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ON NATURAL REDUCTIVITY OF FIVE-DIMENSIONAL COMMUTATIVE SPACES

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INTRODUCTION. The naturally reductive homogeneous spaces have been studied by a number of authors as a natural generalization of Riemannian symmetric spaces.

O. Kowalski and L. Vanhecke drew their attention to the relationship between the naturally reductive spaces and the commutative spaces which are known to generalize symmetric spaces, as well. In this context they suppose that the Riemannian manifolds under consideration are conected, simply connected and complete.

The three-dimensional naturally reductive spaces have been classified by F. Tricerri and L. Vanhecke [14]. O. Kowalski found the same classification in a different context, and he also proved that the naturally reductive spaces and the commutative spaces form the same class in dimension three [9].

In the papers ([11], [12]) O. Kowalski and L. Vanhecke gave the complete classification for naturally reductive spaces as well as for the commutative spaces in dimension four. Once again, they proved that both classes coincide.

In the paper [13] O. Kowalski and L. Vanhecke gave the complete classification for the naturally reductive spaces in dimension five and they have also proved the commutativity of these spaces.

The purpose of this paper is to prove the converse: all five-dimensional commutative spaces are naturally reductive.

In this way we prove that, in dimensions three, four and five the class of naturally reductive spaces coincides with that of commutative spaces.

In dimension $n = 6$, the coincidence of naturally reductive spaces is not true any more. The six-dimensional generalized Heisenberg group with two-dimensional center is a commutative space, but it is not naturally reductive [5]. On the other hand, the homogeneous space of the type $SU(3)/T$, where T is a maximal torus, being endowed with an arbitrary invariant Riemannian metric, is naturally reductive but not commutative [4].

The paper is organized as follows: in the first section we give some necessary definitions and known facts concerning naturally reductive spaces and commutative spaces; in the second section, divided in some subsections, we shall prove the natural reductivity of five-dimensional spaces.

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1. BASIC FACTS ABOUT NATURALLY REDUCTIVE SPACES AND COMMUTATIVE SPACES

In the first place we present some basic facts on naturally reductive spaces ([11], [12]).

Let (M, g) be an n-dimensional homogeneous Riemannian manifold in the sense that the full group $I(M)$ of isometries acts transitively on M. The isotropy subgroup H of $I(M)$ at any fixed point $o \in M$ is compact. Thus (M, g) has at least one representation in the form G/H, where G is a connected Lie subgroup of $I(M)$ acting transitively and effectively on M, H is a conmpact subgroup of G, and g is a G-invariant Riemannian metric on the coset space G/H.

The Lie algebra $\underline{g}$ of G admits an $\mathrm{Ad}(H)$-invariants positive inner product (,). Let us take the orthogonal decomposition $\underline{g} = \underline{m} \oplus \underline{h}$, where $\underline{h}$ is the Lie algebra of H and $\underline{m} = (\underline{h})^{\perp}$ is the orthogonal complement in $\underline{g}$. This decomposition is *reductive* in the sense that $\mathrm{Ad}_G(H)\underline{m} \subseteq \underline{m}$. G/H is called a reductive homogeneous space with respect to the above decomposition. In general we may have more than one representation of (M, g) in the form G/H, and a fixed coset space G/H may admits more than one reductive decomposition.

Definition 1. *The homogeneous Riemannian space (M, g) is said to be* **naturally reductive** *if there exists a representation $(M, g) = G/H$ as above and a reductive decomposition*

$$\underline{g} = \underline{m} \oplus \underline{h}, \quad \mathrm{Ad}_G(H)\underline{m} \subseteq \underline{m}, \tag{1}$$

with the following property

$$\langle [X, Y]_{\underline{m}}, Z \rangle + \langle [X, Z]_{\underline{m}}, Y \rangle = 0, \tag{2}$$

holds for every $X, Y, Z \in \underline{m}$, where $\langle\ ,\ \rangle$ denotes the scalar product on $\underline{m}$ induced by the Riemannian metric g via the natural identification $\underline{m} \equiv T_0 M$, [6].

It is sometimes difficult to decide whether a homogeneous space (M, g) is naturally reductive or not. One has to consider all subgroups $G \subset I(M)$ acting transitively on M, all reductive decomposition (1) and then the condition (2).

From the condition (1) it follows that the Lie group $\mathrm{Ad}_G H$ acts as a group of automorphisms of the Lie algebra $\underline{g}$, and as a group of orthogonal transformation of the subspace $\underline{m}$. Hence the Lie algebra $\mathrm{ad}_{\underline{g}}\underline{h}$ acts on $\underline{g}$ as an algebra of derivations, and on $\underline{m}$ as an algebra of skew-symmetric endomorphisms.

In the sequel, we shall write briefly AX instead of $\mathrm{ad}(A)X$ for $A \in \underline{h}$, $X \in \underline{m}$.

Let $\widetilde{\nabla}$ be the canonical connection of the fixed reductive homogeneous space $(M, g) = G/H$ with the $\mathrm{Ad}(H)$-invariant decomposition $\underline{g} = \underline{m} \oplus \underline{h}$ ([6], p. 150).

Then, at the origin $o \in M$, we have the following formulas for the torsion tensor $\widetilde{T}$ and the curvature tensor $\widetilde{R}$ of:

$$(3) \qquad \begin{cases} \widetilde{T}(X,Y)_0 = -[X,Y]_{\underline{m}}, \\ \widetilde{R}(X,Y)_0 = -\operatorname{ad}([X,Y]_{\underline{h}}), \quad \text{for every } X,Y \in \underline{m}, \end{cases}$$

where we use the canonical identification $\underline{m} \equiv T_0 M$ via the projection $\pi : G \to G/H$, i.e., via the linear map $d\pi_e : T_e(G) \to T_0(G/H)$.

Because any G-invariant tensor field on M is parallel with respect to the connection $\widetilde{\nabla}$, we have

$$(4) \qquad \widetilde{\nabla} g = \widetilde{\nabla}\widetilde{T} = \widetilde{\nabla}\widetilde{R} = 0.$$

From (4) we see that if $A \in \underline{h}$ acts as a derivation on the tensor algebra $\mathcal{T}(\underline{m})$ of $\underline{m}$, then we get (in $\underline{m} \equiv T_0 M$)

$$(5) \qquad A \cdot g = A \cdot \widetilde{T} = A \cdot \widetilde{R} = 0, \quad \text{for every } A \in \underline{h}.$$

Now, using (3), the Jacobi identity on $\underline{g}$, and (4), we get the following reduced Bianchi identities:

(6) $\sigma(\widetilde{R}(X,Y),Z) = \sigma(\widetilde{T}(\widetilde{T}(X,Y),Z))$, (first Bianchi identity),

(7) $\sigma(\widetilde{R}(\widetilde{T}(X,Y),Z)) = 0$, (second Bianchi identity),

for every X, Y, $Z \in \underline{m}$, where the symbol σ denotes the cyclic sum with respect to X, Y, Z.

In terms of the canonical connection $\widetilde{\nabla}$, we can also write the condition (2) of natural reductivity in the form

$$(8) \qquad g(\widetilde{T}(X,Y),Z) + g(\widetilde{T}(X,Z),Y) = 0, \quad \forall X,Y,Z \in \underline{m}.$$

Let us now have a simply connected Riemannian manifold (M, g) with a reductive representation $M = G/H$, $\underline{g} = \underline{m} \oplus \underline{h}$. Then the isotropy subgroup H is connected because M is simply connected and the condition $\operatorname{Ad}(H)\underline{m} \subseteq \underline{m}$ is equivalent to the following condition ([6], p. 178)

$$(9) \qquad [\underline{h}, \underline{m}] \subseteq \underline{m}.$$

Let us notice that the curvature operators $\widetilde{R}_0(X,Y)$ given in (3) generate a subalgebra of the algebra $\operatorname{ad}_{\underline{g}}(\underline{h})$.

We shall also need the following theorem from linear algebra:

Proposition 1.1. *Let* $\mathbf{V}$ *be an* n*-dimensional vector space with a positive inner product, and let* $A : \mathbf{V} \to \mathbf{V}$ *be a skew-symmetric endomorphism. Then the rank of* A *is an even number* $2k \leq n$*, and there is an orthonormal basis* $\{X_1, \ldots, X_n\} \subset \mathbf{V}$ *and real numbers* $\lambda_1, \ldots, \lambda_k$ *such that*

$$(10) \qquad \begin{cases} AX_1 = \lambda_1 X_2, AX_2 = -\lambda_1 X_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ AX_{2k-1} = \lambda_k X_{2k}, AX_{2k} = -\lambda_k X_{2k-1}, \\ AX_{2k+1} = \ldots = AX_n = 0. \end{cases}$$

Here the numbers $\pm i\lambda_j$, $j = 1, \ldots, k$, are non-zero eigenvalues of the endomorphisms A, and $U_j = X_{2j-1} + iX_{2j}$, $U_j = X_{2j-1} - iX_{2j}$ are the corresponding eigenvector of A.

Now, we present the basic facts on commutative spaces ([12], pp. 30-31, [13], pp. 5-6).

Let M be a smooth manifold, $C^\infty(M)$ the algebra of all smooth functions on M, and G a Lie transformation group acting effectively on M, [3].

Definition 2. *A differential operator* $\mathrm{D} : C^\infty(M) \to C^\infty(M)$ *is said to be G-invariant with respect to the group* G *if for any* $f \in C^\infty(M)$ *and any* $g \in G$ *the following relation holds:*

$$(11) \qquad \mathrm{D}(f \cdot \Phi_g) = (\mathrm{D} f) \cdot \Phi_g, \quad f \in C^\infty(M), g \in G,$$

where Φ_g *denotes the action of* $g \in G$ *on* M.

Definition 3. *A homogeneous Riemannian space* (M, g) *is said to be a commutative space if the algebra of all G-invariant differential operators on* M *is commutative for* $G = I^\circ(M)$.

Remark. If M is written in the form $M = G/H$, then the corresponding algebra of G-invariant differential operators is usually denoted by $\mathrm{D}(G/H)$, (see [3]).

Proposition 1.2. *Let* (M, g) *be a homogeneous Riemannian space. Then* (M, g) *is a commutative space if and only if there i s a subgroup* $G \subset I^\circ(M)$ *acting transitively on* M *such that the corresponding algebra* $\mathrm{D}(G/H)$ *is commutative.*

Proof. Denoting $\widetilde{G} = I^\circ(M)$ we get $\mathrm{D}(\widetilde{G}/\widetilde{H}) \subset D(G/H)$.

Proposition 1.3. *Let* G/H *be a reductive homogeneous manifold, where* H *is connected and compact, and* G *acts on* G/H *to the left. Then the algebra* $\mathrm{D}(G/H)$ *of all G-invariant differential operators has a finite number of generators.*

A finite set of generators for $\mathrm{D}(G/H)$ can be found by a purely algebraic method, which we shall describe also briefly [12].

