $P(\pi_2, \pi_2^d)$. Correspondingly, we say that X is a ***weak Hilbert space*** if there is a constant C such that

$$\sigma_{2,\infty}^a(u) \leq C\pi_2^d(u), \ \forall u \in L(\ell_2^n, X), \ \forall n \in \mathbb{N}.$$

For fixed n we let $w\gamma_2^{(n)}$ (X) : = inf C, the infimum being extended over all C as above, so that X is a weak Hilbert space if and only if $w\gamma_2(X) := \sup_{n\in\mathbb{N}} w\gamma_2^{(n)}(X) < \infty$. A wealth of chamcterizations and results about weak Hilbert spaces is to be found in G. Pisier's paper [43], among which the fact that X is a weak Hilbert space if and only if it verifies the weak analogue of Kwapied's result (cf. (0.33′)), more precisely, *if and only if it has (simultaneously) weak type 2 and weak cotype 2.*

Here we supplement this by an observation on Orlicz spaces which allows us to solve in the negative the «three space problem» for weak Hilbert spaces: ***given a subspace Y of X such that both Y and X/Y are weak Hilbert spaces, does itfollow that X is a weak Hilbert space, too?*** If we read «isomorphic to a Hilbert space» instead of «weak Hilbert space», the answer is «no», as it was first proved in [5]. Later on, another counterexample was provided by N.J. Kalton and N.T. Peck [15]; we will show that this solves in the negative the «three space problem» for weak Hilbert spaces, too.

Proposition 4.1. ***Let ℓ_M be an Orlicz sequence space. Then ℓ_M is a weak Hilbert space if and only if it is isomorphic to ℓ_2, i.e. if and only if $M(\epsilon)$ is equivalent to ϵ^2.***

Proof. Let ℓ_M be a weak Hilbert space. Since ℓ_M has weak cotype 2, by proposition 2.15 there is a constant C_1 such that $M(\epsilon) \geq C_1\epsilon^2$ for all ϵ close to 0 , but this already means that ℓ_M embeds (continuosly) into ℓ_2 in the canonica1 way. Further, since clearly ℓ_M does not contain subspaces isomorphic to ℓ_∞, by [25] 4.a.4 and 4.b.1, $(\ell_M)^*$ and ℓ_{M^*} are isomorphic, ***M* being*** the Orlicz function complementary to ***M*** (cf. [25] 4.b.l). Since ℓ_M has weak type 2, ℓ_{M^*} has weak cotype 2 by 3.3 and thus, by the same argument as above, there is a constant C_2 such that $M^*(\epsilon) \geq C_2\epsilon^2$ for all ϵ close to 0. Since for all $\alpha = (\alpha_k) \in \ell_M$ we have (see [25] 4.b).

$$\begin{aligned}||\alpha||_{\ell_M} &\leq \sup\left\{\sum_k \alpha_k\beta_k : \sum_k M^*(|\beta_k|) \leq 1\right\}\\ &\leq \sup\left\{\sum_k \alpha_k\beta_k : \sum_k C_2\beta_k^2 \leq 1\right\}\\ &= C_2^{-1/2}\left(\sum_k \alpha_k^2\right)^{1/2},\end{aligned}$$

we also get that ℓ_2 canonically embeds into ℓ_M. It follows that ℓ_M and ℓ_2 coincide as sets and have equivalent norms, so that ***M(E)*** must be quivalent to ϵ^2. □

Kalton and Peck [15] defined the space Z_2 of all sequences $((a_n, b_n))_{n\in\mathbb{N}}$ of pairs of real numbers such that

$$\beta := \left(\sum_{n=1}^{\infty} b_n^2 \right)^{1/2} < \infty$$

and

$$||((a_n, b_n))_{n \in \mathbb{N}}|| := \beta + \left[\sum_{n=1}^{\infty} (a_n - b_n \log[|b_n|\beta^{-1}])^2 \right]^{1/2} < \infty.$$

The latter expression is equivalent to a norm, and Z_2 is a Banach space. One of the significant features of Z_2 is that it is not isomorphic to a Hilbert space, since it contains the Orlicz space ℓ_N, where $N(\epsilon) := \epsilon^2 (\log \epsilon)^2$ for ϵ close to 0. Since Z_2 is also known to contain a subspace Y such that both Y and Z_2/Y are isometric to ℓ_2, Z_2 provides an example to show that being isomorphic to a Hilbert space is not a «three space property» [15]. But ℓ_N even fails to be a weak Hilbert space, by 4.1, so the same is true for Z_2 as well (although it has cotype $2 + \epsilon$ and type $2 - \epsilon$ for all positive ϵ, by a general result proved in [5]). Hence we have the following

Corollary ***4.2. Being a weak Hilbert space is not a «three space property».***

We prove now a proposition which clarifies the connection between $w\gamma_2^{(n)}$ (X) and the so-called ***Grothendieck numbers*** /c,(X) for a Banach space X . Recall that, for all $n \in \mathbb{N}$,

$$k_n(X) := \sup\{|\det((x_i, x_j^*)_{i,j=1}^n)| : x_i \in B_X, x_j^* \in B_{X^*}\}.$$

A recent account of the theory of Grothendieck numbers is given in [9]. They were originally introduced by A. Grothendieck [10] and fist used by G. Pisier [43] to characterize weak Hilbert spaces.

