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A FACTORIZATION FORMULA FOR SOME ENTROPY IDEALS

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ABSTRACT: We establish a factorization theorem for entropy ideals generated by Lorentz-Marcinkiewicz sequence spaces $\lambda^{\infty}(\varphi)$.

0. INTRODUCTION

Entropy ideals generated by Lorentz-Marcinkiewicz sequence spaces $\lambda^{q}(\varphi)$ have been considered in [2] and [3], where some of their properties have been derived. These ideals play an important role in order to characterize the degree of compactness of weakly singular integral operators (see [4]). In this paper we obtain factorization formulae for entropy ideals of the type $\lambda^{\infty}(\varphi)$.

To establish such factorization, we shall use some techniques developed by A. Pietsch [10] for the case of entropy ideals generated by ℓ_p spaces (see also [6]) combined with the real method of intepolaion with a function parameter (cf., e.g., [5] and [8]). In the process, we will also obtain some information on the behaviour of entropy numbers under interpolation with function parameter.

1. PRELIMINARIES

We will use throughout this paper standard operator ideal notation, as may be found for example in [9]. Concerning intepolation theory, we refer to [1] and [5].

The class of all (bounded linear) operators between arbitrary Banach spaces is denoted by $\mathcal{L}$, while $\mathcal{L}(E, F)$ stands for the collection of those operators acting from E into F. For the closed unit ball of E we use the symbol U_E.

If $T \in \mathcal{L}(E, F)$ and $n = 1, 2, \ldots$, then the nth entropy number $e_n(T)$ is defined as the infimum of all $\epsilon > 0$ such that there are $y_1, y_2, \ldots, y_q \in F$ with $q \leq 2^{n-1}$ and

$$T(U_E) \subset \bigcup_{1 \leq j \leq q} \{y_j + \epsilon U_F\}.$$

Let $[\mathcal{U}, A]$ and $[\mathcal{V}, B]$ be quasi-normed operator ideals. The component $\mathcal{U} \cdot \mathcal{V}(E, F)$ of the product $\mathcal{U} \cdot \mathcal{V}$ consists of all opertors $T \in \mathcal{L}(E, F)$ which can be factorized in the form $T = SR$ with $S \in \mathcal{U}(M, F)$ and $R \in \mathcal{V}(E, M)$. Here, M is a suitable Banach space. We put

$$A \cdot B(T) = \inf [A(S) B(R)],$$

where the infimum is taken over all possible factorizations as above. Then $[\mathcal{U} \cdot \mathcal{V}, A \cdot B]$ is a quasi-normed operator ideal (see [9], Thm. 7.1.2).

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2. FUNCTION PARAMETER AND INTERPOLATION

The function $\varphi\colon (0\,,\infty) \to (0\,,\infty)$ belongs to the calss $\mathcal{B}$ if and only if φ is continuous, $\varphi(1) = 1$ and

$$\overline{\varphi}(t) = \sup_{s>0} \left(\frac{\varphi(st)}{\varphi(s)}\right) < \infty, \quad \text{for every } t > 0\,.$$

If $\varphi \in \mathcal{B}$ then $\overline{\varphi}$ is submultiplicative (i.e., $\overline{\varphi}(ts) \le \overline{\varphi}(t)\overline{\varphi}(s)$) and Lebesgue measurable. Moreover, the so called Boyd indices, $\alpha_{\overline{\varphi}}$ and $\beta_{\overline{\varphi}}$, of the function $\overline{\varphi}$ are well defined by

$$\alpha_{\overline{\varphi}} = \inf_{1<t<\infty} \left(\frac{\log \overline{\varphi}(t)}{\log t}\right) = \lim_{t\to\infty} \left(\frac{\log \overline{\varphi}(t)}{\log t}\right)$$

$$\beta_{\overline{\varphi}} = \sup_{0<t<1} \left(\frac{\log \overline{\varphi}(t)}{\log t}\right) = \lim_{t\to\infty} \left(\frac{\log \overline{\varphi}(t)}{\log t}\right)$$