Let $\underline{g} = \underline{m} \oplus \underline{h}$ be a reductive decomposition of the Lie algebra of G. Let $S(\underline{m})$ be the algebra of all polynomial functions on the dual space $\underline{m}^*$. For any basis $\{X_1, \ldots, X_n\}$ of $\underline{m}$, $S(\underline{m})$ can be identified with the polynomial ring $R[X_1, \ldots, X_n]$, i.e., with the symmetric subalgebra of the tensor algebra $\mathcal{T}(\underline{m})$ of the vector space $\underline{m}$.

Denote by $U(\underline{g})$ the universal enveloping algebra of $\underline{g}$, i.e., $U(\underline{g})$ is the factor algebra $\mathcal{T}(\underline{g})/\mathcal{N}(\underline{g})$, where $\mathcal{T}(\underline{g})$ is the tensor algebra over $\underline{g}$ and $\mathcal{N}(\underline{g}) \subset \mathcal{T}(\underline{g})$, is the ideal generated by all elements of the form $X \otimes Y - Y \otimes X - [X, Y]$, $X, Y \in \underline{g}$. There is a canonical injection $j : \underline{g} \to U(\underline{g})$, and $U(\underline{g})$ can be identified with the algebra $\mathrm{D}(G)$ of all G_L-invariant differential operators on G, where G_L is the group of the left translation on G.

Now, we introduce the «symmetrization map» $\lambda : S(\underline{m}) \to U(\underline{g})$ as follows. We choose a basis $\{X_1, \ldots, X_n\}$ of $\underline{m}$, and for any finite sequence $Y_1, \ldots, Y_k$ selected from the set $\{X_1, \ldots, X_n\}$ put

$$\lambda(Y_1 Y_2 \ldots Y_k) = \frac{1}{k!} \sum_{\sigma \in S_k} Y_{\sigma(1)} \cdot Y_{\sigma(2)} \cdot \ldots \cdot Y_{\sigma(k)} \in U(\underline{g}). \tag{12}$$

where the dots mark the multiplication in the algebra $U(\underline{g})$.

Then extending this map λ by lineartiy to the whole $S(\underline{m})$, it can be proved that the map λ does not depend on the choise of the basis in $\underline{m}$, and λ is not homomorphism of the algebras, in general.

For any $A \in S(\underline{m})$, $\lambda(A)$ is G_L-invariant differential operators on G, and A determines also a differential operator on G/H, if we restrict ourselves to the functions from $C^\infty(M)$ which are constant along the fibres of the bundle $G \to G/H$. The last operator is not necessarily G-invariant.

Next, let $I(\underline{m}) \subset S(\underline{m})$ denote the subring of all $\mathrm{Ad}(H)$-invariant polynomials in $S(\underline{m})$. Then λ gives a bijection between $I(\underline{m})$ and $D(G/H)$. Using the Hilberts «Basissatz» and the Haar measure on H, one can see that there is a finite number of generators in $I(\underline{m})$, say $P_1, \ldots, P_r$. Then the corresponding elements $\lambda(P_1), \ldots, \lambda(P_r) \in U(\underline{g})$ interpreted as differential operators on G/H form a complete system of generators of $\mathrm{D}(G/H)$.

For our calculations in the sequel we need the following algebraic criterion for the commutativity or non-commutativity of the algebra $\mathrm{D}(G/H)$, ([3], pp. 389-396).

Proposition 1.4. *Let $\{X_1, \ldots, X_n\} \subset \underline{m}$ and $\{A_1, \ldots, A_s\} \subset \underline{h}$ be vector bases. Each element of $U(\underline{g})$ can be expressed in a unique way as a «polynomial» with real coefficients:*

$$\sum c_{i_1, \ldots i_n, j_1, \ldots j_s} (X_1)^{i_1} \cdot \ldots \cdot (X_n)^{i_n} \cdot (A_1)^{j_1} \cdot \ldots \cdot (A_s)^{j_s}. \tag{13}$$

Further, an element $\mathrm{D} \in \lambda(I(\underline{m})) \subset U(\underline{g})$ defines a non-zero differential operator in $\mathrm{D}(G/H)$ if and only if, when expressed in the form (13) it possesse at least one non-zero coefficient $c_{i_1, \ldots, i_n, 0, \ldots, 0}$.

Finally, the algebra $D(G/H)$ is commutative if and only if, given a set of generators $P_1, \dots, P_r \subset I(\underline{m})$, all commutators $[\lambda(P_i), \lambda(P_j)] \in U(\underline{g})$ vanish as differential operators on G/H.

2. THE NATURAL REDUCTIVITY OF FIVE-DIMENSIONAL COMMUTATIVE SPACE

In this section we shall prove our main theorem.

Theorem 2.1. *Let (M, g) be a simply connected five-dimensional Riemannian homogeneous space. If (M, g) is commmutative, then it is also naturally reductive.*

Proof. Let us write $(M, g) = G/H$, where $G \subset I^\circ(M)$ and let us have a reductive decomposition

(14) $$\underline{g} = \underline{m} \oplus \underline{h}, \quad \mathrm{Ad}(H)\underline{m} \subseteq \underline{m}.$$

Here $\underline{h}$ can be identified with the Lie algebra $\underline{h}^* = \mathrm{ad}_{\underline{g}}(\underline{h})$ of skew-symmetric endomorphism of $\underline{m}$, i.e., $\underline{h}^* \subset \underline{so}(5)$.

From ([12], Proposition 3) it follows that, if $G \subset I^\circ(M)$ is big enough, then

(15) $$\dim \underline{h} \geq 1.$$

From ([15], p. 137) it follows that if

(A) $\underline{m}$ is $\mathrm{Ad}(H)$-irreducible, (i.e., if G/H is isotropy irreducible), then (M, g) is naturally reductive.

Suppose now that there is given an orthonormal basis $\{X_1, \dots, X_5\} \subset \underline{m}$. Then any skew-symmetric endomorphism of $\underline{m}$ is a linear combination of the elementary endomorphisms A_{ij}, $i, j = 1, \dots, 5$ definied as follows

(16) $$A_{ij}X_i = X_j, \quad A_{ij}X_j = -X_i, \quad A_{ij}X_k = 0,$$

for every $i, j, k = 1, \dots, 5$, $i < j$, $k \neq i, j$.

Choosing properly the basis of $\underline{m}$, we have the following remaining possibilities for the action of $\mathrm{Ad}(H)$ on $\underline{m}$.

(B) There is a 4-dimensional $\mathrm{Ad}(H)$-irreducible subspace $\mathrm{Span}(X_1, \dots, X_4) \subset \underline{m}$, and $\mathrm{Ad}(H)$ acts trivially on X_5.

(C) There is a 3-dimensional $\mathrm{Ad}(H)$-irreducible subspace $\mathrm{Span}(X_1, X_2, X_3) \subset \underline{m}$. In this case there are two subcases:

(C_1) $\mathrm{Ad}(H)$ acts trivially on $\mathrm{Span}(X_4, X_5)$;

(C_2) $\mathrm{Ad}(H)$ acts non-trivially on $\mathrm{Span}(X_4, X_5)$.

(D) There are two 2-dimensional $\mathrm{Ad}(H)$-irreducible subspaces $\mathrm{Span}(X_1, X_2)$, $\mathrm{Span}(X_3, X_4) \subset \underline{m}$, and $\mathrm{Ad}(H)$ acts trivially on X_5. We have again two subcases:

(D_1) $\mathrm{Ad}(H)$ is 2-dimensional;

(D_1) $\mathrm{Ad}(H)$ is 1-dimensional.

(E) There is a 2-dimensional $\mathrm{Ad}(H)$-irreducible subspace $\mathrm{Span}(X_1, X_2) \subset \underline{m}$, and $\mathrm{Ad}(H)$ acts triavially on $\mathrm{Span}(X_3, X_4, X_5) \subset \underline{m}$.

We shall now make the proof of Theorem 2.1. for all cases (B)-(E), step by step.

Case B. From Lemma 3.2. of the paper [10] (see also [2], p. 31) it follows that $\mathrm{Ad}(H)$ which is isomorphic to a subgroup of $SO(4)$ contains a subgroup H which is equivalent as transformation group to the group $Sp(1)$ of the unit quaternions acting on $Q \simeq R^4$.

Here H acts transitively on sphere $S^3 \subset \mathrm{Span}(X_1, \ldots, X_4)$ and hence each $\mathrm{Ad}(H)$-invariant polynomial on $\mathrm{Span}(X_1, \ldots, X_4)$ is of the form $P(X_1^2 + X_2^2 + X_3^2 + X_4^2)$, where $P = P(t)$ is a polynomial of one variable. The other $\mathrm{Ad}(H)$-invariant polynomial on $\underline{m}$ is X_5. Thus $X_1^2 + X_2^2 + X_3^2 + X_4^2$, and X_5 form a set of generators of $I(\underline{m})$. Further $\mathrm{D}(G/H)$ is generated as algebra by the differential operators $X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3 + X_4 \cdot X_4$, X_5 (see Formula (12)).

Further, we have

$$\underline{h}^* \subset \mathrm{Span}(A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}), \tag{17}$$

and $\hat{H} \subset H$ is equivalent to $Sp(1) \subset SO(4)$.

Then, for a proper choice of the orthonormal basis $\{X_1, \ldots, X_n\}$ the Lie subalgebra $\widehat{\underline{h}} \subset \underline{h}^*$ can be written in the form

$$\begin{cases} \widehat{\underline{h}} = \mathrm{Span}(A_{12} + A_{34}, A_{14} + A_{23}, A_{13} - A_{24}), \\ \text{and either } \underline{h}^* = \widehat{\underline{h}}, \\ \text{or } \underline{h}^* = \mathrm{Span}(A_{12}, A_{34}, A_{14} + A_{23}, A_{13} - A_{24}), \\ \text{or } \underline{h}^* = \underline{so}(4). \end{cases} \tag{18}$$

Let us consider the subcase B_1 : $\underline{h}^* = \widehat{\underline{h}}$.

