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STÄCKEL SYSTEMS AND KILLING TENSORS *

S. BENENTI **

1. INTRODUCTION

The problem of finding the geodesic of a Riemannian manifold can be solved by the method of Jacobi. With this method the problem is turned to the integration of a partial differential equation, the Hamilton-Jacobi equation of the geodesics. Once a complete integral of this equation has been found, the solutions of the ordinary differential geodesic equations are obtained by a straightforward process of differentiation and substitution. This method has been applied in Analytical Mechanics to find the dynamical trajectories of holonomic systems and also in General Relativity to find the trajectories of test particles in some exact solutions of Einstein equations. In fact, in almost all the cases in which this method can be applied with success the complete integral in a sum of functions each one depending on one coordinate only and the Hamilton-Jacobi equation splits into separated ordinary differential equations. Thus we say that the Hamilton-Jacobi equation is integrable by separation of variables.

Coordinate systems which allow the integration by separation of variables of the geodesic Hamilton-Jacobi equation are called *separable*. Separable coordiantes have some importance in Mathematical Physics, since they allow the integration by separation of variables of various second order field equations, related to a Riemannian metric and to a potential function (Laplace, Helmholtz, Schrödinger, etc.), provided that the potential and the Ricci tensor satisfy suitable conditions.

We know that the existence of separable coordinates on a Riemannian manifold is related to the existence of Killing vectors (i.e. linear first integrals or isometries) and of Killing tensors of order 2 (i.e. quadratic first integrals). The geometrical conditions imposed to a manifold for the existence of separable coordinates are very nice but actually very restrictive. However, the manifolds with constant curvature abound of separable coordinates, since they have the greatest possible number of Killing vectors and tensors one can find on a manifold [12]. Separable coordinates are fully classified for Euclidean spaces, for spheres and for pseudospheres (see [8], also for an essential hystorical account on the problem). Furthermore, as we have mentioned, many interesting examples of separable coordinates come from General Relativity (see for instance [5], [17]).

This lecture is devoted to the geometrical characterization of *orthogonal* separable coordinates. The characterization of non-orthogonal separable coordinates would need a longer

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treatment (see for instance [10],). We shall see that the existence of separable coordinates is characterized by purely algebraic conditions on sets of Killing tensors of order 2. We shall call such sets *Stäckel systems.* We begin with an outline of the Jacobi method and a theorem on the separation of variables which will be used at the end. In Section 3 basic definitions and theorems concerning the algebra of contravariant symmetric tensors and Killing tensors are presented. In Section 4 we introduce a modified version of the equations written by Eisenhart [7] for the Killing tensors of order 2, by considering a non-holonomic frame made of eigenvectors. They will be used to prove a theorem which is a modified version of previous theorems due to Eisenhart [6], Woodhouse [17] and Kalnins and Miller [9].

2. THE METHOD OF JACOBI

Let Q be a differential manifold of dimension n. Let $\pi_Q : T^*Q \to Q$ be the cotangent fibration of Q. Local coordinates of Q will be denoted by $q = (q^i)$ (latin indices range from 1 to n). The corresponding canonical coordinates on T^*Q will be denoted by (q^i, p_i). Coordinates (p_i) are called *momenta.* We shall use the canonical symplectic structure of the cotangent bundle T^*Q generated by the *foundamental form* Θ_Q (the *Liouville 1-form*), locally defined by

$$\Theta_Q = p_i dq^i.$$

With a C^∞ real function H on T^*Q we associate a C^∞ vector field $\mathbf{X}_H$ on T^*Q defined by equation

$$i_{\mathbf{X}_H} d\Theta_Q = -dH. \tag{1}$$

Here the symbol i denotes the inner product of a vector field by a differential form. The function H is called the *Hamiltonian* of the vector field $\mathbf{X}_H$. In canonical coordinates equation (1) is represented by the *Hamilton equations*:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2}$$

The integral curves of the vector field $\mathbf{X}_H$ are the solutions of this differential system.

The symplectic structure on T^*Q generates a Poisson structure on the space of the C^∞ real functions on T^*Q. The Poisson bracket $\{F, G\}$ of two functions is defined by equation

$$\{F, G\} = \langle \mathbf{X}_F \wedge \mathbf{X}_G, d\theta_Q \rangle. \tag{3}$$

In canonical coordinates

$$\{F, G\} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}. \tag{3'}$$

(This definition differs in sign from that of Classical Mechanics). Two functions F and G are said to be in *involution* if $\{F, G\} = 0$. A function F is said to be a *first integral* of the vector field $\mathbf{X}_H$ (or of the Hamiltonian H) if it is constant along the integral curves of $\mathbf{X}_H$. A function F is a first integral of $\mathbf{X}_H$ if and only if $\{F, H\} = 0$. Hence the hamiltonian itself is a first integral (the *energy integral*).