They are real numbers, satisfying $-\infty < \beta_{\overline{\varphi}} \le \alpha_{\overline{\varphi}} < \infty$ and the following holds

$$\alpha_{\overline{\varphi}} < 0 \quad \text{if and only if} \quad \int_1^\infty \overline{\varphi}(t)\,\frac{\mathrm{d}t}{t} < \infty;$$

$$\beta_{\overline{\varphi}} > 0 \quad \text{if and only if} \quad \int_0^1 \overline{\varphi}(t)\,\frac{\mathrm{d}t}{t} < \infty.$$

Important examples of functions belonging to $\mathcal{B}$ are

$$\varphi(t) = t^{1/p}(1 + |\log t|)^{\gamma}, \quad \text{for } 0 < p \le \infty \text{ and } -\infty < \gamma < \infty.$$

In this case,

$$\overline{\varphi}(t) = t^{1/p}(1 + |\log t|)^{|\gamma|},$$

its indices being $\beta_{\overline{\varphi}} = \alpha_{\overline{\varphi}} = 1/p$.

Two positive functions φ and ρ are referred to as equivalent if there are two positive constants c_1 and c_2 such that

$$c_1\rho(t) \le \varphi(t) \le c_2\rho(t)\,, \quad t > 0\,.$$

In order to prove the factorization formula, we shall need the two following essentially known facts on function parameters. For the sake of completeness we give their proofs.

Lemma 2.1. *Let φ, $\chi \in \mathcal{B}$ with $\beta_{\overline{\chi}} > 0$, and let ρ be the function defined by $\rho(t) = \dfrac{\varphi(t)}{\chi(\varphi(t))}$. Then ρ belongs to $\mathcal{B}$.*

Proof . Since $\beta_{\overline{\chi}} > 0$, the function χ is equivalent to an increasing function (see [8], Prop. 4). Hence, there exists a constant $c > 0$, such that

$$\overline{\chi}(t_1) \leq c\overline{\chi}(t_2), \quad \text{if } t_1 \leq t_2 .$$

Consequently,

$$\begin{aligned}
\overline{\rho}(t) &= \sup_{s>0} \left(\frac{\chi(\varphi(s))\,\varphi(ts)}{\chi(\varphi(ts))\,\varphi(s)} \right) \\
&\leq \overline{\varphi}(t) \sup_{s>0} \left(\overline{\chi} \left(\frac{\varphi(s)}{\varphi(ts)} \right) \right) \\
&= \overline{\varphi}(t) \sup_{s>0} \left(\overline{\chi} \left(\frac{\varphi(st^{-1})}{\varphi(s)} \right) \right) \\
&\leq c\overline{\varphi}(t)\overline{\chi}(\overline{\varphi}(t^{-1})),
\end{aligned}$$

which shows that ρ belongs to $\mathcal{B}$. □

Lemma 2.2. *Let $\varphi \in \mathcal{B}$ be an increasing function with $\beta_{\overline{\varphi}} > 0$. Then $\psi = \varphi^{-1}$ also belongs to $\mathcal{B}$, and its indices are $\beta_{\overline{\psi}} = 1/\alpha_{\overline{\varphi}}$, $\alpha_{\overline{\psi}} = 1/\beta_{\overline{\varphi}}$.*

Proof . Given $\epsilon > 0$, according to the definition of $\beta_{\overline{\varphi}}$, we can find $\delta > 0$ such that for any $s > 0$ and any $t < \delta$ we have

$$\varphi(st) \leq t^{\mu}\varphi(s),$$

where $\mu = \beta_{\overline{\varphi}} - \epsilon$. Thus,

$$st \leq \varphi^{-1}(t^{\mu}\varphi(s)).$$

Set $u = t^{\mu}\varphi(s)$ and $v = t^{-\mu}$. It follows that for any $u > 0$ and any $v > \delta^{-\mu}$ we get

$$\varphi^{-1}(uv) \leq v^{1/\mu}\varphi^{-1}(u).$$

Whence, we conclude that $\psi = \varphi^{-1}$ belongs to $\mathcal{B}$ and that $\alpha_{\overline{\psi}} \leq 1/\beta_{\overline{\varphi}}$. By using a similar argument, one can easily show that $\beta_{\overline{\psi}} \geq 1/\alpha_{\overline{\varphi}}$. If we now interchange the roles of φ and ψ we shall have $\alpha_{\overline{\varphi}} \leq 1/\beta_{\overline{\psi}}$ and $\beta_{\overline{\varphi}} \geq 1/\alpha_{\overline{\psi}}$. This, together with the above estimates, give the desired results. □

We close this section with some definitions from intepolation theory (see [1], [5] and [8]).