For the torsion tensor $\widehat{T}$ at the origin $o \in M$ we have from (5):

$$P \cdot \widetilde{T} = 0, \quad \text{for every } P \in \widehat{\underline{h}}, \tag{19}$$

or equivalently

$$P(\widetilde{T}(X_i, X_j)) = \widetilde{T}(PX_i, X_j) + \widetilde{T}(X_i, PX_j), \ i, j = 1, \ldots, 5. \tag{20}$$

Let us put

$$(21)\qquad \widetilde{T}(X_i, X_j) = \sum_{k=1}^{5} t_{ij}^k X_k, \quad t_{ij}^k = -t_{ij}^k, \quad i, j = 1, \dots, 5.$$

From (20) we get, after a lengthy but routine calculations, for $P = A_{12} + A_{34}$:

$$(22)\qquad \begin{cases} \widetilde{T}(X_1, X_2) = t_{12}^5 X_5, & \widetilde{T}(X_2, X_3) = -t_{14}^5 X_5, \\ \widetilde{T}(X_1, X_3) = t_{13}^5 X_5, & \widetilde{T}(X_2, X_4) = \ \ t_{13}^5 X_5, \\ \widetilde{T}(X_1, X_4) = t_{14}^5 X_5, & \widetilde{T}(X_3, X_4) = \ \ t_{34}^5 X_5, \\ \widetilde{T}(X_1, X_5) = t_{15}^1 X_1 + t_{15}^2 X_2 + t_{15}^3 X_3 + t_{15}^4 X_4, \\ \widetilde{T}(X_2, X_5) = -t_{15}^2 X_1 + t_{15}^1 X_2 - t_{15}^4 X_3 + t_{15}^3 X_4, \\ \widetilde{T}(X_3, X_5) = t_{35}^1 X_1 + t_{35}^2 X_2 + t_{35}^3 X_3 + t_{35}^4 X_4, \\ \widetilde{T}(X_4, X_5) = -t_{35}^2 X_1 + t_{35}^1 X_2 - t_{35}^4 X_3 + t_{35}^3 X_4. \end{cases}$$

In the formulas (22) there are 12 independent parameters $t_{ij}{}^k \in R$.

Acting by $P = A_{14} + A_{23}$ on the both sides of the formulas (22), and using also (16) and (20), we obtain the following relations for the coefficients $T_{ij}{}^k$:

$$(23)\qquad \begin{cases} t_{34}^5 = -t_{12}^5, \\ t_{35}^1 = -t_{15}^3, \\ t_{35}^2 = t_{15}^4, \\ t_{35}^3 = t_{15}^1, \\ t_{35}^4 = -t_{15}^2, \end{cases}$$

On the basis of (22) and (23), and introducing new notations for t_{ij}^k, we obtain finally for $\widetilde{T}(X_i, X_j)$:

$$(24)\qquad \begin{cases} \widetilde{T}(X_1, X_2) = aX_5, & \widetilde{T}(X_2, X_3) = -cX_5, \\ \widetilde{T}(X_1, X_3) = bX_5, & \widetilde{T}(X_2, X_4) = \ \ bX_5, \\ \widetilde{T}(X_1, X_4) = cX_5, & \widetilde{T}(X_3, X_4) = -aX_5, \\ \widetilde{T}(X_1, X_5) = dX_1 + fX_2 + gX_3 + hX_4, \\ \widetilde{T}(X_2, X_5) = -fX_1 + dX_2 - hX_3 + gX_4, \\ \widetilde{T}(X_3, X_5) = -gX_1 + hX_2 + dX_3 - fX_4, \\ \widetilde{T}(X_4, X_5) = -hX_1 - gX_2 + fX_3 + dX_4, \end{cases}$$

Now, we shall use the commutativity of the algebra $D(H/G)$ which is generated by the differential operators $X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3 + X_4 \cdot X_4$, X_5. We shall calculate the corresponding commutators in $U(\underline{g})$, and we shall express this commutator in the form of Proposition 1.4.

We see easily that

$$[X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3 + X_4 \cdot X_4, X_5] = \tag{25}$$

$$= \sum_{k=1}^{4} \{X_k \cdot [X_k, X_5] + [X_k, X_5] \cdot X_k\}.$$

and due to (3) we have

$$[X_k, X_5] = -\widetilde{T}(X_k, X_5) - B_k, \tag{26}$$

where $B_k \in \widehat{\underline{h}}$ for $k = 1, \ldots, 4$.

On the basis of (24)-(26), and (9) we get finally

$$\left\{\begin{aligned} D &= [X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3 + X_4 \cdot X_4, X_5] = \\ &= -2d(X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3 + X_4 \cdot X_4) + \\ &+ \sum_{k=1}^{4} [B_k, X_k] = \\ &= -2d(X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3 + X_4 \cdot X_4) + \\ &+ \sum_{k=1}^{4} \alpha_k X_k. \end{aligned}\right. \tag{27}$$

From our assumption about the commutativity of the algebra $D(G/H)$ we get $d = 0$. Then (24) reduce to

$$\left\{\begin{array}{ll} \widetilde{T}(X_1, X_2) = aX_5, & \widetilde{T}(X_2, X_3) = -cX_5, \\ \widetilde{T}(X_1, X_3) = bX_5, & \widetilde{T}(X_2, X_4) = bX_5, \\ \widetilde{T}(X_1, X_4) = cX_5, & \widetilde{T}(X_3, X_4) = -aX_5, \\ \widetilde{T}(X_1, X_5) = fX_2 + gX_3 + hX_4, & \\ \widetilde{T}(X_2, X_5) = -fX_1 - hX_3 + gX_4, & \\ \widetilde{T}(X_3, X_5) = -gX_1 + hX_2 - fX_4, & \\ \widetilde{T}(X_4, X_5) = -hX_1 - gX_2 + fX_3, & \end{array}\right. \tag{28}$$

The formula (28) contains 6 independent parameters a, b c, f, g, $h \in R$.

In order to simplify (28) we use the following skew-symmetric transformation of the subspace $\mathrm{Span}(X_1, \ldots, X_4)$:

$$FX_i \equiv \tilde{T}(X_i, X_5), \quad i = 1, \ldots, 4, \tag{29}$$

with the matrix

$$F \equiv \begin{bmatrix} 0 & f & g & h \\ -f & 0 & -h & g \\ -g & h & 0 & -f \\ -h & -g & f & 0 \end{bmatrix}. \tag{30}$$

The corresponding eigenvalues of F are

$$\begin{cases} \lambda_1 = i\rho, & \lambda_2 = -i\rho \\ \lambda_3 = i\rho, & \lambda_4 = -i\rho, \end{cases} \qquad \text{where } \rho = \sqrt{f^2 + g^2 + h^2} \tag{31}$$

Then there is an orthonormal basis $\{X_1', \ldots, X_4'\}$ of $\mathrm{Span}(X_1, \ldots, X_4)$ such that

$$\begin{cases} F(X_1' + iX_2') = i\rho(X_1' + iX_2'), \\ F(X_3' + iX_4') = i\rho(X_3' + iX_4'). \end{cases} \tag{32}$$

With respect to the new orthogonal basis $\{X_1', \ldots, X_4', X_5\}$ of $\underline{m}$ (where we write again X_i instead of X_i') we have

$$\begin{cases} \tilde{T}(X_1, X_2) = \tilde{a}X_5, & \tilde{T}(X_2, X_3) = -\tilde{c}X_5, \\ \tilde{T}(X_1, X_3) = \tilde{b}X_5, & \tilde{T}(X_2, X_4) = \tilde{b}X_5, \\ \tilde{T}(X_1, X_4) = \tilde{c}X_5, & \tilde{T}(X_3, X_4) = -\tilde{a}X_5, \\ \tilde{T}(X_1, X_5) = -\rho X_2, & \tilde{T}(X_2, X_5) = \rho X_1, \\ \tilde{T}(X_3, X_5) = \rho X_4, & \tilde{T}(X_4, X_5) = -\rho X_3. \end{cases} \tag{33}$$

In the sequel we shall omit the tilda $\sim$. Because our new orthonormal basis is equally oriented with the old one, we still have

$$\begin{cases} \underline{h}^* = \mathrm{Span}(A, B, C), \\ \text{where } A = A_{12} + A_{34}, B = A_{14} + A_{23}, C = A_{13} - A_{24}. \end{cases} \tag{34}$$

Recall that the following relations hold:

$$(35)\qquad [A,B]=-2C,\quad [B,C]=-2A,\quad [C,A]=-2B.$$

In that follows we shall use the reduced first and second Bianchi identity (6), (7), and also the third identity from (5).

The third identity in formula (5), i.e., $P\cdot\widetilde{R}=0$, $P\in\underline{\widehat{h}}$, is equivalent to the following relation:

$$(36)\qquad \begin{cases} P(\widetilde{R}(X,Y)Z)= \\ =\widetilde{R}(PX,Y)Z+\widetilde{R}(X,PY)Z+\widetilde{R}(X,Y)PZ, \\ \text{for every } X,Y,Z\in\underline{m}, \text{ and every } P\in\underline{\widehat{h}}. \end{cases}$$

Thus, for $X=X_1$, $Y=X_2$, $Z=X_i$, $i=1,\ldots,5$, and $P=A$ we get from (36) and (16) that $A(\widetilde{R}(X_1,X_2)X_i)=\widetilde{R}(AX_1,X_2)X_i+\widetilde{R}(X_1,AX_2)X_i+\widetilde{R}(X_1,X_2)AX_i=\widetilde{R}(X_1,X_2)AX_i$, $i=1,\ldots,5$, hence $[\widetilde{R}(X_1,X_2),A]=0$. Because $\widetilde{R}(X_1,X_2)\in\underline{h}^*$, it follows that $\widetilde{R}(X_1,X_2)=\lambda_{12}A$, $\lambda_{12}\in R$.

Proceeding analogously with the other cases we obtain finally the following relations:

$$(37)\qquad \begin{cases} \widetilde{R}(X_1,X_2)=\lambda_{12}A,\ \widetilde{R}(X_1,X_3)=\lambda_{13}C,\ \widetilde{R}(X_1,X_4)=\lambda_{14}B, \\ \widetilde{R}(X_3,X_4)=\lambda_{34}A,\ \widetilde{R}(X_2,X_4)=\lambda_{24}C,\ \widetilde{R}(X_2,X_3)=\lambda_{23}B, \end{cases}$$

where $\lambda_{12},\ldots,\lambda_{34}\in R$.

The remaining $\widetilde{R}(X_i,X_5)$ are of the form

$$(38)\qquad \widetilde{R}(X_i,X_5)=a_{i5}A+b_{i5}B+c_{i5}C,\quad i=1,\ldots,4,$$

where a_{i5}, b_{i5}, $c_{i5}\in R$.

Now, substituting in (36) different values $X=X_i$, $Y=X_j$, $Z=X_k$, $i,j,k=1,\ldots,4$ and acting successively on both sides of (36) by $P=A$, $P=B$ or $P=C$ we obtain easily (omitting the long elementary calculations):

$$(39)\qquad \begin{cases} 2\lambda_{12}+\lambda_{24}-\lambda_{13}=0, & 2\lambda_{24}+\lambda_{14}+\lambda_{23}=0, \\ 2\lambda_{12}-\lambda_{23}-\lambda_{14}=0, & 2\lambda_{24}+\lambda_{12}+\lambda_{34}=0, \\ 2\lambda_{34}+\lambda_{24}-\lambda_{13}=0, & 2\lambda_{14}+\lambda_{24}-\lambda_{13}=0, \\ 2\lambda_{34}-\lambda_{14}-\lambda_{23}=0, & 2\lambda_{14}-\lambda_{12}-\lambda_{34}=0, \\ 2\lambda_{13}-\lambda_{14}-\lambda_{23}=0, & 2\lambda_{23}-\lambda_{13}-\lambda_{24}=0, \\ 2\lambda_{13}-\lambda_{12}-\lambda_{34}=0, & 2\lambda_{23}-\lambda_{12}-\lambda_{34}=0. \end{cases}$$

Solving this system we get that all $\lambda_{ij} = 0$, $i, j = 1, \ldots, 4$, hence from (37) we have

$$(40) \qquad \widetilde{R}(X_i, X_j) = 0, \quad \text{for } i, j = 1, \ldots, 4.$$

Now, we shall prove that $\widetilde{R}(X_i, X_5) = 0$, for $i = 1, \ldots, 4$.