Besides the Hamilton equations (2) a Hamiltonian H generates a first order partial differential equation, the *(reduced) Hamilton-Jacobi equation,*

$$H\left(q^i, \frac{\partial W}{\partial q^j}\right) = h, \tag{4}$$

where h is a real parameter (the *energy*). A *complete integral* of this equation is a solution

$$W = W(q^i, c_a) \tag{5}$$

depending on n real constants of integration $(c_a)(a = 1, \ldots, n)$ such that

$$det\left(\frac{\partial^2 W}{\partial q^i \partial c_a}\right) \neq 0 \tag{6}$$

everywhere. Usually one of the constans (c_a), say c_n, is the energy h. When such a complete integral is known then the solutions of the Hamilton equations (2) are determined by the following equations:

$$b^\alpha + \frac{\partial W}{\partial c_\alpha} = 0 \qquad (\alpha = 1, \ldots, n-1), \tag{7}$$

$$b^n + \frac{\partial W}{\partial h} = t, \tag{8}$$

$$p_i = \frac{\partial W}{\partial q^i}, \tag{9}$$

where (b^α, b^n) are further n real parameters. Equations (7) define a system of unparametrized curves on Q. These curves are the projections to Q of the integral curves of $\mathbf{X}_H$.

Example. Let $(Q, \mathbf{g})$ be a Riemannian manifold. (We use this term in a general sense: the metric $\mathbf{g}$ can be positive-definite or semi-definite). Let us consider the *geodesic Hamiltonian*

$$H = \frac{1}{2} g^{ij} p_i p_j, \tag{10}$$

where (g^{ij}) are the contravariant components of the metric tensor $\mathbf{g}$. The vector field $\mathbf{X}_H$ on T^*Q is the *geodesic field.* The integral curves of $\mathbf{X}_H$ are indeed projected by π_Q onto the geodesic of $(Q, \mathbf{g})$. Hence, the unparametrized geodesics are given by equations (7), where W is a complete integral of the *geodesic Hamilton-Jacobi equation*

$$g^{ij} \frac{\partial W}{\partial q^i} \frac{\partial W}{\partial q^j} = 2h. \tag{11}$$

From a geometrical point of view a complete integral of the Hamilton-Jacóbi equation (2) is a local *Lagrangian transversal foliation* of T^*Q, *compatible with the Hamiltonian* H. This means that: (i) For each admissible value of the constants $\mathbf{c} = (c_a)$, equations (9) define a Lagrangian submanifold $L_{\mathbf{c}}$ of T^*Q which is trasversal to the fibres (i.e. it is the image of a section of T^*Q). We recall that a Lagrangian submanifold is an isotropic manifold of maximal dimension n. Isotropic means that the pull-back to this submanifold of the canonical symplectic form is the zero form. (ii) Lagrangian submanifolds corresponding to different values of $\mathbf{c}$ have empty intersection. This is due to the completeness condition (6). (iii) The Hamiltonian H is constant on each submanifold of the foliation. This is expressed by equation (2).

There is an alternative representation of such a foliation, by means of first integral in involution. Let (F_a) be n real differentiable independent functions on T^*Q. The equations

$$F_a = c_a, \tag{12}$$

where (c_a) are real parameters, define a local foliation of T^*Q. This foliation is transversal to the fibres fo T^*Q if and only if

$$det \left(\frac{\partial F_a}{\partial p_j} \right) \neq 0. \tag{13}$$

The foliation is Lagrangian if and only if the functions (F_a) are in involution,

$$\{F_a, F_b\} = 0. \tag{14}$$

The hamiltonian H is constant with respect to this foliation if and only if

(15) $$\{H, F_a\} = 0,$$

i.e. if and only if the functions F_a are first integrals. The system of equations (15) is equivalent to the Hamilton-Jacobi equation (2). We pass from the first representation of the foliation to the second one by solving equations (9) with respect to (c_a); then we get equations (12). Conversely, if we solve equations (12) with respect to the momenta, we find a system of fucntions $p_i = W_i(q^i, c_a)$. Because of conditions (14) the 1-form $W_i dq^i$ is integrable and an integral function W of this form is a complete integral. Equations (15) imply that the Hamiltonian H is functionally dependent on the functions (F_a). The choice $c_n = h$ is equivalent to $F_n = H$.

Definition 1. *Local coordinates (q^i) of the manifold Q are called separable with respect to a Hamiltonian H if the Hamilton-Jacobi equation (2) has a complete integral of the form*

(16) $$W = W_1(q^1, c_a) + W_2(q^2, c_a) + \ldots + W_n(q^n, c_a),$$

i.e. such that

(17) $$\partial_i \partial_j W = 0 \quad for \quad i \neq j.$$

For simplicity, we shall use the notation:

$$\partial_i = \frac{\partial}{\partial q^i}, \quad \partial^i = \frac{\partial}{\partial p_i}.$$

Levi-Civita [13] gave a criterion for the separability of a coordinate system. The following equations must be identically satisfied (no sum with respect to the distinct indices i and j):

(18) $$\partial^i H \partial^j H \partial_i \partial_j H + \partial_i H \partial_j H \partial^i \partial^j H - \partial^i H \partial_j H \partial_i \partial^j H - \partial_i H \partial^j H \partial^i \partial_j H = 0 \quad (i \neq j).$$

Definition 2. *Two functions F and G on T^*Q are said to be in separable involution with respect to a coordinate system (q^i) if*

(19) $$\{F, G\}_i \doteq \partial^i F \partial_i G - \partial_i F \partial^i G = 0.$$

for each index i (no sum with respect to the index i).