An interpolation couple (E_0, E_1) consists of two Banach spaces E_0 and E_1 which are continuously embedded into a Hausdorff topological vector space Z. We can endow $E_0 + E_1$ with the norm $K(1, x)$, where

$$K(t,x) = \inf\{||x_0||_{E_0} + t||x_1||_{E_1} : x = x_0 + x_1\},$$

and $E_0 \cap E_1$ with the norm $J(1, x)$, where

$$J(t,x) = \max\{||x||_{E_0}, t||x||_{E_1}\}.$$

A Banach space E is called an intermediate space between E_0 and E_1 if $E_0 \cap E_1 \subset E \subset E_0 + E_1$ and the corresponding embedding maps are continuous.

Definition 2.3. *Let $\varphi \in \mathcal{B}$ and let (E_0, E_1) be an interpolation couple. Suppose that E is an intermediate space between E_0 and E_1. Then, we say that*
i) E is of K-type φ if $K(t, x) \leq c\varphi(t)||x||_E$, $t > 0$, $x \in E$;
ii) E is of J-type φ if $||x||_E \leq cJ(t, x)/\varphi(t)$, $t > 0$, $x \in E_0 \cap E_1$.

In order to give examples of such a spaces, we recall the definition of real intepolation space with a function parameter. Let (E_0, E_1) be an intepolation couple, let $1 \leq q \leq \infty$ and $\varphi \in \mathcal{B}$. The space $(E_0, E_1)_{\varphi,q:K}$ consists of all $x \in E_0 + E_1$ for which the following functional is finite:

$$||x||_{\varphi,q:K} = \begin{cases} \left(\int_0^\infty \left(\frac{K(t,x)}{\varphi(t)}\right)^q \frac{dt}{t}\right)^{1/q}, & \text{if } 1 \leq q < \infty \\ \sup_{t>0} \left(\frac{K(t,x)}{\varphi(t)}\right), & \text{if } q = \infty. \end{cases}$$

Example 2.4. Let $\varphi \in \mathcal{B}$ with $0 < \beta_{\overline{\varphi}} \leq \alpha_{\overline{\varphi}} < 1$ and let (E_0, E_1) be an interpolation couple. Then, for every $1 \leq q \leq \infty$, $(E_0, E_1)_{\varphi,q:K}$ is of K-type φ and J-type φ (see [5], Lemma 2.1).

3. ENTROPY IDEALS

Definition 3.1. *Given $\varphi \in \mathcal{B}$, we define*

$$\mathcal{E}_{\varphi,\infty} = \{T \in \mathcal{L} : E_{\varphi,\infty}(T) = \sup_{n\geq 1}(\varphi(n)e_n(T)) < \infty\}.$$

It is well known that the classes $\mathcal{E}_{\varphi,\infty}$ are quasi-normed operator ideals (see [3], §2). Observe that $E_{\varphi,\infty}(T) = ||(e_n(T)||_{\varphi,\infty}$, where $||\cdot||_{\varphi,\infty}$ is the quasi-norm in the Lorentz-Marcinkiewicz sequence space $\lambda^\infty(\varphi)$ (see [2], §2).