Substituting into the first Bianchi identity (6) $\widetilde{T}(X_i, X_j)$ and $\widetilde{R}(X_i, X_j)$ from the formula (33) and (38) for $X = X_i$, $Y = X_j$, $Z = X_k$, $i < j < k$, $i, j, k = 1, \ldots, 5$ and after tedious but routine calculations (which we omit) we get the following relations:

$$(41) \qquad \begin{cases} a_{i5} = b_{i5} = c_{i5} = 0, & i = 1, \ldots, 4, \\ \rho a = \rho b = \rho c = 0. \end{cases}$$

Thus, we have that

$$(42) \qquad \widetilde{R}(X_i, X_5) = 0, \quad \text{for } i, j = 1, \ldots, 4.$$

On the basis of (40) and (42) we see that all the curvature transformations $\widetilde{R}(X_i, X_j)$ vanish, $i, j = 1, \ldots, 5$

Hence, according to (3), $\underline{m} \subset \underline{g}$ is a *Lie subalgebra*.

In that follows we shall distinguish to subcases with respect to ρ:

$$(43) \qquad I) \rho = 0, \qquad II) \rho \neq 0.$$

In the first subcase $I(\rho = 0)$, the formula (33) reduces to

$$(44) \qquad \begin{cases} \widetilde{T}(X_1, X_2) = aX_5, & \widetilde{T}(X_2, X_3) = -cX_5, \\ \widetilde{T}(X_1, X_3) = bX_5, & \widetilde{T}(X_2, X_4) = bX_5, \\ \widetilde{T}(X_1, X_4) = cX_5, & \widetilde{T}(X_3, X_4) = -aX_5, \\ \widetilde{T}(X_i, X_5) = 0, & \text{for } i = 1, \ldots, 4. \end{cases}$$

Now, if $a = b = c = 0$ in the formula (44), then $\widetilde{T}(X_i, X_j) = 0$ $i, j = 1, \ldots, 5$, the condition (8) is satisfied identically, and hence our space $(M, g) = G/H$ is naturally reductive.

In that follows we suppose that

$$(45) \qquad a^2 + b^2 + c^2 > 0.$$

Using a symplectic transformation of Span$(X_1, \ldots, X_4)$ we can find a new orthonormal basis for which $a \neq 0$, and $b = c = 0$.

Then the Lie algebra multiplication on $\underline{m}$ is given (see (3)) by

$$\text{(46)} \qquad \begin{cases} [X_1, X_2] = -aX_5, & [X_2, X_3] = 0, a \neq 0, \\ [X_1, X_3] = 0, & [X_2, X_4] = 0, \\ [X_1, X_4] = 0, & [X_3, X_4] = aX_5, \\ [X_i, X_5] = 0, & \text{for } i = 1, \ldots, 4. \end{cases}$$

According to ([13], p. 455) our space (M, g) is a group space and it can be identified with the Heisenberg group H^5 which can be also identified the Cartesian space $R^5(x, y, z, u, v)$:

$$\text{(47)} \qquad G = \left\{ \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ u & v & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

The Heisenberg group H^5 is naturally reductive space and has a left-invariant Riemannian metric with one parameter ρ:

$$\text{(48)} \qquad g = \frac{1}{\rho}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}u^2 + \mathrm{d}v^2) + (u\,\mathrm{d}x + v\,\mathrm{d}y - \mathrm{d}z)^2, \quad \rho \neq 0.$$

Now, we shall consider the second subcase $II(\rho \neq 0)$.

Then (41) implies

$$\text{(49)} \qquad a = b = c = 0.$$

Thus, using (32), (42) and (49) we have

$$\text{(50)} \qquad \begin{cases} \widetilde{R}(X_i, X_j) = 0, i, j = 1, \ldots, 5, \\ \widetilde{T}(X_i, X_j) = 0, \text{ for } (i,j) \neq (k,5),\ k = 1, \ldots, 4, \\ \widetilde{T}(X_1, X_5) = -\rho X_2, \widetilde{T}(X_2, X_5) = \rho X_1,\ \rho \neq 0, \\ \widetilde{T}(X_3, X_5) = \rho X_4, \widetilde{T}(X_4, X_5) = -\rho X_3. \end{cases}$$

Now, A_{12}, A_{34} are skew-symmetric derivations of the corresponding Lie algebra $\underline{m}$. This means that (M, g) admits an isometry group $\widetilde{G} \supset G$ such that the Lie algebra of $\mathrm{Ad}(\widetilde{H}) \supset \mathrm{Ad}(H)$ is isomorphic to $\underline{\widehat{h}} = \mathrm{Span}(A_{12}, A_{34}, B, C)$. The Lie algebra of G

is then isomorphic to $\underline{g} = \underline{m} \oplus \underline{h}$, which gives a *reductive* decomposition with respect to $\mathrm{Ad}(H)$.

In order to prove that our space $(M, g) = G/H$ is naturally reductive we shall replace our reductive decomposition $\underline{g} = \underline{m} \oplus \underline{h}$ by a new reductive decomposition

$$\underline{\tilde{g}} = \underline{m'} \oplus \underline{h}, \quad \underline{m'} = \mathrm{Span}(X_1', \ldots, X_5'), \tag{51}$$

where

$$\begin{cases} X_i' = X_i, \quad i = 1, \ldots, 4, \\ X_5' = X_5 + \rho(A_{12} - A_{34}). \end{cases} \tag{52}$$

Here, $\underline{m'} \subset \underline{\tilde{g}}$ is an abelian subalgebra.

We identify again canonically $\underline{m'} \equiv T_0 M$ via the linear isomorphism:

$$\begin{cases} \varphi : \underline{m} \to m', \\ \varphi\left(\sum_{i=1}^{5} a_i X_i\right) = \sum_{i=1}^{5} a_i X_i'. \end{cases} \tag{53}$$

Then the canonical scalar product $\langle\ ,\ \rangle'$ on $\underline{m'}$ is defined by the formula

$$\langle X', \quad Y' \rangle' = \langle X, Y \rangle, \qquad X, Y \in \underline{m}, \quad X', Y' \in \underline{m'}. \tag{54}$$

Now, using formula (2) for the new reductive decomposition, we see that our space (M, g) is naturally reductive. (In fact, it appears that our space is the Euclidean space E^5).

Let us consider the subspace B_2: $\underline{h}^* \supset \underline{\hat{h}}$ and, $\underline{h}^* = \mathrm{Span}(A_{12}, A_{34}, B, C)$.

We can use now the formulas

$$P \cdot \tilde{T} = 0, \quad P \in \underline{h}^*. \tag{55}$$

We can start again with the formulas (24), and in addition, we can act by $P = A_{12}$ on all these formulas, using (20).

After elementary calculations we obtain the following conditions for the unknown coefficients:

$$b = c = h = g = 0. \tag{56}$$

Next, using the commutativity condition for the algebra $D(G/H)$, as before, we shall abtain the same condition $d = 0$.

Hence, in the considering subscase (B_2) we have now the following form of the tensor $\widetilde{T}(X_i, X_j)$:

$$
(57)\qquad \begin{cases} \widetilde{T}(X_1,X_2) = aX_5, & \widetilde{T}(X_2,X_3) = 0, \\ \widetilde{T}(X_1,X_3) = 0, & \widetilde{T}(X_2,X_4) = 0, \\ \widetilde{T}(X_1,X_4) = 0, & \widetilde{T}(X_3,X_4) = -aX_5, \\ \widetilde{T}(X_1,X_5) = fX_2, \\ \widetilde{T}(X_2,X_5) = -fX_1, \\ \widetilde{T}(X_3,X_5) = -fX_4, \\ \widetilde{T}(X_4,X_5) = fX_3, \end{cases}
$$

We see that (57) is a special case of (33), but now $\widetilde{R}(X_i, X_j) \in \underline{h}^* \supset \widehat{\underline{h}}$, $i,j = 1,\dots,5$. We shall prove again that $\widetilde{R}(X_i, X_j) = 0$, for all $i, j = 1, \dots, 5$. From the basic formula (36) we obtain first the following relations for $\widetilde{R}(X_i, X_j)$, which are analogous to (37):

$$
(58)\qquad \begin{cases} \widetilde{R}(X_1,X_2) = a_{12}A_{12} + b_{12}B_{34}, \\ \widetilde{R}(X_3,X_4) = a_{34}A_{12} + b_{34}B_{34}, \\ \widetilde{R}(X_1,X_3) = a_{13}(A_{12} - A_{34}) + c_{13}C, \\ \widetilde{R}(X_2,X_4) = a_{24}(A_{12} - A_{34}) + c_{24}C, \\ \widetilde{R}(X_1,X_4) = a_{14}(A_{12} - A_{34}) + c_{14}B, \\ \widetilde{R}(X_2,X_3) = a_{23}(A_{12} - A_{34}) + c_{23}B, \end{cases}
$$

By the further use of (36) we obtain finally

$$
(59)\qquad \begin{cases} \widetilde{R}(X_1,X_2) = a_{12}(A_{12} - A_{34}), \\ \widetilde{R}(X_3,X_4) = -a_{12}(A_{12} - A_{34}), \\ \widetilde{R}(X_1,X_3) = \widetilde{R}(X_2,X_4) = a_{13}(A_{12} - A_{34}), \\ \widetilde{R}(X_1,X_4) = -\widetilde{R}(X_2,X_3) = a_{14}(A_{12} - A_{34}), \end{cases}
$$

where a_{12}, a_{13}, a_{14} are arbitrary real parmeters.

The remaining operators $\widetilde{R}(X_i, X_5)$ are of the form

$$
(60)\qquad \widetilde{R}(X_i,X_5) = a_{i5}A_{12} + b_{i5}A_{34} + c_{i5}B + d_{i5}C, \quad i = 1,\dots,4.
$$

Now, we shall use the first Bianchi identity (6). Substituting (57) into (6) we see that $\widetilde{T}(\widetilde{T}(X_i, X_j), X_5) = 0$, and hence

$$
(61)\qquad \widetilde{R}(X_i,X_j)X_5 = 0, \quad \text{for } i,j = 1,\dots,4.
$$

Next, we substitute (60) into (61), then we obtain, after some lengthy calculations, that

$$\widetilde{R}(X_i, X_5) = 0, \quad i = 1, \ldots, 4. \tag{62}$$

Finally, using (6), (57) and (59) for X_i, X_j, $X_k \neq X_5$ we get

$$fa = a_{12} = a_{13} = a_{14} = 0. \tag{63}$$

Then, from (59), (62) and (63) we have

$$\begin{cases} \widetilde{R}(X_i, X_j) = 0, & \text{for } i, j = 1, \ldots, 5, \\ fa = 0. \end{cases} \tag{64}$$

Now, we shall consider two subcases.