In Section 5 we shall use the following theorem, which was already stated in [2].

Theorem 1. *Local coordinates* (q^i) *are separable with respect to a Hamiltonian* H *if and only if there exist* n *independent first integrals* (F_a) *in separable involution with respect to* (q^i).

Proof. By combining equations (9) and (12) we get the identities

$$F_a(q^i, \partial_j W) = c_a.$$

By differentiating them with respect to a coordinate q^i, we get

$$\partial_i F_a + \partial^j F_a \Gamma_{ij} = 0, \quad \Gamma_{ij} = \partial_i \partial_j W. \tag{20}$$

We must show that condition

$$\Gamma_{ij} = 0, \quad i \neq j, \tag{21}$$

is equivalent to

$$\{F_a, F_b\}_i = 0 \tag{22}$$

If (21) holds, then (20) implies

$$\partial_i F_a + \partial^i F_a \Gamma_{ii} = 0 \tag{23}$$

(no sum w.r. to the index i).

If we write the similar equation for a different function F_b and take the difference of the two equations, we obtain:

$$\Gamma_{ii}\{F_b, F_a\}_i = 0.$$

If $\Gamma_{ii} \neq 0$, then (22) follows. If $\Gamma_{ii} = 0$, then (23) shows that $\partial_i F_a = 0$ for all indices $a = 1, \ldots, n$, but this again implies (22). Conversely, assume that (22) holds. The matrix $(\partial^i F_a)$ is regular (condition (13)). Let us denote by (A_i^a) the inverse matrix: $\partial^j F_a A_i^a = \delta_i^j$. Let us solve the system (20) with respect to the functions Γ_{ij},

$$\Gamma_{ji} = -A_j^a \partial_i F_a.$$

Let us multiply this equation by $\partial^i F_b \neq 0$ (there is at least one index b for which this element is different from zero), without summing with respect to the index i, and use condition (22). We get

$$\Gamma_{ji} \partial^i F_b = -A_j^a \partial^i F_a \partial_i F_b = -\delta_j^i \partial_i F_b.$$

This shows that $\Gamma_{ji} = 0$ for $j \neq i$. ∎

3. KILLING TENSORS

Let $S^k(Q)$ be the space of C^∞ contravariant symmetric tensor fields of order k on the manifold Q. By definition $S^1(Q)$ is the space of vector fields, and $S^0(Q) = C^\infty(Q, \mathbf{R})$ is the space of differentiable real functions on Q. With each $\mathbf{K} \in S^k(Q)$ we associate a C^∞ real function $E_\mathbf{K}$ on T^*Q defined by equation

$$E_\mathbf{K}(p) = \frac{1}{k!} \langle \mathbf{K}(\pi_Q(p)), p^k \rangle, \quad p \in T^*Q,$$

where $p^k = p \otimes \ldots \otimes p$, k times. For $k = 0$, $\mathbf{K}$ is a function on Q and we define

$$E_\mathbf{K} = \mathbf{K} \cdot \pi_Q = \pi_Q^* \mathbf{K}.$$

For $k \neq 0$ the local canonical coordinate representation of $E_\mathbf{K}$ is a homogeneous polynomial function of degree k of the momenta:

$$E_\mathbf{K} = \frac{1}{k!} \mathbf{K}^{i_1 \ldots i_k} p_{i_1} \ldots p_{i_k}.$$

There is a natural identification between these functions on T^*Q and the symmetric contravariant tensors on Q.

The *symmetric product* of two contravariant tensors $\mathbf{K} \in S^k(Q)$ and $\mathbf{L} \in S^l(Q)$ is the symmetric tensor of order $k + l$ denoted by $\mathbf{K} \cap \mathbf{L}$ and defined by equation

$$E_{\mathbf{K} \cap \mathbf{L}} = E_\mathbf{K} E_\mathbf{L}. \tag{1}$$

The *Lie bracket* of two contravariant tensors $\mathbf{K} \in S^k(Q)$ and $\mathbf{L} \in S^l(Q)$ is the symmetric tensor $[\mathbf{K}, \mathbf{L}] \in S^{k+l-1}(Q)$ defined by equation

$$E_{[\mathbf{K}, \mathbf{L}]} = \{E_\mathbf{K}, E_\mathbf{L}\}, \tag{2}$$

where $\{,\}$ is the canonical Poisson bracket. The bracket [,] can be linearly extended to the direct sum

$$S(Q) = \oplus_{k=0}^{+\infty} S^k(Q).$$

Then this space becomes a Lie algebra. The Lie bracket is a derivation with respect to the symmetric product, i.e. it is bilinear and the Leibniz rule holds:

$$[\mathbf{K}, \mathbf{L} \cap \mathbf{M}] = [\mathbf{K}, \mathbf{L}] \cap \mathbf{M} + [\mathbf{K}, \mathbf{M}] \cap \mathbf{L}.$$