We will also need the following two Propositions

Proposition 3.2. *Let $\varphi, \chi \in \mathcal{B}$ with $0 < \beta_{\overline{\chi}} \leq \alpha_{\overline{\chi}} < 1$ and let E be an intermediate space between E_0 and E_1 having K-type χ. If $T \in \mathcal{E}_{\varphi,\infty}(E_0, F)$ and $T \in \mathcal{L}(E_1, F)$, then we have $T \in \mathcal{E}_{\rho,\infty}(E, F)$, where $\rho(t) = \dfrac{\varphi(t)}{\chi(\varphi(t))}$.*

Proof. First we notice that Lemma 2.1 implies $\rho \in \mathcal{B}$. Denote by T_i the operator T acting from E_i into $F (i = 0, 1)$. Taking into account that $\lim_{t\to 0} t\chi(1/t) = 0$ and proceeding as in [9], Prop. 12.1.11, it is not hard to check that

$$(1) \qquad e_{n_0+n_1-1}(T : E \to F) \leq 2ce_{n_0}(T_0)\chi\left(\frac{e_{n_1}(T_1)}{e_{n_0}(T_0)}\right).$$

Here, $e_{n_0+n_1-1} = 0$ if $e_{n_0}(T_0) = 0$ or $e_{n_1}(T_1) = 0$.

From (1) and the fact $\overline{\rho}$ is bounded on every compact set contained in $(0, \infty)$ (see [7], p. 241), we can easily see that there are two positive constants c_1 and c_2 (independent of T) such that

$$\begin{aligned}
\sup_{n\geq 1}(\rho(n)e_n(T)) &\leq c_1 \sup_{n\geq 1}\left(\rho(n)e_n(T_0)\chi\left(\frac{e_n(T_1)}{e_n(T_0)}\right)\right) \\
&\leq c_1 \sup_{n\geq 1}\left(\varphi(n)e_n(T_0)\overline{\chi}\left(\frac{e_n(T_1)}{\varphi(n)e_n(T_0)}\right)\right) \\
&\leq c_2\overline{\chi}(\|T_1\|)\sup_{n\geq 1}\left(\varphi(n)e_n(T_0)\overline{\chi}\left(\frac{1}{\varphi(n)e_n(T_0)}\right)\right)
\end{aligned}$$

This last expression is finite because the sequence $(\varphi(n)e_n(T_0))$ is bounded and the function $t \to t\overline{\chi}(1/t)$ has lower Boyd index greater thant zero. □

Proposition 3.3. *Let $\varphi, \chi \in \mathcal{B}$ with $0 < \beta_{\overline{\chi}} \leq \alpha_{\overline{\chi}} < 1$ and let F be an intermediate space between F_0 and F_1 having J-type χ. If $T \in \mathcal{L}(E, F_0)$ and $T \in \mathcal{E}_{\varphi,\infty}(E, F_1)$, then we have $T \in \mathcal{E}_{\tau,\infty}(E, F)$, where $\tau(t) = \chi(\varphi(t))$.*

Proof. Let T_i denote the operator T acting from E into $F_i (i = 0, 1)$. A similar reasoning to that in [9], Prop. 12.1.12, allows us to obtain

$$(2) \qquad e_{n_0+n_1-1}(T : E \to F) \leq 2ce_{n_0}(T_0)\left[\chi\left(\frac{e_{n_0}(T_0)}{e_{n_1}(T_1)}\right)\right]^{-1}.$$

Here, $e_{n_0+n_1-1} = 0$ if $e_{n_0}(T_0) = 0$ or $e_{n_1}(T_1) = 0$.