For $f = 0$ we have from (57) and (64) the following situation

$$\begin{cases} \widetilde{T}(X_1, X_2) = aX_5, & \widetilde{T}(X_3, X_4) = -aX_5, \ a \in R, \\ \widetilde{T}(X_i, X_j) = 0, & \text{for } (i, j) \neq (1,2), (3,4), \\ \widetilde{T}(X_k, X_l) = 0, & \text{for } k, l = 1, \ldots, 5. \end{cases} \tag{65}$$

This coincides with the case (46), which is naturally reductive.

For $f \neq 0$, we have from (64) that $a = 0$, and taking also into account (57) we get finally

$$\begin{cases} \widetilde{T}(X_1, X_5) = fX_2, \\ \widetilde{T}(X_2, X_5) = -fX_1, \\ \widetilde{T}(X_3, X_5) = -fX_4, \\ \widetilde{T}(X_4, X_5) = fX_3, \\ \widetilde{T}(X_i, X_j) = 0, & \text{for } (i, j) \neq (k, 5), k = 1, \ldots, 4, \\ \widetilde{T}(X_k, X_l) = 0, & \text{for } k, l = 1, \ldots, 5. \end{cases} \tag{66}$$

This is the same case as (50), which proves that the corresponding commutative space (M, g) is naturally reductive (and, in fact, equal to the Euclidean space E^5).

Let us consider the subcase B_3 : $\underline{\widehat{h}} \subset \underline{h}^*$, and $\underline{h}^* = \underline{so}(4)$.

Having the tensor $\widetilde{T}(X_i, X_j)$, $i, j = 1, \ldots, 5$ in the form (57) we can now act on both sides by $A_{13} \in \underline{so}(4)$, and we get

$$a = f = 0. \tag{67}$$

Thus, we have from (57)

$$\widetilde{T}(X_i, X_j) = 0, \quad \text{for } i, j = 1, \ldots, 5. \tag{68}$$

Hence, we see from (8) that our commutative space (M, g) is naturally reductive.

Case (C). In the subcase (C_1) the group $\mathrm{Ad}(H)$ is equivalent to $SO(3)$ as a transformation group. Here $\mathrm{Ad}(H)$ acts transitively on the sphere $S^2 \subset \mathrm{Span}\ (X_1, X_2, X_3)$ and hence each $\mathrm{Ad}(H)$-invariant polynomial on Span (X_1, X_2, X_3) is of the form $P(X_1^2 + X_2^2 + X_3^2)$, where $P = P(t)$ is a polynomial of one variable. The other $\mathrm{Ad}(H)$-invariant polynomial are X_4, X_5. Thus, $X_1^2 + X_2^2 + X_3^2$, and X_4, X_5 form a set of generators of $I(\underline{m})$. Further, $D(G/H)$ is generated by the differential operators $X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3$, X_4, X_5.

The Lie algebra $\widehat{\underline{h}}$ of $\mathrm{Ad}(H)$ is given by

$$\underline{h}^* = \mathrm{Span}(A_{12}, A_{23}, A_{31}). \tag{69}$$

Now, we shall prove, as before, that in the subcase (C_1) our commutative space $(M, g) = G/H$ is naturally reductive.

Acting successively by $P = A_{12}, A_{23}, A_{31}$ on (20), we get, after routine calculations, the following form of the tensor $\widetilde{T}(X_i, X_j)$:

$$\begin{cases} \widetilde{T}(X_1, X_2) = aX_3, & \widetilde{T}(X_1, X_3) = -aX_2, \quad \widetilde{T}(X_2, X_3) = aX_1, \\ \widetilde{T}(X_1, X_4) = bX_1, & \widetilde{T}(X_2, X_4) = bX_2, \quad \widetilde{T}(X_3, X_4) = bX_3, \\ \widetilde{T}(X_1, X_5) = cX_1, & \widetilde{T}(X_2, X_5) = cX_2, \quad \widetilde{T}(X_3, X_5) = cX_3, \\ \widetilde{T}(X_4, X_5) = dX_4 + eX_5. \end{cases} \tag{70}$$

Now, we shall use the commutativity of the algebra $D(G/H)$ which is generated by the differential operators $X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3$, X_4, X_5. Thus, we check their commutators in $U(\underline{g})$ using Proposition 1.4.

We introduce the following notations:

$$\begin{cases} E_{i4} = -\widetilde{R}(X_i, X_4), \quad i = 1, 2, 3, \\ E_{i5} = -\widetilde{R}(X_i, X_5), \\ E_{45} = -\widetilde{R}(X_4, X_5). \end{cases} \tag{71}$$

Hence, from (3), (70) and (71) we abtain

$$\begin{cases} [X_i, X_4] = -bX_i + E_{i4}, \qquad i = 1, 2, 3, \\ [X_i, X_5] = -cX_i + E_{i5}, \\ [X_4, X_5] = -dX_4 - eX_5 + E_{45}. \end{cases} \tag{72}$$

Now, we get, after routine calculations, the following form of the main commutators:

$$(73)\qquad \begin{cases} D_1 &= [X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3, X_4] = \\ &= \sum_{i=1}^{3}\{X_i \cdot [X_i \cdot X_4] + [X_i, X_4] \cdot X_i\} = \\ &= \sum_{i=1}^{3}\{-2b(X_i \cdot X_i) + [E_{i4}, X_i] + 2(X_i \cdot E_{i4})\}; \\ D_2 &= [X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3, X_5] = \\ &= \sum_{i=1}^{3}\{X_i \cdot [X_i \cdot X_5] + [X_i, X_5] \cdot X_i\} = \\ &= \sum_{i=1}^{3}\{-2c(X_i \cdot X_i) + [E_{i5}, X_i] + 2(X_i \cdot E_{i5})\}; \end{cases}$$

Applying Proposition 1.4. to (72)-(73), the commutativity of $D(G/H)$ implies

$$(74)\qquad b = c = d = e = 0.$$

In this way we get from (70) that

$$(75)\qquad \begin{cases} \widetilde{T}(X_1, X_2) = aX_3, \widetilde{T}(X_1, X_3) = -aX_2, \widetilde{T}(X_2, X_3) = aX_1, \\ \widetilde{T}(X_i, X_j) = 0, \quad \text{for } (i,j) \neq (1,2), (1,3), (2,3). \end{cases}$$

The condition (8) is easily verified and hence our commutative space (M, g) is naturally reductive.

We can show in this case that $(M, g) \approx S^3 \times R^2$.

In the subcase (C_2) the group $\mathrm{Ad}(H)$ is equivalent to $SO(3) \times SO(2)$ as a transformation group.

Here $SO(3)$ acts transitively on the sphere $S^2 \subset \mathrm{Span}(X_1, X_2, X_3)$ and $SO(2)$ acts transitively on the circle $S^1 \subset \mathrm{Span}(X_4, X_5)$.

The corresponding Lie algebra $\underline{h}^* = \underline{so}(3) \oplus \underline{so}(2)$ has the form

$$(76)\qquad \underline{h}^* = \mathrm{Span}(A_{12}, A_{23}, A_{31}, A_{45}).$$

Comparing the form of algebra $\underline{h}^*$ presented in (69) with (76) we see that we can start from the tensor $\widetilde{T}(X_i, X_j)$ presented by formula (70).

Acting on both sides of (70) by A_{45} and using (20) we get after simple calculations, that

$$(77)\qquad b = c = d = e = 0.$$

Hence, the tensor $\widetilde{T}(X_i, X_j)$ takes the form (75), and our commutative space $(M, g) = G/H$ is again naturally reductive. In this subcase the commutativity condition was not needed to prove the natural reductivity. We note that in this case $(M, g) \approx S^3 \times M_2$, where M_2 is a 2-dimensional space form (R^2, or S^2, or $SL(2,R)/SO(2)$).

Case ($\mathbf{D_1}$). Here the group $\mathrm{Ad}(H)$ is equivalent to $SO(2) \times SO(2)$ as a transformation group. Hence, $\mathrm{Ad}(H)$ acts transitively on the circle $S^1 \subset \mathrm{Span}(X_1, X_2)$, and independently on the circle $S^1 \subset \mathrm{Span}(X_3, X_4)$.

Obviously, $X_1^2+X_2^2$, $X_3^2+X_4^2$, X_5 form a set of generators of $I(\underline{m})$. Further, $\mathrm{D}(G/H)$ is generated by the differential operators $X_1 \cdot X_1 + X_2 \cdot X_2 + X_3 \cdot X_3 + X_4 \cdot X_4$, X_5.

The Lie algebra of $\mathrm{Ad}(H)$ acting on $\underline{m}$ has the form

$$\underline{h}^* = \mathrm{Span}(A_{12}, A_{34}). \tag{78}$$

Using the identities $A_{12} \cdot \widetilde{T} = 0$, $A_{34} \cdot \widetilde{T} = 0$ we get, after elementary culculation, that the tensor $\widetilde{T}(X_i, X_j)$ has the following form:

$$\begin{cases} \widetilde{T}(X_1,X_2) = t_{12}^5 X_5, & \widetilde{T}(X_2,X_3) = 0, \\ \widetilde{T}(X_1,X_3) = 0, & \widetilde{T}(X_2,X_4) = 0, \\ \widetilde{T}(X_1,X_4) = 0, & \widetilde{T}(X_3,X_4) = t_{34}^5 X_5, \\ \widetilde{T}(X_1,X_5) = t_{15}^1 X_1 + t_{15}^2 X_2, & \\ \widetilde{T}(X_2,X_5) = -t_{15}^2 X_1 + t_{15}^1 X_2, & \\ \widetilde{T}(X_3,X_5) = t_{35}^3 X_3 + t_{35}^4 X_4, & \\ \widetilde{T}(X_4,X_5) = -t_{35}^4 X_3 + t_{35}^3 X_4. & \end{cases} \tag{79}$$

Now, we shall use the commutativity of the algebra $\mathrm{D}(G/H)$, and we shall calculate successively the three commutators:

$$\begin{cases} D_1 = [X_1 \cdot X_1 + X_2 \cdot X_2, X_5], \\ D_2 = [X_3 \cdot X_3 + X_4 \cdot X_4, X_5], \\ D_3 = [X_1 \cdot X_1 + X_2 \cdot X_2, X_3 \cdot X_3 + X_4 \cdot X_4]. \end{cases} \tag{80}$$

Realizing the same calculations as for obtaining the formulas (25)-(27) we can write modulo $\underline{h}$

$$\begin{cases} D_1 \equiv -2t_{15}^1 (X_1 \cdot X_1 + X_2 \cdot X_2) + \alpha_1 X_1 + \alpha_2 X_2, \\ D_2 \equiv -t_{35}^3 (X_3 \cdot X_3 + X_4 \cdot X_4) + \beta_1 X_3 + \beta_2 X_4, \\ D_3 \equiv 0. \end{cases} \tag{81}$$