This last property directly follows from definitions (1) and (2) and from the analogous property of the Poisson bracket. It can be shown that when $\mathbf{K}$ is a vector field ($k = 1$), then $[\mathbf{K}, \mathbf{L}]$ is the Lie derivative of $\mathbf{L}$ with respect to $\mathbf{K}$. In local coordinates the components of $[\mathbf{K}, \mathbf{L}]$ are defined by

$$[\mathbf{K}, \mathbf{L}]^{i\ldots j} = \frac{(k+l-1)!}{k!\,l!}\,(k K^{h(i\ldots}\partial_h L^{\ldots j)} - l L^{h(i\ldots}\partial_h K^{\ldots j)}),$$

where $(\ldots)$ is the symmetrization operator over the indices and ∂_h denotes the partial derivative with respect to the coordinate q^h.

Two symmetric tensors $\mathbf{K}$ and $\mathbf{L}$ are said to *commute* or to be *in involution* if $[\mathbf{K}, \mathbf{L}] = 0$.

If Q is a Riemannian manifold then there is a natural identification between covariant and contravariant tensors. This identification will be understood throughout this lecture.

Definition 1. *Let $(Q, \mathbf{g})$ be a Riemannian manifold. A Killing tensor is a symmetric tensor field $\mathbf{K}$ which commutes with the metric tensor: $[\mathbf{K}, \mathbf{g}] = 0$.*

Killing tensors form a subalgebra of $S(Q)$. A function F on T^*Q is a first integral of the geodesic field if and only if $\{F, H\} = 0$. Since $[\mathbf{K}, \mathbf{g}] = \{E_\mathbf{K}, E_\mathbf{g}\} = 2\{E_\mathbf{K}, H\}$ where H is the geodesic Hamiltonian, it follows that:

Proposition 1. *A symmetric tensor $\mathbf{K}$ is a Killing tensor if and only if the function $E_\mathbf{K}$ is a first integral of the geodesic flow.*

As a consequence Killing tensors are identified with polynomial first integrals of the geodesic flow. It can also be proved that

Proposition 2. *A symmetric tensor $\mathbf{K}$ is a Killing tensor if and only if $(\nabla \mathbf{K}) = \mathbf{0}$, where ∇ is the covariant derivative with respect to the Levi-Civita connection and the brackets $(\ldots)$ denote the symmetrization operator.*

This property is usually assumed as a definition of Killing tensor. For our purposes Definition 1 is preferable, since the Levi-Civita connection is not involved.

Killing functions ($k = 0$) are locally constant functions. For *Killing vectors* ($k = 1$) we have the following characteristic properties.

Proposition 3. *A vector field $\mathbf{K}$ is a Killing vector if and only if the corresponding flow is isometric (the Lie derivative of the metric tensor is zero).*

Proposition 4. *A vector field $\mathbf{K}$ is a Killing vector if and only if, as a derivation on functions, it commutes with the Laplacian Δ: $\mathbf{K}(\Delta\psi) - \Delta(\mathbf{K}\psi) = 0$.*

The concept of Killing tensor is an extension of the original concept of Killing vector in the direction of symmetric tensors. There is also an extension towards the anti-symmetric tensors (see for instance [18]). In the following discussion we shall deal with Killing tensors of order 2 only, so that "Killing tensor" will mean "Killing tensor of order 2".

Symmetric tensors of order two can be interpreted as linear operators on vector fields and 1-forms. We denote by $\mathbf{K} \cdot \mathbf{X}$ and by $\mathbf{K} \cdot \phi$ the value of the tensor $\mathbf{K}$ on the vector field $\mathbf{X}$ and on the 1-form ϕ. We recall that a vector field $\mathbf{X}$ (respectively a 1-form ϕ) is an *eigenvector* (resp. an *eigenform*) of $\mathbf{K}$ associated with the *eigenvalue* λ of a (symmetric) tensor $\mathbf{K}$ of order two if $\mathbf{K} \cdot \mathbf{X} = \lambda \mathbf{X}$, or $\mathbf{K} \cdot \phi = \lambda \phi$.

4. THE INTRINSIC KILLING EQUATIONS

A *frame* on a manifold Q is a set of n independent C^∞ vector fields $(\mathbf{X}_i)$. Frames can be defined only locally, unless Q is parallelizable. The commutation relations

$$(1) \qquad [\mathbf{X}_i, \mathbf{X}_j] = \Omega_{ij}^h \mathbf{X}_h$$

define a set of functions (Ω_{ij}^h). These functions are not all independent because of the anti-commutativity of the Lie bracket and the Jacobi identity. A frame is said to be *holonomic* if the vectors commute, i.e. if $\Omega_{ij}^h = 0$. In this case there exist local coordinates (q^i) such that $\mathbf{X}_i = \partial/\partial q^i$.