Consequently, we have

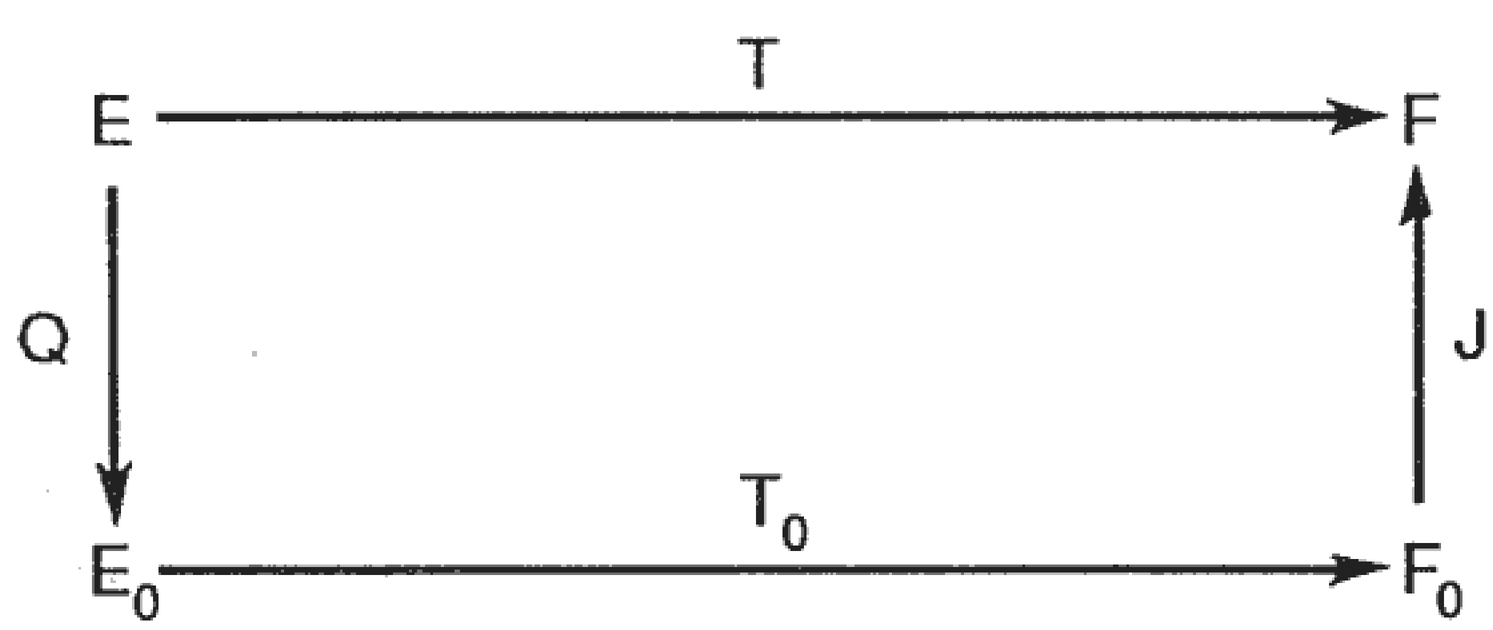
$$\sup_{n\geq 1}(\tau(n)e_n(T)) \leq c_1 \sup_{n\geq 1}\left(\tau(n)e_n(T_0)\left[\chi\left(\frac{e_n(T_0)}{e_n(T_1)}\right)\right]^{-1}\right)$$
$$\leq c_1 \sup_{n\geq 1}\left[e_n(T_0)\overline{\chi}\left(\frac{\varphi(n)e_n(T_1)}{e_n(T_0)}\right)\right]$$
$$\leq c_1 \sup_{n\geq 1}\left[e_n(T_0)\overline{\chi}\left(\frac{1}{e_n(T_0)}\right)\right]\sup_{n\geq 1}[\overline{\chi}(\varphi(n)e_n(T_1))] < \infty. \qquad \square$$

Now we are in a position to state the factorization formula.

Theorem 3.4. *Let* $\varphi_i \in \mathcal{B}$ *with* $\beta_{\overline{\varphi}_i} > 0 (i = 0, 1)$ *and* $\alpha_{\overline{\varphi}_0} - \beta_{\overline{\varphi}_0} < \beta_{\overline{\varphi}_1}$ *or* $\alpha_{\overline{\varphi}_1} - \beta_{\overline{\varphi}_1} < \beta_{\overline{\varphi}_0}$. *If* $\varphi = \varphi_0\varphi_1$, *then*

$$\mathcal{E}_{\varphi_1,\infty} \cdot \mathcal{E}_{\varphi_0,\infty} = \mathcal{E}_{\varphi,\infty}.$$

Proof. Suppose first $\alpha_{\overline{\varphi}_0} - \beta_{\overline{\varphi}_0} < \beta_{\overline{\varphi}_1}$ and let $T \in \mathcal{E}_{\varphi,\infty}(E, F)$. Since $\beta_{\overline{\varphi}} \geq \beta_{\overline{\varphi}_0} + \beta_{\overline{\varphi}_1} > 0$, we may assume without loss of generality that φ is increasing (see [8], Prop. 4). In order to factorize T, we proceed similarly as in [10], Thm. 3. Let $E_0 = E/\ker(T)$ and $F_0 = \overline{T(E)}$. Then, the following diagram commutes



Here Q denotes the canonical surjection form E into E_0, J denotes the canonical injection form F_0 into F, and T_0 is a one-to-one operator. Moreover, (E_0, F_0) forms an interpolation couple, the embedding being T_0.