From the commutativity it follows that $D_1 = D_2 = 0$, hence

$$t^1_{15} = t^3_{35} = 0 \tag{82}$$

Taking into account (82), the formulas (79) reduce to the following ones:

$$\begin{cases} \widetilde{T}(X_1,X_2) = aX_5, & \widetilde{T}(X_2,X_3) = 0, \\ \widetilde{T}(X_1,X_3) = 0, & \widetilde{T}(X_2,X_4) = 0, \\ \widetilde{T}(X_1,X_4) = 0, & \widetilde{T}(X_3,X_4) = bX_5, \\ \widetilde{T}(X_1,X_5) = cX_2, & \widetilde{T}(X_3,X_5) = dX_4, \\ \widetilde{T}(X_2,X_5) = -cX_1, & \widetilde{T}(X_4,X_5) = -dX_3. \end{cases} \tag{83}$$

The curvature transformations have the following forms (see (78))

$$\widetilde{R}(X_i,X_j) = a_{ij}A_{12} + b_{ij}A_{34}, \quad i<j, \quad i,j=1,\dots,5. \tag{84}$$

From (36) and (84) we obtain easily

$$\widetilde{R}(PX,Y) + \widetilde{R}(X,PY) = 0, \quad \text{for } P = A_{12}, \text{ and } P = A_{34}. \tag{85}$$

In particular $\widetilde{R}(A_{12}X\,,\,X_5) = \widetilde{R}(A_{34}X\,,\,X_5) = 0$ which implies $\widetilde{R}(X_i\,,\,X_5) = 0$, for $i = 1\,,\,\dots,\,4$.

Hence, from (85) we get easily that $\widetilde{R}(X_k\,,\,X_l) = 0$, for $(k,\,l) \neq (1,2),(3,4)$, $k,l = 1,\dots,4$.

Thus, we have obtained finally

$$\begin{cases} \widetilde{R}(X_1,X_2) = a_{12}A_{12} + b_{12}B_{34}, \\ \widetilde{R}(X_3,X_4) = a_{34}A_{12} + b_{34}B_{34}, \quad a_{kl}, b_{kl} \in R, \\ \widetilde{R}(X_i,X_j) = 0, \quad \text{for } (i,j) \neq (1,2),(3,4). \end{cases} \tag{86}$$

Now, we shall replace the old reductive decomosition $\underline{g} = \underline{m} \oplus \underline{h}$, $[\underline{h},\,\underline{m}] \subseteq \underline{m}$ by the new reductive decomposition

$$\underline{g} = \underline{m}' \oplus \underline{h}, \quad [\underline{h}, \underline{m}'] \subseteq \underline{m}', \tag{87}$$

where

$$(88)\qquad \begin{cases} \underline{m}' = \mathrm{Span}(X_1', \ldots, X_5'), \\ X_i' = X_i, \quad \text{for } i = 1, \ldots, 4, \\ X_5' = X_5 - cA_{12} - dA_{34}. \end{cases}$$

We can calculate the Lie brackets in $\underline{g}$ using (3), (83) and (86). We obtain easily

$$(89)\qquad \begin{cases} [X_1', X_2']_{\underline{m}'} = -aX_5', \\ [X_3', X_4']_{\underline{m}'} = -bX_5', \\ [X_i', X_j'] = 0 \text{ otherwise}. \end{cases}$$

According to formula (32) from (13) this gives the Heisenberg group H^5 with a left-invariant metric (see the case (B_1))

$$(90)\qquad g = \frac{1}{a}(\mathrm{d}u^2 + \mathrm{d}x^2) + \frac{1}{b}(\mathrm{d}v^2 + \mathrm{d}y^2) + (u\,\mathrm{d}x + v\,\mathrm{d}y - \mathrm{d}z)^2 \quad a, b \neq 0.$$

Thus, our commutative space (M, g) is naturally reductive.

Remark. If only one of the parameters a, b is different from zero, then our space (M, g) is the direct product of the Heisenberg group H^3 and R^2, hence on the basis of (Lemma 1.2, [13]) our space is naturally reductive, as well.

Case $(\mathbf{D}_2)$. In this case we can suppose that

$$(91)\qquad \underline{h}^* = \mathrm{Span}(A_{12} + \lambda A_{34}), \quad \text{where } \lambda \geq 1.$$

We shall distinguish later three subcases:

a) For $\lambda = 1$, the generators for $I(\underline{m})$ are exactly $\mathrm{Ad}(H)$-invariant polynomials $X_1^2 + X_2^2$, $X_3^2 + X_4^2$, $X_1X_3 + X_2X_4$, $X_1X_4 - X_2X_3$, X_5.

b) For $\lambda = 2$, the generators for $I(\underline{m}^c)$ (the complexification of $I(\underline{m})$) are the $\mathrm{Ad}(H)$-invariant polynomials $U_1\overline{U}_1$, $U_2\overline{U}_2$, $U_1^2\overline{U}_2$, $\overline{U}_1^2 U_2$, X_5, where $U_1 = X_1 + iX_2$, $U_2 = X_3 + iX_4$.

c) For $1 < \lambda < 2$, or $\lambda > 2$, we always have $\mathrm{Ad}(H)$-invariant polynomials $X_1^2 + X_2^2$, $X_3^2 + X_4^2$, X_5, and possibly other $\mathrm{Ad}(H)$-invariant polynomials, as generators for $I(\underline{m})$.

The corresponding calculations can be made easily using the complexification of the space $\underline{m}$ and the complex form of the action $\exp[t(A_{12} + A_{34})]$.

At first we shall present some common fact for a general $\lambda \geq 1$.

Using the identity $(A_{12} + \lambda A_{34}) \cdot \widetilde{T} = 0$, we get as before, the following form for $\widetilde{T}(X_i, X_j)$:

$$(92)\quad \begin{cases} \tilde{T}(X_1,X_2) = t_{12}^5 X_5, \quad \tilde{T}(X_3,X_4) = t_{34}^5 X_5, \\ \tilde{T}(X_1,X_3) = t_{13}^1 X_1 + t_{13}^2 X_2 + t_{13}^5 X_5, \\ \tilde{T}(X_1,X_4) = -t_{13}^2 X_1 + t_{13}^1 X_2 + t_{14}^5 X_5, \\ \tilde{T}(X_2,X_3) = t_{13}^2 X_1 - t_{13}^1 X_2 - t_{14}^5 X_5, \\ \tilde{T}(X_2,X_4) = t_{13}^1 X_1 + t_{13}^2 X_2 + t_{13}^5 X_5, \\ \tilde{T}(X_1,X_5) = t_{15}^1 X_1 + t_{15}^2 X_2 + t_{13}^5 X_3 + t_{15}^4 X_4, \\ \tilde{T}(X_2,X_5) = -t_{15}^2 X_1 + t_{15}^1 X_2 - t_{15}^4 X_3 + t_{15}^3 X_4, \\ \tilde{T}(X_3,X_5) = t_{35}^1 X_1 + t_{35}^2 X_1 + t_{35}^3 X_3 + t_{35}^4 X_4, \\ \tilde{T}(X_4,X_5) = -t_{35}^2 X_1 + t_{13}^1 X_2 - t_{35}^4 X_3 + t_{35}^3 X_4, \end{cases}$$

with the additional relations:

$$(93)\quad \begin{cases} (\lambda-1)t_{13}^5 = 0, \quad (\lambda-1)t_{14}^5 = 0, \\ (\lambda-1)t_{15}^3 = 0, \quad (\lambda-1)t_{15}^4 = 0, \\ (\lambda-1)t_{35}^1 = 0, \quad (\lambda-1)t_{35}^2 = 0, \\ (\lambda-2)t_{13}^1 = 0, \quad (\lambda-2)t_{13}^2 = 0. \end{cases}$$

Subcases ($\mathbf{D_2}$) *a)* For $\lambda = 1$ we get from (93):

$$(94)\quad t_{13}^1 = t_{13}^2 = 0.$$

Thus, the tensor $\tilde{T}(X_i, X_j)$ has the form

$$(95)\quad \begin{cases} \tilde{T}(X_1,X_2) = t_{12}^5 X_5, \quad \tilde{T}(X_2,X_3) = -t_{14}^5 X_5, \\ \tilde{T}(X_1,X_3) = t_{13}^5 X_5, \quad \tilde{T}(X_2,X_4) = t_{13}^5 X_5, \\ \tilde{T}(X_1,X_4) = t_{14}^5 X_5, \quad \tilde{T}(X_3,X_4) = t_{34}^5 X_5, \\ \tilde{T}(X_1,X_5) = t_{15}^1 X_1 + t_{15}^2 X_2 + t_{15}^3 X_3 + t_{15}^4 X_4, \\ \tilde{T}(X_2,X_5) = -t_{15}^2 X_1 + t_{15}^1 X_2 - t_{15}^4 X_3 + t_{15}^3 X_4, \\ \tilde{T}(X_3,X_5) = t_{35}^1 X_1 + t_{35}^2 X_2 + t_{35}^3 X_3 + t_{35}^4 X_4, \\ \tilde{T}(X_4,X_5) = -t_{35}^2 X_1 + t_{35}^1 X_2 - t_{35}^4 X_3 + t_{35}^3 X_4. \end{cases}$$

The symmetrization map $\lambda\colon I(\underline{m}) \to U(\underline{g})$ gives the generators $X_1 \cdot X_1 + X_2 \cdot X_2$, $X_3 \cdot X_3 + X_4 \cdot X_4$, $X_1 \cdot X_3 + X_2 \cdot X_4 + Z_1$, $X_1 \cdot X_4 - X_2 \dot{X}_3 + Z_2$, X_5 of $\mathrm{D}(G/H)$, where $Z_1, Z_2 \in \underline{m}$.

Now, we shall calculate the corresponding commutators of algebra $U(\underline{g})$.