Let $\mathbf{K}$ and $\mathbf{L}$ be two contravariant symmetric tensor fields or order 2. Let $(\mathbf{X}_i)$ be a frame. Let K^{ij} and L^{ij} be the components of $\mathbf{K}$ and $\mathbf{L}$ with respect to the frame $(\mathbf{X}_i)$. By a straightforward calculation, based on formula (3) of Section 3, we get the components of $[\mathbf{K}, \mathbf{L}]$ with respect the frame:

$$(2) \qquad [\mathbf{K}, \mathbf{L}]^{hij} = K^{kh}\mathbf{X}_k L^{ij} - L^{kh}\mathbf{X}_k K^{ij} + (K^{rh}L^{si} + K^{sh}L^{hi})\Omega_{rs}^j + \dots$$

Here, periods denote the two terms obtained from the first one by cyclic permutation of the indices (h, i, j).

Suppose that $\mathbf{K}$ and $\mathbf{L}$ are "diagonalized" in the frame $(\mathbf{X}_i)$, i.e. that $K^{ij} = L^{ij} = 0$ for $i \neq j$. From formula (2), written for different indices (h, i, j) (when $n \geq 3$), for $h = i \neq j$ and for $h = i = j$ respectively, it follows that equation $[\mathbf{K}, \mathbf{L}] = 0$ is equivalent to equations

$$(3) \qquad \begin{cases} \Omega_{hi}^j(K^hL^i - K^iL^h) + \Omega_{ij}^h(K^iL^j - K^jL^i) + \Omega_{jh}^i(K^jL^h - K^hL^j) = 0, \\ K^i\mathbf{X}_i L^j - L^i\mathbf{X}_i K^j + 2(K^iL^j - K^jL^i)\Omega_{ij}^j = 0, \\ K^i\mathbf{X}_i L^i - L^i\mathbf{X}_i K^i = 0, \end{cases}$$

where $K^i = K^{ii}$ and $L^i = L^{ii}$ (no sum with respect to the repeated indices).

Proposition 1. *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be two symmetric tensors on a Riemannian manifold* $(Q, \mathbf{g})$ *of dimension* n. *Assume that they have a common set of* n *independent real orthogonal eigenvectors* $(\mathbf{X}_i)$. *Let* ρ_i *and* σ_i *be the real eigenvalues of* $\mathbf{K}$ *and* $\mathbf{L}$ *respectively, corresponding to* $\mathbf{X}_i$. *Then* $\mathbf{K}$ *and* $\mathbf{L}$ *commute, i.e.* $[\mathbf{K}, \mathbf{L}] = 0$, *if and only if*

(4)
$$\begin{cases} \Omega_{hij}(\rho_h\sigma_i - \rho_i\sigma_h) + \Omega_{ijh}(\rho_i\sigma_j - \rho_j\sigma_i) + \Omega_{jhi}(\rho_j\sigma_h - \rho_h\sigma_j) = 0, & (h, i, j \neq) \\ \sigma_i \mathbf{X}_i \rho_j - \rho_i \mathbf{X}_i \sigma_j = (\rho_i\sigma_j - \rho_j\sigma_i)(\mathbf{X}_i \log|\varepsilon^j| + 2\varepsilon^j \Omega_{ijj}), \end{cases}$$

where

$$\Omega_{hij} = [\mathbf{X}_h, \mathbf{X}_i] \cdot \mathbf{X}_j, \quad \varepsilon^i = g^{ii} = (\mathbf{X}_i \cdot \mathbf{X}_i)^{-1}. \tag{5}$$

Proof. We take the set $(\mathbf{X}_i)$ as a frame and substitute

$$K^i = \varepsilon^i \rho_i, \quad L^i = \varepsilon^i \sigma_i, \quad \Omega_{ij}^h = \varepsilon^h \Omega_{ijh},$$

in equation (3). ∎

If we choose $\mathbf{L} = \mathbf{g}$ (so that $\sigma_i = 1$), we have:

Proposition 2. *Let* $\mathbf{K}$ *be a symmetric tensor of order 2 on a Riemannian manifold* $(Q, \mathbf{g})$. *Let the eigenvalues* (ρ_i) *of* $\mathbf{K}$ *be real (this assumption is satisfied if the metric tensor is positive-definite) and let* $(\mathbf{X}_i)$ *be a frame made of orthogonal eigenvectors of* $\mathbf{K}$. *Then* $\mathbf{K}$ *is a Killing tensor, i.e.* $[\mathbf{K}, \mathbf{g}] = 0$, *if and only if the following equations are satisfied, where* $2\Omega_{h(ij)} = \Omega_{hij} + \Omega_{hji}$:

$$\begin{cases} \rho_h \Omega_{h(ij)} + \rho_i \Omega_{i(hj)} + \rho_j \Omega_{j(hi)} = 0 \qquad (h, i, j \neq), \\ \mathbf{X}_i \rho_j = (\rho_i - \rho_j)(\mathbf{X}_i \log|\varepsilon^j| + 2\varepsilon^j \Omega_{ijj}). \\ \mathbf{X}_i \rho_i = 0. \end{cases} \tag{6}$$