Write $\chi(t) = \varphi_0(\varphi^{-1}(t))$. According to Lemma 2.2 and the assumption on the indices of $\overline{\varphi}_0$ and $\overline{\varphi}_1$, we have

$$\alpha_{\overline{\chi}} \leq \alpha_{\overline{\varphi}_0} \cdot \alpha_{\overline{\varphi^{-1}}} = \frac{\alpha_{\overline{\varphi}_0}}{\beta_{\overline{\varphi}}} \leq \frac{\alpha_{\overline{\varphi}_0}}{(\beta_{\overline{\varphi}_0} + \beta_{\overline{\varphi}_1})} < 1$$

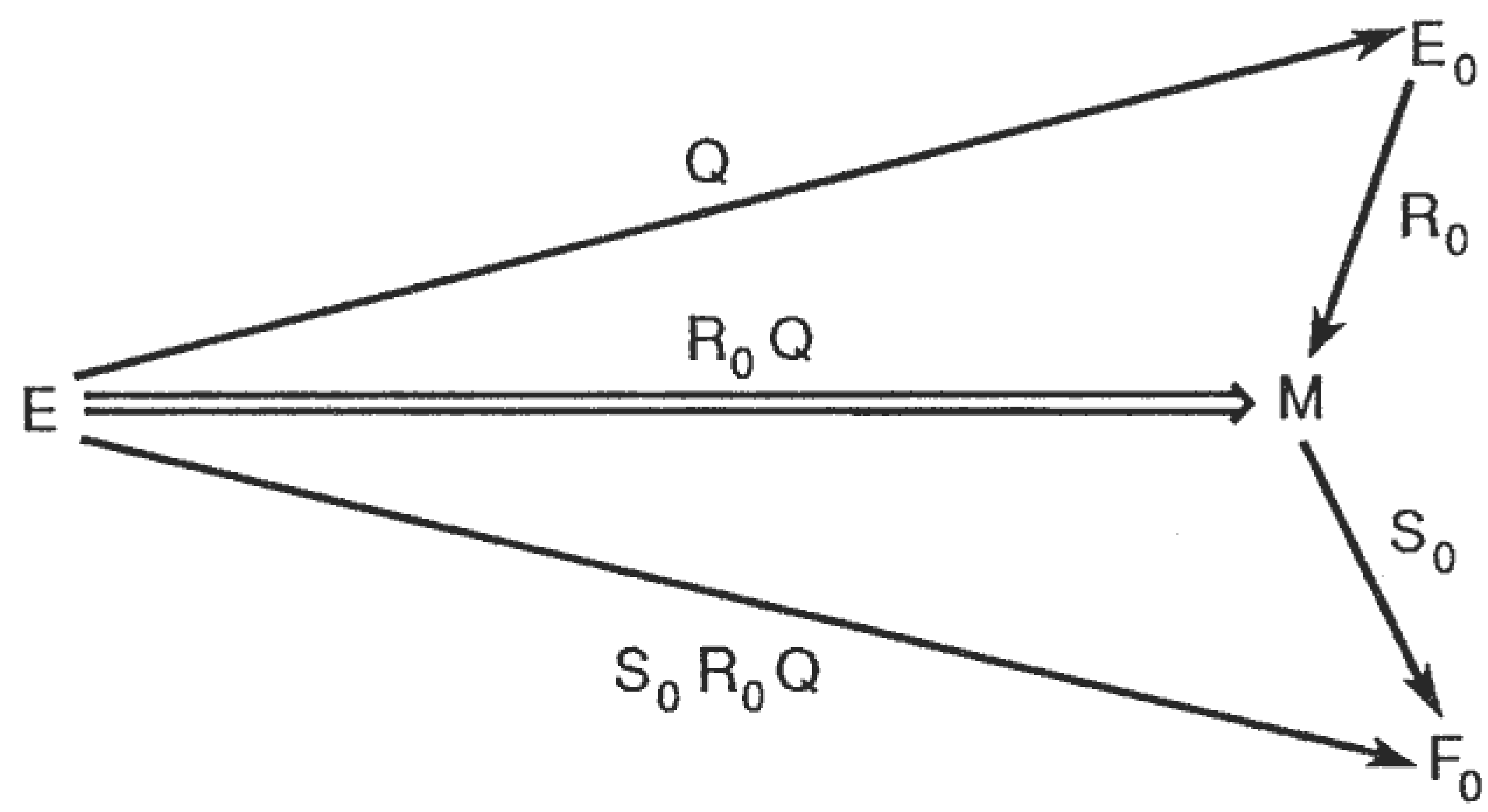
and