We introduce the following short notations:

(96) $$\widetilde{R}(X_i, X_j) = r_{ij}P, \quad P = A_{12} + A_{34}, \quad i,j = 1,\ldots,5.$$

Thus, we get modulo $\underline{h}$:

(97) $$\begin{cases} D_1 = & [X_1 \cdot X_1 + X_2 \cdot X_2, X_5] \equiv \\ \equiv & -2t_{15}^1(X_1 \cdot X_1 + X_2 \cdot X_2) - \\ & -2t_{15}^3(X_1 \cdot X_3 + X_2 \cdot X_4) + \\ & -2t_{15}^4(X_1 \cdot X_4 - X_2 \cdot X_3) + r_{25}X_1 - r_{15}X_2 + \\ & -2(t_{13}^5 t_{15}^3 + t_{14}^5 t_{15}^4)X_5. \end{cases}$$

From the commutativity of the algebra $\mathrm{D}(G/H)$ we get immediately

(98) $$t_{15}^1 = t_{15}^3 = t_{15}^4 = 0.$$

Next, we have

(99) $$\begin{cases} D_2 = & [X_3 \cdot X_3 + X_4 \cdot X_4, X_5] \equiv \\ \equiv & -2t_{35}^3(X_3 \cdot X_3 + X_4 \cdot X_4) - \\ & -2t_{35}^1(X_1 \cdot X_3 + X_2 \cdot X_4) + \\ & +2t_{35}^2(X_1 \cdot X_4 - X_2 \cdot X_3) + r_{45}X_3 - r_{35}X_4 + \\ & -2(t_{13}^5 t_{35}^1 - t_{14}^5 t_{35}^2)X_5. \end{cases}$$

Hence, we obtain the relations

(100) $$t_{35}^1 = t_{35}^2 = t_{35}^3 = 0.$$

In that follows we shall calculate explicitly only the leading terms (of the highest degree) of our commutators.

First, we have

$$\begin{aligned} D_3 = & [X_1 \cdot X_3 + X_2 \cdot X_4 + Z_1, X_1 \cdot X_4 - X_2 \cdot X_3 + Z_2] \equiv \\ & \equiv - t_{34}^5(X_1 \cdot X_1 + X_2 \cdot X_2) \cdot X_5 - \\ & - t_{12}^5(X_3 \cdot X_3 + X_4 \cdot X_4) \cdot X_5 + \\ & + \text{lower terms}. \end{aligned}$$

Thus, we get the relations

(101) $$t_{12}^5 = t_{34}^5 = 0 .$$

Thus, we have

$$\begin{aligned} D_4 =& [X_1 \cdot X_1 + X_2 \cdot X_2 , X_1 \cdot X_3 + X_2 \cdot X_4 + Z_1] \equiv \\ &\equiv - 2 t_{13}^5 (X_1 \cdot X_1 + X_2 \cdot X_2) \cdot X_5 + \text{lower terms}. \end{aligned}$$

Thus, we obtain

(102) $$t_{13}^5 = 0 .$$

Furhter, we have

$$\begin{aligned} D_5 =& [X_1 \cdot X_1 + X_2 \cdot X_2 , X_1 \cdot X_4 - X_2 \cdot X_3 + Z_2] \equiv \\ &\equiv - 2 t_{14}^5 (X_1 \cdot X_1 + X_2 \cdot X_2) \cdot X_5 + \text{lower terms}. \end{aligned}$$

and hence

(103) $$t_{14}^5 = 0 .$$

Finally, we calculate

$$\begin{aligned} D_6 =& [X_1 \cdot X_3 + X_2 \cdot X_4 + Z_1 , X_5] \equiv \\ &\equiv (t_{15}^2 + t_{35}^4)(X_1 \cdot X_3 + X_2 \cdot X_4) + \text{lower terms}. \end{aligned}$$

and hence

(104) $$t_{35}^4 = -t_{15}^2 .$$

From the formulas (98) and (100)-(104) we obtain the following form of the tensor $\widetilde{T}(X_i, X_j)$:

(105) $$\begin{cases} \widetilde{T}(X_1, X_5) = t_{15}^2 X_2 , & \widetilde{T}(X_2, X_5) = -t_{15}^2 X_1 , \\ \widetilde{T}(X_3, X_5) = -t_{15}^2 X_4 , & \widetilde{T}(X_4, X_5) = t_{15}^2 X_3 , \\ \widetilde{T}(X_i, X_j) = 0 \text{ otherwise}. & \end{cases}$$

We can easily prove from the Bianchi identity (6) that

$$\widetilde{R}(X_i, X_j) = 0, \quad \text{for } i, j = 1, \ldots, 5. \tag{106}$$

Thus, we have the same case as in (50), and our space is naturally reductive.

Subcase ($\mathbf{D_2}$) *b)* For $\lambda = 2$ we get from (93):

$$t_{13}^5 = t_{14}^5 = t_{15}^4 = t_{35}^1 = t_{35}^2. \tag{107}$$

Thus, it follows from (93) and (107) that the tensor $\widetilde{T}(X_i, X_j)$ has the following form:

$$\begin{cases} \widetilde{T}(X_1, X_2) = t_{12}^5 X_5, \\ \widetilde{T}(X_1, X_3) = t_{13}^1 X_1 + t_{13}^2 X_2, \\ \widetilde{T}(X_1, X_4) = -t_{13}^2 X_1 + t_{13}^1 X_2, \\ \widetilde{T}(X_1, X_5) = t_{15}^1 X_1 + t_{15}^2 X_2, \\ \widetilde{T}(X_2, X_3) = t_{13}^2 X_1 - t_{13}^1 X_2, \\ \widetilde{T}(X_2, X_4) = t_{13}^1 X_1 + t_{13}^2 X_2, \\ \widetilde{T}(X_2, X_5) = -t_{15}^2 X_1 + t_{15}^1 X_2, \\ \widetilde{T}(X_3, X_4) = t_{34}^5 X_5, \\ \widetilde{T}(X_3, X_5) = t_{35}^3 X_3 + t_{35}^4 X_4, \\ \widetilde{T}(X_4, X_5) = -t_{35}^4 X_3 + t_{35}^3 X_4, \end{cases} \tag{108}$$

In order to simplify the calculations we shall use the complexification $\underline{m}^c$ of $\underline{m}$ and the complex form of the action of $\exp[t(A_{12} + 2A_{34})]$, $t \in R$.

We put

$$U_1 = X_1 + iX_2, \quad U_2 = X_3 + iX_4. \tag{109}$$

For the operator $A = A_{12} + 2A_{34}$ we get

$$AU_1 = -iU_1, \quad AU_2 = -2iU_2, \quad AX_5 = 0. \tag{110}$$

With respect to the basis $\{U_1, \overline{U}_1, U_2, \overline{U}_2, X_5\}$, the representation of $\mathrm{Ad}(H)$ on $\underline{m}^c$ is given by the formulas:

$$\begin{cases} \mathrm{Ad}(h)U_1 = e^{-it}U_1, & \text{for } h \in H, \quad t \in R, \\ \mathrm{Ad}(h)U_2 = e^{-2it}U_2, & \mathrm{Ad}(h)X_5 = X_5. \end{cases} \tag{111}$$

From (111) we see easily that the $\mathrm{Ad}(H)$-invariant polynomials $U_1\overline{U}_1$, $U_2\overline{U}_2$, $U_1^2\overline{U}_2$, $\overline{U}_1^2 U_2$, X_5 on $I(\underline{m}^c)$ form a set of generators, and the symmetrization map gives the following generators for $\mathrm{D}(G/H)$ the complexification of $\mathrm{D}(G/H)$: $U_1 \cdot \overline{U}_1 + \overline{U}_1 \cdot U_1$, $U_2 \cdot \overline{U}_2 + \overline{U}_2 \cdot U_2$, $U_1 \cdot U_1 \cdot \overline{U}_2 + U_1 \cdot \overline{U}_2 \cdot U_1 + \overline{U}_2 \cdot U_1 \cdot U_1$, $\overline{U}_1 \cdot \overline{U}_1 \cdot U_2 + \overline{U}_1 \cdot U_2 \cdot \overline{U}_1 + U_2 \cdot \overline{U}_1 \cdot \overline{U}_1$, X_5.

In the complex form, the formulas (108) can be rewritten as follows:

$$(112)\qquad \begin{cases} \tilde{T}(U_1,\overline{U}_1) = -2\,it_{12}^5 X_5, \\ \tilde{T}(U_1,U_2) = 0, \\ \tilde{T}(U_1,\overline{U}_2) = -2(t_{13}^1 + it_{13}^2)\overline{U}_1 \\ \tilde{T}(U_1,X_5) = (t_{15}^1 - it_{15}^2)U_1, \\ \tilde{T}(U_2,\overline{U}_2) = -2\,it_{34}^5 X_5, \\ \tilde{T}(U_2,X_5) = (t_{35}^2 - it_{35}^4)U_2, \end{cases}$$

(we omit the relations obtained by the complex conjugation).

Now, we obtain, after routine calculations, the following form of the Lie brackets in $\mathrm{D}(G/H)^c$ modulo the subalgebra $\underline{h}^c$:

$$(113)\qquad \begin{cases} D_1 = & [U_1 \cdot \overline{U}_1 + \overline{U}_1 \cdot U_1, X_5] \equiv \\ \equiv & -4\,t_{15}^1(U_1 \cdot \overline{U}_1) + \text{lower terms}; \\ D_2 = & [U_2 \cdot \overline{U}_2 + \overline{U}_2 \cdot U_2, X_5] \equiv \\ \equiv & -4\,t_{35}^3(U_2 \cdot \overline{U}_2) + \text{lower terms}; \\ D_3 = & -8(t_{13}^1 + it_{13}^2)(\overline{U}_1 \cdot \overline{U}_1 \cdot U) + \text{lower terms}. \end{cases}$$

Thus, from the commutativity of the algebra $\mathrm{D}(G/H)^c$ we get the relations:

$$(114)\qquad t_{13}^1 = t_{13}^2 = t_{15}^1 = t_{35}^3 = 0.$$

The last commutators have the form

$$(115)\qquad \begin{cases} D_4 = & [U_1 \cdot U_1 \cdot \overline{U}_2 + U_1 \cdot \overline{U}_2 \cdot U_1 + \overline{U}_2 \cdot U_1 \cdot U_1, X_5] \equiv \\ \equiv & 3i(2t_{15}^2 - t_{35}^4)(U_1 \cdot U_1 \cdot \overline{U}_2) + \text{lower terms}; \\ D_5 = & [U_1 \cdot U_1 \cdot \overline{U}_2 + U_1 \cdot \overline{U}_2 \cdot U_1 + \overline{U}_2 \cdot U_1 \cdot U_1, \\ & \overline{U}_1 \cdot \overline{U}_1 \cdot U_2 + \overline{U}_1 \cdot U_2 \cdot \overline{U}_1 + U_2 \cdot \overline{U}_1 \cdot \overline{U}_1] \equiv \\ \equiv & -18\,it_{34}^5(U_1 \cdot U_1 \cdot \overline{U}_1 \cdot \overline{U}_1 \cdot X_5) + \\ & +36\,it_{12}^5(U_1 \cdot \overline{U}_1 \cdot U_2 \cdot \overline{U}_2 \cdot X_5) + \text{lower terms}; \end{cases}$$

From the commutativity condition we get the relations:

$$(116)\qquad t_{12}^5 = t_{34}^5 = 0, \quad \text{and} \quad t_{35}^4 = 2t_{15}^2.$$

The remaining commutators give not any conditions for t_{ij}^k.