Remark 1. *Similar equations has been written by Eisenhart [7], with respect to an orthogonal unitary frame* $(\mathbf{X}_i)(\varepsilon_i = \pm 1)$,

$$\begin{cases} (\rho_h - \rho_i)\gamma_{hij} + (\rho_i - \rho_j)\gamma_{ijh} + (\rho_j - \rho_h)\gamma_{jhi} = 0, \\ \mathbf{X}_i \rho_j = 2(\rho_j - \rho_i)\varepsilon^j \gamma_{ijj}, \qquad \mathbf{X}_i \rho_i = 0, \end{cases} \tag{7}$$

where

$$\gamma_{hij} = \mathbf{X}_i \cdot \nabla_{\mathbf{X}_j} \mathbf{X}_h \tag{8}$$

are called coefficients of rotation. These coefficients have been sistematically used in Riemannian geometry to study the behaviour of an orthogonal unitary frame. However, it is better for our purposes to use equations (6) instead of equations (7). Indeed, since we deal with purely differential properties of the frame $(\mathbf{X}_i)$ *it is more convenient to use, instead of the coefficients of rotation, the commutation coefficients* (Ω_{ij}^h) *or* (Ω_{ijh}) *whose definition does not involve a redundant structure as the Levi-Civita connection.*

Remark 2. If the frame is holonomic and $\mathbf{X}_i = \partial_i$, then the first equations in (6) are identically satisfied and the second ones reduce to

$$\partial_i \rho_j = (\rho_i - \rho_j)\partial_i \log |g^{jj}|. \tag{9}$$

Proposition 3. *If two Killing tensors* $\mathbf{K}$ *and* $\mathbf{L}$ *are both diagonalized in an orthogonal holonomic frame, then they are in involution.*

Proof. The frame is made of eigenvectors and equations $(6)_2$ hold for $\mathbf{K}$ and $\mathbf{L}$. These equations imply equations $(4)_2$. Equations $(4)_1$, just like equations $(6)_1$, are identically satisfied since the frame is holonomic. ■

5. STÄCKEL SYSTEMS

Orthogonal systems of separable coordinates have been called "Stäckel systems" (see [6]) in honour of P. Stäckel, who established the first fundamental results on separation of variables (see Theorem 2 below). However, for our purposes it is more convenient to consider Stäckel systems and orthogonal separable coordinate systems as two distinct concepts. Indeed, our discussion will be based on the following two definitions.

Definition 1. *A Stäckel system on a Riemannian manifold* $(Q, \mathbf{g})$ *of dimension* n *is a set* $\{\mathbf{K}_a; a = 1, 2, \ldots, n\}$ *of* n *Killing tensors on an open submanifold* U *of* Q *such that:*

(i) They are independent at each point of U.

(ii) They have n independent orthogonal eigenvector fields in common.

(iii) The common eigenvector fields are normal.

A vector field is called *normal* if the orthogonal distribution is completely integrable. Two Stäckel systems are said to be *equivalent* if they are related by a linear transformation with constant coefficients.

Two Stäckel systems are said to be *equivalent* if they are related by a linear transformation with constant coefficients.

For a definite-positive metric the requirement "real" in condition (ii) is redundant.

Definition 2. *An orthogonal separable system on a Riemannian manifold* (Q, g) *is a system of local coordinates* (q^i) *on* Q *such that*

$$g^{ij} = dq^i \cdot dq^j = 0, \quad for \quad i \neq j,$$

and the corresponding geodesic Hamilton-Jacobi equation

$$g^{ii} \left(\frac{\partial W}{\partial q^i} \right)^2 = 2h, \tag{1}$$

has a complete integral of the form

$$W = W_1(q^1, c) + W_2(q^2, c) + \ldots + W_n(q^n, c). \tag{2}$$

Two such systems (q^i) and (r^i) are said to be *equivalent* if they are related by transformations of the kind $q^1 = q^1(r^1), \ldots, q^n = q^n(r^n)$ (they have the same *coordinate surfaces*).

Our aim is to prove that:

Theorem 1. *There is bijective correspondence between equivalence classes of Stäckel systems and equivalence classes of orthogonal separable systems.*

We need a third definition.

Definition 3. *A Stäckel matrix is a* $n \times n$ *regular matrix of functions* $[\phi_i^{(j)}]$ *of* n *variables* (q^i) *such that each element* $\phi_i^{(j)}$ *is a function of the variable* q^i *alone (i.e. of the variable corresponding to the lower index). We denote by* $[\phi^i_{(a)}]$ *the inverse of a Stäckel matrix* $[\phi_i^{(a)}]$.

Stäckel matrices are recurrent objects in the theory of separation of variables. How to pass from orthogonal separable systems to Stäckel systems is shown by the following theorem.

Theorem 2. *(P. Stäckel). If* (q^i) *is an orthogonal separable system, then there exists a Stäckel matrix* $[\phi_i^{(a)}]$ *such that*

$$g^{ii} = \phi^i_{(n)}. \tag{3}$$

The symmetric tensors $\{\mathbf{K}_a; a = 1, \ldots, n\}$, whose components are defined by

$$K_a^{ii} = \phi^i_{(a)}, \quad K_a^{ij} = 0 \quad for \quad i \neq j, \tag{4}$$

from a Stäckel system (with $\mathbf{K}_n = \mathbf{g}$).