$$\beta_{\overline{\chi}} \geq \beta_{\overline{\varphi}_0} \cdot \beta_{\overline{\varphi^{-1}}} = \frac{\beta_{\overline{\varphi}_0}}{\alpha_{\overline{\varphi}}} \geq \frac{\beta_{\overline{\varphi}_0}}{(\alpha_{\overline{\varphi}_0} + \alpha_{\overline{\varphi}_1})} > 0.$$

Hence, we can find an intermediate space M between E_0 and F_0 which has K-type χ and J-type χ. (Take, for example, $M = (E_0, F_0)_{\chi,1}$).

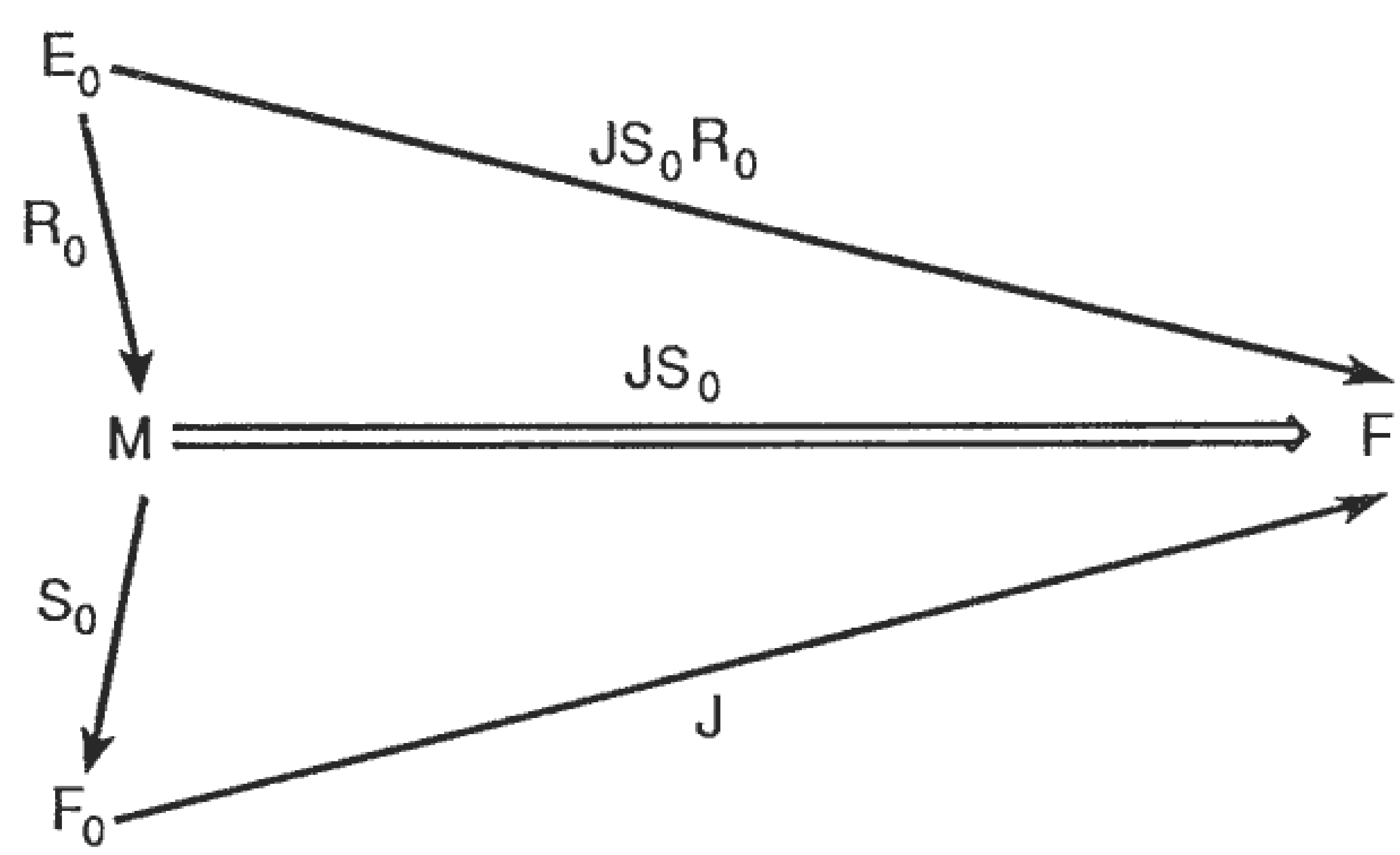
Let us denote by $R_0 \in \mathcal{L}(E_0, M)$ and $S_0 \in \mathcal{L}(M, F_0)$ the corresponding embedding maps.