Proposition ***4.3. Let X be a Banach space. Then***

$$\frac{1}{6} \sup_{t \le n} k_t(X)^{1/t} \le w\gamma_2^{(n)}(X) \le 3e^2 \sup_{t \le n} k_t(X)^{1/t}.$$

Proof. Let $u \in L(\ell_2^n, X)$, $v \in L(X, \ell_2^n)$. It is known (cf. [17] l.b.4 and l.b.2) that

$$\sigma_{1,\infty}^a(vu) = \sup_{t \le n} t|\lambda_t(|vu|)|,$$

where $|vu| := ((vu)^* vu)^{1/2}$ and $(\lambda_t(w))_{t \in \mathbb{N}}$ is sequence of all eigenvalues of a given operator $w \in \boldsymbol{L}(\ell_2^n)$, repeated according to multiplicity and arranged in nonincreasing order.

By polar decomposition, there exists a partial isometry $i \in L(\ell_2^n)$ such that $|vu| = i^* vu$. Further, by [38] 27.3.3,

$$\lambda_t(|vu|) = \lambda_t(i^* vu) = \lambda_t(u i^* v), \quad \forall t \in \mathbb{N},$$

and so

$$\sigma_{1,\infty}^a(vu) = \sup_{t \leq n} t|\lambda_t(u i^* v)| \leq e^2 (\sup_{t \leq n} k_t(X)^{1/t}) \gamma_2^*(u i^* v),$$

where the inequality is taken from [9] 2.2.2. By another result of Kwapień (see e.g. [38] 17.4.3 and 19.3.10),

$$\gamma_2^*(u i^* v) \leq \pi_2^d(u) \pi_2(i^* v) \leq \pi_2^d(u) \pi_2(v),$$

since $||i|| \leq 1$. It follows that

$$\sigma_{1,\infty}^a(vu) \leq e^2 (\sup_{t \leq n} k_t(X)^{1/t}) \pi_2^d(u) \pi_2(v),$$

hence, by definition,

$$w\pi_2(u) \leq e^2 (\sup_{t \leq n} k_t(X)^{1/t}) \pi_2^d(u).$$

By the proof of proposition 1.4 and by (0.32), it follows then that $\sigma_{2,\infty}^a(u) \leq 3\, w\pi_2(u)$, and thus the right hand inequality is proved.

To prove the left hand one, let $u \in L(\ell_2^n, X)$, $v \in L(X, \ell_2^n)$. By [9] 1.1.10, we have

$$k_n(vu) = \prod_{k=1}^{n} a_k(vu)$$

so that, by (0.28),

$$k_n(vu)^{1/n} \leq \frac{1}{(n!)^{1/n}} \sigma_{1,\infty}^a(vu) \leq \frac{2}{(n!)^{1/n}} \sigma_{2,\infty}^x(u) \sigma_{2,\infty}^x(v).$$

Further, since $\sigma_{2,\infty}^x(v) \leq \pi_2(v)$ and by the well-known inequality $(n!)^{-1/n} \leq 3/n$, we get

$$k_n(vu)^{1/n} \leq \frac{6}{n} w\gamma_2^{(n)}(X) \pi_2^d(u) \pi_2(v).$$

Now proceeding as in the proof of [9] 1.6.2 and using [9] 1.6.3 we get the inequality

$$k_n(X)^{1/n} \leq 6\, w\gamma_2^{(n)}(X).$$

Since this is true for all n and since the sequence $(w\gamma_2^{(n)})_{n \in \mathbb{N}}$ is increasing, the desired result follows. □

Corollary 4.4. ***(Pisier [43]) X is a weak Hilbert space if and only if $(k_n(X)^{1/n})_{n\in\mathbb{N}}$ is bounded.***

With the aid of proposition 4.3 we are able to give among others an improvement of some results of S. Geiss (see [9] 2.3.4):

Theorem 4.5. ***Let $q \in [2, \infty)$ and X be a Banach space. The following conditions are equivalent:***

(a) There is a constant C such that

$$w\gamma_2^{(n)}(X) \leq Cn^{1/2-1/q}, \quad \forall n \in \mathbb{N}.$$

(b) There is a constant C such that

$$k_n(X)^{1/n} \leq Cn^{1/2-1/q}, \quad \forall n \in \mathbb{N}.$$

(c) $\Gamma_1(.,X) \subset S^x_{q,\infty}(.,X)$.