For the proof we can refer to the original papers by Stäckel [15], [16], or, for instance, to [8]. In fact, Stäckel proved that the tensors $\mathbf{K}_a$ defined by (4) are in involution. Since $\mathbf{K}_n$ is the metric, they are Killing tensors. Since these tensors are diagonalized in the coordinates (q^i) , they commute as linear operators, hence they form a Stäckel system. Once the Stäckel matrix $[\phi_i^{(a)}]$ is found (this is a purely "algebraic" problem) the sum of the functions

$$W_i(q^i, c) = \int \sqrt{2 c_a \phi_i^{(a)}} dq^i$$

gives a complete integral of the form (2), with $c_n = h$. Indeed, equation (1) becomes

$$\frac{1}{2} \phi^i_{(n)} p_i^2 = h = c_n.$$

We can look at this equation as one of the following system

$$\frac{1}{2} \phi^i_{(a)} p_i^2 = c_a,$$

and this system can be solved with respect the momenta,

$$p_i^2 = 2 c_a \phi_i^{(a)}.$$

Incidentally we notice that the constants of integration are just the values of the first integrals corresponding to the Killing tensors

$$c_a = \frac{1}{2} E_{\mathbf{K}_a}.$$

Hence, the fact that the Killing tensors are in involution is a consequence of the theorem of Jacobi, which states that the constants of a complete integral correspond to first integral in involution.

Eisenhart [6] [7], Woodhouse [17], Kalnins and Miller [9] gave geometrical characterizations of orthogonal separable systems in terms of Killing tensors. In Eisenhart's theorem [6] separable coordinates are generated by n common closed eigenforms ϕ^a of n independent Killing tensors (quadratic first integrals) $\mathbf{K}_a$ in involution (one of them, $\mathbf{K}_n$, is assumed to be the metric tensor). It is also assumed that the eigenvalues ρ_{ai} of $\mathbf{K}_a$ are simple and that $det(\rho_{ai} - \rho_{aj}) \neq 0$ for $i \neq j$ and $a = 1, \ldots, n-1$. In [17] t he Killing tensorts are assumed to have common closed eigenforms only. In [9] it is assumed that the Killing tensors are in involution, that they commute as linear operators and that at least one of them has all simple eigenvalues. In [1] [2] the Killing tensors are assumed to be in involution and to have common eigenvectors in involution.

The proof of Theorem 1 will follow from the discussion of some equivalent definitions of Stäckel system (Definition 1).

Proposition 1. *In the case of a positive definite metric conditions (i) - (ii) in Definition 1 are equivalent to (i) - (ii'), where*

(ii') The Killing tensors commute as linear operators: $\mathbf{K}_a \cdot \mathbf{K}_b, = \mathbf{K}_b \cdot \mathbf{K}_a$.

When this conditions holds the common eigenvector fields are uniquely determined (up to a factor).

Proof. The implication from (ii) to (ii') is obvijous. To prove the converse, let us consider a finite-dimensional real vector space E with a metric tensor g (i.e. with a scalar product $\mathbf{u} \cdot \mathbf{v}$). Let $\mathbf{K}$ be a linear symmetric operator on E. Symmetric means that $\mathbf{v} \cdot \mathbf{K} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{K} \cdot \mathbf{v}$ for each pair of vectors $(\mathbf{u}, \mathbf{v})$. It is known that eigenvectors of $\mathbf{K}$ corresponding to different eigenvalues are orthogonal: $\mathbf{K} \cdot \mathbf{u} = \rho\mathbf{u}, \mathbf{K} \cdot \mathbf{v} = \sigma\mathbf{v}$ and $\rho \neq \sigma$ imply $\mathbf{u} \cdot \mathbf{v} = 0$. It is known that each eigenvalues ρ of $\mathbf{K}$ generates a maximal linear subspace of eigenvectors S_ρ whose dimension is equal to the multiplicity of ρ as a root of the characteristic equation. If L is a symmetric linear operator commuting with $\mathbf{K}, \mathbf{K} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{K}$, then S_ρ is an invariant subspace of $\mathbf{L}$. Indeed we have: $\mathbf{v} \in S_\rho \Rightarrow \mathbf{K} \cdot \mathbf{v} = \rho\mathbf{v} \Rightarrow \mathbf{L} \cdot \mathbf{K} \cdot \mathbf{v} = \rho\mathbf{L} \cdot \mathbf{v} \Rightarrow \mathbf{K} \cdot \mathbf{L} \cdot \mathbf{v} = \rho\mathbf{L} \cdot \mathbf{v} \Rightarrow \mathbf{L} \cdot \mathbf{v} \in S_\rho$. If $dim(S_\rho) = 1$(ρ is a simple root), then $\mathbf{L} \cdot \mathbf{v} = \lambda\mathbf{v}$, and $\mathbf{v}$ is also an eigenvector of L. Let us consider a set $\{\mathbf{K}_a; a = 1, \ldots, n\}$ of n independent and commuting symmetric linear operators on E. With each simple eigenvalue of one of them it corresponds a unique common eigendirection. An eigenvalue of multiplicity m of one of the operators generates an eigenspace of dimension m of that operator which is an invariant subspace of the remaining operators. By considering in the order all the operators and picking out the eigenspaces of dimension l, we reduce the space E to a direct sum of mutually orthogonal subspaces