Next, consider the diagram



The operator $S_0 R_0 Q$ belongs to $\mathcal{E}_{\varphi,\infty}(E, F_0)$ because $T = J\ S_0\ R_0\ Q$ and J is a metric injection. Therefore, Proposition 3.3 yields that $R = R_0\ Q \in \mathcal{E}_{\varphi_0,\infty}\ (E, M)$.

On the other hand, since Q is a metric surjection we have that $J S_0\ R_0 \in \mathcal{E}_{\varphi,\infty}\ (E_0, F)$. Whence, we can use the following diagram



and Proposition 3.2 to get that $S = JS_0 \in \mathcal{E}_{\varphi_1,\infty}(M, F)$.

This shows that

$$T = SR \in \mathcal{E}_{\varphi_1,\infty} \cdot \mathcal{E}_{\varphi_0,\infty}(E, F).$$

The case $\alpha_{\overline{\varphi}_1} - \beta_{\overline{\varphi}_1} < \beta_{\overline{\varphi}_0}$ can be treated in the same way, now setting $\chi(t) = \dfrac{t}{\varphi_1(\varphi^{-1}(t))}$.

Finally, the inclusion $\mathcal{E}_{\varphi_1,\infty} \cdot \mathcal{E}_{\varphi_0,\infty} \subset \mathcal{E}_{\varphi,\infty}$ follows by using the multiplicativity property of entropy numbers and the fact that $\overline{\varphi}$ is bounded on every compact subset of $(0\,,\,\infty)$. □

We end the paper with a conseguence of Theorem 3.4. Let us first recall that given $0 < p < \infty$ and $-\infty < \gamma < \infty$, the Lorentz-Zygmund entropy ideal $\mathcal{E}_{p,\infty,\gamma}$ is formed by all $T \in \mathcal{L}$ such that

$$E_{p,\infty,\gamma}(T) = \sup_{n\geq 1}[\,n^{1/p}(1+\log n)^{\gamma} e_n(T)\,] < \infty.$$

This is nothing else but the ideal $\mathcal{E}_{\varphi,\infty}$ with $\varphi(t) = t^{1/p}(1+|\log t|)^{\gamma}$.

As we mentioned before, we have in this case $\alpha_{\overline{\varphi}} = \beta_{\overline{\varphi}} = 1/p$ and, therefore, according to the preceding theorem we obtain the following

Corollary 3.5. *Assume that* $0 < p_0\,,\,p_1 < \infty$, $-\infty < \gamma_0\,,\,\gamma_1 < \infty$, $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_0}$ *and* $\gamma = \gamma_1 + \gamma_0$. *Then*

$$\mathcal{E}_{p_1,\infty,\gamma_1} \cdot \mathcal{E}_{p_0,\infty,\gamma_0} = \mathcal{E}_{p,\infty,\gamma}.$$

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