(d) $\Pi_2^d(.,X) \subset S^x_{q,\infty}(.,X)$. ***In other words, there is a constant C such that***

$$\sigma^a_{q,\infty}(u) \leq C\pi_2^d(u), \ \forall u \in L(\ell_2^n, X), \ \forall n \in \mathbb{N}.$$

(e) $\Pi_{2,2,2}(.,X) \subset S^x_{q,\infty}(.,X)$, where $\Pi_{2,2,2} := \Gamma_2^{-1} \cdot \Pi_2 \cdot \Gamma_2^{-1}$.

If $q > 2$, then conditions (a)-(e) above are equivalent to each of the following statements:

(f) There is a constant C such that, for every $n \in \mathbb{N}$ and every n-dimensional subspace E of X, there exists a projection p of X onto E such that

$$\gamma_2(p) \leq Cn^{1/2-1/q}.$$

(g) There is a constant C such that, for every subspace Z of X, every Banach space Y and every operator $v \in L(Z,Y)$ with rank ***$(v) \leq n$, there exists an extension $w \in L(X,Y)$ of v such that***

$$\gamma_2(w) \leq Cn^{1/2-1/q}\|v\|.$$

Further, if $q \in [2, \infty)$, each of the conditions above implies that X has weak cotype q and weak type q^*; in particular they characterize weak Hilbert spaces if $q = 2$.

Remark: if $q = 2$, characterization (c) of weak Hilbert spaces may be considered as a «weak analogue» of Grothendieck's theorem (0 .13′).

Proof . By 4.3, (a) and (b) are equivalent. Let us now prove the equivalence of (c), (d) and (e):

(c) $\Rightarrow$ (d) follows from $\Pi_2^d = \Gamma_1 \cdot \Gamma_2 \subset \Gamma_1$.

(d) $\Rightarrow$ (e) is consequence of the straightforward identities $\Pi_{2,2,2} = \Pi_2^d \cdot \Gamma_2^{-1}$ and $S_{q,\infty}^x = S_{q,\infty}^x . \Gamma_2^{-1}$.

(e) $\Rightarrow$ (c). By Grothendieck's Theorem (0.13') , for every $\mathcal{L}_1$ -space the identity operator is contained in $\Pi_{2,2,2}$, so that $\Gamma_1 \subset \Pi_{2,2,2}$.

Conditions (a)-(e) are equivalent if we can show (a) $\Leftrightarrow$ (d). If g = 2 , both (a) and (d) characterize weak Hilbert spaces (by definition), so we assume g > 2.

(a) $\Rightarrow$ (d). Let $k \leq n$ and $u \in L(\ell_2^n, X)$. By (a),

$$k^{1/2} a_k(u) \leq w\gamma_2^{(n)}(X)\pi_2^d(u) \leq Cn^{1/2-1/q}\pi_2^d(u).$$

Letting $k = [n/2]$ and using (0.32) we see that

$$\sigma_{q,\infty}^a(u) \leq C'\pi_2^d(u)$$

for some constant C' depending only on C and g , which gives at once (d).

(d) $\Rightarrow$ (a). By (0.31) and (0.29), there is a constant C such that

$$\sigma_{2,\infty}^a(u) \leq Cn^{1/2-1/q}\sigma_{q,\infty}^a(u)$$

for all $u \in L(\ell_2^n, X)$, and so (a) follows directly form the definition of $w\gamma_2^{(n)}$ (X) .

To prove the assertion about conditions (f) and (g), we use the proof of (1, g) $\Rightarrow$ (3 , g) $\Leftrightarrow$ (4, g) of prop. 6 in [16] (after substituting everywhere $\sigma_{q,\infty}^a$ for $\pi_{q,2}$).

It remains to prove that if either of the conditions (a)-(e) is fullfilled, then X has weak cotype g and weak type g* . Since Π_γ extends the Hilbert-Schmidt operators, and since $\Pi_{2,2,2}$ is the largest such extension, we have $\Pi_\gamma \subset \Pi_{2,2,2}$, so (e) together with proposition 2.1 show that X has weak cotype g . Let us now prove that X has weak type g* if it satisfies (c). Here we may assume g > 2 (since in the case g = 2 conditions (a)-(e) are even equivalent to X being a weak Hilbert space). By (c) there is a constant C such that, for all $u \in L(\ell_2^n, X)$,

$$\sigma_{q,\infty}^a(u) \leq C\gamma_1(u),$$

hence, by (0.31), there is a constant C' such that

$$\pi_2(u) < C'n^{1/2-1/q}\gamma_1(u).$$

Now, the inclusions $\Pi_2 \subset \Pi_\gamma$, $\Pi_2^d \subset \Gamma_1$ and the corresponding inequalities between the ideal norms show that there is a constant C" such that

$$\pi_\gamma(u) \leq C''n^{1/2-1/q}\pi_2^d(u), \quad \forall u \in L(\ell_2^n, X).$$

The latter is easily seen to imply that

$$T_2(X, n) \leq C'' n^{1/2 - 1/q} = C'' n^{1/q^* - 1/2}.$$

By theorem 3.4, X has weak type q^*.

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