Thus, on the basis of (108), (112) and (116) the tensor $\widetilde{T}$ has the following form (in real domain):

$$(117) \qquad \begin{cases} \widetilde{T}(X_1, X_5) = aX_2, & \widetilde{T}(X_2, X_5) = -aX_1, \\ \widetilde{T}(X_3, X_5) = 2aX_4, & \widetilde{T}(X_4, X_5) = -2aX_3, \\ \widetilde{T}(X_i, X_j) = 0 & \text{otherwise, } a \in R. \end{cases}$$

We can easily prove from the Bianchi identity (6) that

$$(118) \qquad \widetilde{R}(X_i, X_j) = 0, \quad \text{for } i, j = 1, \ldots, 5.$$

Thus, we have a special case of (83) and (86), and our space is naturally reductive.

Subcase ($\mathbf{D_2}$) *c)* For $1 < \lambda < 2$ or $\lambda > 2$ we see from (93) that the tensor $\widetilde{T}(X_i, X_j)$ has the following form:

$$(119) \qquad \begin{cases} \widetilde{T}(X_1, X_2) = t_{12}^5 X_5, & \widetilde{T}(X_2, X_3) = 0, \\ \widetilde{T}(X_1, X_3) = 0, & \widetilde{T}(X_2, X_4) = 0, \\ \widetilde{T}(X_1, X_5) = 0, & \widetilde{T}(X_3, X_4) = t_{34}^5 X_5, \\ \widetilde{T}(X_1, X_5) = t_{15}^1 X_1 + t_{15}^2 X_2, & \\ \widetilde{T}(X_2, X_5) = -t_{15}^2 X_1 + t_{15}^1 X_2, & \\ \widetilde{T}(X_3, X_5) = t_{35}^3 X_3 + t_{35}^4 X_4, & \\ \widetilde{T}(X_4, X_5) = -t_{35}^4 X_3 + t_{35}^3 X_4, & \end{cases}$$

Now, the elements $X_1 \cdot X_1 + X_2 \cdot X_2$, $X_3 \cdot X_3 + X_4 \cdot X_4$, $X_5 \in U(\underline{g})$ define operators from $\mathrm{D}(G/H)$.

The commutators have the following form (modulo the subalgebra $\underline{h}$):

$$(120) \qquad \begin{cases} D_1 = & [X_1 \cdot X_1 + X_2 \cdot X_2, X_5] \equiv \\ \equiv & -2t_{15}^1 (X_1 \cdot X_1 + X_2 \cdot X_2) + \text{ lower terms;} \\ D_2 = & [X_3 \cdot X_3 + X_4 \cdot X_4, X_5] \equiv \\ \equiv & -2t_{35}^3 (X_3 \cdot X_3 + X_4 \cdot X_4) + \text{ lower terms;} \\ D_3 = & (X_1 \cdot X_1 + X_2 \cdot X_2, X_3 \cdot X_3 + X_4 \cdot X_4] \equiv 0. \end{cases}$$

Hence, from the commutativity condition we have the relations:

$$(121) \qquad t_{15}^1 = t_{35}^2 = 0.$$

Thus, the tensor $\widetilde{T}(X_i, X_j)$ has the form:

$$(122)\qquad \begin{cases} \widetilde{T}(X_1,X_2) = t_{12}^5 X_5, & \widetilde{T}(X_2,X_3) = 0, \\ \widetilde{T}(X_1,X_3) = 0, & \widetilde{T}(X_2,X_4) = 0, \\ \widetilde{T}(X_1,X_4) = 0, & \widetilde{T}(X_3,X_4) = t_{34}^5 X_5, \\ \widetilde{T}(X_1,X_5) = t_{15}^2 X_2, & \widetilde{T}(X_3,X_5) = t_{35}^4 X_4, \\ \widetilde{T}(X_2,X_5) = -t_{15}^2 X_1 & \widetilde{T}(X_4,X_5) = -t_{35}^4 X_3. \end{cases}$$

We can easily prove from the Bianchi identity (6) that

$$(123)\qquad \widetilde{R}(X_i, X_j) = 0, \quad \text{for } i, j = 1, \ldots, 5.$$

Thus, we have a special case of (83), and our space is naturally reductive.

Case (E). In this case the group $\mathrm{Ad}(H)$ is equivalent to $SO(2)$ as a transformation group. Here our group $SO(2)$ acts transitively on the circle $S^1 \subset \mathrm{Span}(X_1, X_2)$, and acts trivially on $\mathrm{Span}(X_3, X_4, X_5)$.

Obviously, $X_1^2 + X_2^2$, X_3, X_4, X_5 form a set of generators, of $I(\underline{m})$.

Further, $\mathrm{D}(G/H)$ is generated by the differential operators $X_1 \cdot X_1 + X_2 \cdot X_2$, X_3, X_4, X_5.

The Lie algebra of $\mathrm{Ad}(H)$ acting on $\underline{m}$ is

$$(124)\qquad \underline{h}^* = \mathrm{Span}(A_{12}).$$

Using the identity $A_{12} \cdot \widetilde{T} = 0$ we get after elementary calculations, that the tensor $\widetilde{T}(X_i, X_j)$ has the following form:

$$(125)\qquad \begin{cases} \widetilde{T}(X_1,X_2) = t_{12}^3 X_3 + t_{12}^4 X_4 + t_{12}^5 X_5, \\ \widetilde{T}(X_1,X_3) = t_{13}^1 X_1 + t_{13}^2 X_2, \\ \widetilde{T}(X_1,X_4) = t_{14}^1 X_1 + t_{14}^2 X_2, \\ \widetilde{T}(X_1,X_5) = t_{15}^1 X_1 + t_{15}^2 X_2, \\ \widetilde{T}(X_2,X_3) = -t_{13}^2 X_1 + t_{13}^1 X_2, \\ \widetilde{T}(X_2,X_4) = -t_{14}^2 X_1 + t_{14}^1 X_2, \\ \widetilde{T}(X_2,X_5) = -t_{15}^2 X_1 + t_{15}^1 X_2, \\ \widetilde{T}(X_3,X_4) = t_{34}^3 X_3 + t_{34}^4 X_4 + t_{34}^5 X_5, \\ \widetilde{T}(X_3,X_5) = t_{35}^3 X_3 + t_{35}^4 X_4 + t_{35}^5 X_5, \\ \widetilde{T}(X_4,X_5) = t_{45}^3 X_3 + t_{45}^4 X_4 + t_{45}^5 X_5, \end{cases}$$

Now, we shall calculate the commutators, in $U(\underline{g})$ for the generators $X_1 \cdot X_1 + X_2 \cdot X_2$, X_3, X_4, $X_5 \in D(G/H)$. Realizing this, we obtain modulo $\underline{h}$ using (3) and (125):

$$(126)\qquad \begin{cases} D_1 = & [X_3, X_4] \equiv -t_{34}^3 X_3 - t_{34}^4 X_4 - t_{34}^5 X_5, \\ D_2 = & [X_3, X_5] \equiv -t_{35}^3 X_3 - t_{35}^4 X_4 - t_{35}^5 X_5, \\ D_3 = & [X_1 \cdot X_1 + X_2 \cdot X_2, X_3] \equiv \\ \equiv & -2t_{13}^1 (X_1 \cdot X_1 + X_2 \cdot X_2) + \alpha_1 X_1 + \alpha_2 X_2, \\ D_4 = & [X_1 \cdot X_1 + X_2 \cdot X_2, X_4] \equiv \\ \equiv & -2t_{14}^1 (X_1 \cdot X_1 + X_2 \cdot X_2) + \beta_1 X_1 + \beta_2 X_2, \\ D_5 = & [X_1 \cdot X_1 + X_2 \cdot X_2, X_5] \equiv \\ \equiv & -2t_{15}^1 (X_1 \cdot X_1 + X_2 \cdot X_2) + \gamma_1 X_1 + \gamma_2 X_2, \end{cases}$$

where $\alpha_1, \ldots, \gamma_2 \in R$.

From the commutativity condition we obtain the following relations:

$$(127)\qquad \begin{cases} t_{13}^1 = t_{14}^1 = t_{15}^1 = 0, \\ t_{34}^3 = t_{34}^4 = t_{34}^5 = 0, \\ t_{35}^3 = t_{35}^4 = t_{35}^5 = 0, \\ t_{45}^3 = t_{45}^4 = t_{45}^5 = 0. \end{cases}$$

From (125) and (127) we get using shorter notations:

$$(128)\qquad \begin{cases} \widetilde{T}(X_1, X_2) = aX_3 + bX_4 + cX_5, \\ \widetilde{T}(X_3, X_4) = \widetilde{T}(X_3, X_5) = \widetilde{T}(X_4, X_5) = 0, \\ \widetilde{T}(X_1, X_3) = dX_2, \quad \widetilde{T}(X_2, X_3) = -dX_1, \\ \widetilde{T}(X_1, X_4) = fX_2, \quad \widetilde{T}(X_2, X_4) = -fX_1, \\ \widetilde{T}(X_1, X_5) = gX_2, \quad \widetilde{T}(X_2, X_5) = -gX_1. \end{cases}$$

Further, we have from (124)

$$(129)\qquad \widetilde{R}(X_i, X_j) = r_{ij} A_{12}, \quad r_{ij} \in R, \quad i, j = 1, \ldots, 5.$$

Substituting (128), and (129) in the Bianchi identity (6) we get

$$(130)\qquad \widetilde{R}(X_1, X_2) = r_{12} A_{12}, \quad \widetilde{R}(X_i, X_j) = 0 \text{ otherwise}.$$

To prove the natural reductivity of our space we shall change again the reductive decomposition $\underline{g} = \underline{m} \oplus \underline{h}^*$, $[\underline{h}^*, \underline{m}] \subseteq \underline{m}$ into the new reductive decomposition

$$(131)\qquad \underline{g} = \underline{m}' \oplus \underline{h}^*, \quad [\underline{h}^*, \underline{m}'] \subseteq \underline{m}',$$

where

$$
(132) \qquad \begin{cases} \underline{m}' = \operatorname{Span}(X_1', \ldots, X_5'), \\ X_i' = X_i, \quad i = 1, 2, 3, \\ X_3' = X_3 - (a + d) A_{12}, \\ X_4' = X_4 - (b + f) A_{12}, \\ X_5 = X_5 - (c + g) A_{12}. \end{cases}
$$

The Lie algebra brackets in $\underline{g}$ are, due to (128), (130), (132), and (3) equal to

$$
(133) \qquad \begin{cases} [X_1', X_2'] = -aX_3 - bX_4 - cX_5 + \alpha A_{12}, \quad \alpha \in R, \\ [X_1', X_3'] = aX_2, \quad [X_2', X_3'] = -aX_1, \\ [X_1', X_4'] = bX_2, \quad [X_2', X_4'] = -bX_1, \\ [X_1', X_5'] = cX_2, \quad [X_2', X_5'] = -cX_1, \\ [X_3', X_4'] = [X_3', X_5'] = [X_4', X_5'] = 0. \end{cases}
$$

From (133) and (2) it follows the natural reductivity of our commutative space (M, g) in the case (E).

This completes the proof of Theorem 2.1.

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