$$E = V_1 \oplus V_2 + \ldots \oplus V_k \oplus S_1 \oplus S_2 \oplus \ldots \oplus S_l, \tag{6}$$

where $V_1, V_2, \ldots, V_k$ are the common eigendirections defined by simple eigenvalues and $S_1, S_2, \ldots, S_l$ are subspaces of dimension ≥ 2 made of common eigenvectors: the restriction on $\mathbf{K}_a$ to each one of these subspaces is the multiplication by a real number. We can consider an orthogonal basis on each one of these subspaces. We get an orthogonal basis of E made of common eigenvectors, $(\mathbf{v}_i) = (\mathbf{v}_l, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$ where vectors $(\mathbf{v}_l, \ldots, \mathbf{v}_k)$ belongs to $(V_l, \ldots, V_k)$ respectively. Let us denote by ρ_{ai} the eigenvalue of $\mathbf{K}_a$ corresponding to the vector $\mathbf{v}_i$. For each index $a = 1, \ldots, n$ and for $i, j > k$ such that $\mathbf{v}_i$ and $\mathbf{v}_j$ belong to the same space of the kind S, we have $\rho_{ai} = \rho_{aj}$. It follows that the $n \times n$ matrix $[\rho_{ai}]$ is singular. On the other hand, if we set

$$K_{aij} = \mathbf{v}_i \cdot \mathbf{K}_a \cdot \mathbf{v}_j, \quad g_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j,$$

it follows from equations $\mathbf{K}_a \cdot \mathbf{v}_i = \rho_{ai}\mathbf{v}_i$ that

$$K_{aii} = g_{ii}\rho_{ai}, \quad K_{aij} = 0 \quad for \quad i \neq j. \tag{7}$$

Since the operators $\mathbf{K}_a$ are independent, the $n \times n$ matrix $[K_{aii}]$ is regular. The first equations (7) show that also the matrix $[\rho_{ai}]$ is regular: contradiction. Hence, the spaces of kind S cannot exists and the decomposition (6) reduces to

$$E = V_1 + V_2 + \ldots + V_n. \tag{8}$$

This shows that there is a unique decomposition of E in a direct sum of one-dimensional orthogonal subspaces made of common eigenvectors. ∎

Proposition 2. *The following condition can be substituted for (iii) in Definition 1:*

(iii') The Killing tensors are in involution: $[\mathbf{K}_a, \mathbf{K}_b] = 0$.

Proof. Assume that (i)-(ii)-(iii') hold. Let us consider a local frame $\{X_i; i = 1, \ldots, n\}$ made of common eigenvectors of the Stäckel system. Let ρ_{ai} be the eigenvalue of K_a corresponding to the eigenvector X_i. We have $\det[\phi_{ai}] \neq 0$. Because of Proposition 2 of Section 4, for each index $a = 1, \ldots, n$ the following equations hold (see equation $(4)_1$ in Section 4):

$$\Omega_{ijh}(\rho_{ai}\rho_{bj} - \rho_{aj}\rho_{bi}) + \Omega_{jhi}(\rho_{aj}\rho_{bh} - \rho_{ah}\rho_{bj}) + \Omega_{hij}(\rho_{ah}\rho_{bi} - \rho_{ai}\rho_{bh}) = 0 \ (i, j, h \neq),$$

which can be written (see [9]):

$$\left\| \begin{matrix} \Omega_{ijh} & \Omega_{jhi} & \Omega_{huj} \\ \rho_{ah} & \rho_{ai} & \rho_{aj} \\ \rho_{bh} & \rho_{bi} & \rho_{bj} \end{matrix} \right\| = 0.$$

These equations imply $det[\rho_{ai}] = 0$, unless

$$\Omega_{hij} = 0 \quad (h, i, j \neq).$$

But this means that each X_i is a normal vector field, i.e. that the orthogonal distribution of each X_i is completely integrable. Conversely, if (iii) holds, these distributions define families of orthogonal submanifolds of codimension 1. Hence, there exist orthogonal coordinates (q^i) which have these submanifolds as coordinate surfaces. The vector fields $\partial_i = \partial / \partial q^i$ are common eigenvectors of the Killing tensors. In other words, there exist set of common eigenvectors X_i which commute (i.e. which form a holonomic frame), and there exist local coordinates (q^i) such $X_i = \partial_i$. Two coordinate systems defined in this way are equivalent in the sense of Definition 2. Then (iii') is a consequence of Proposition 3, Section 4. ∎

Proposition 3. *Let* $\{\mathbf{K}_a; a = 1, \dots, n\}$ *be a Stäckel system on an open set* U. *Then:*

(i) There exists a set $\{\mathbf{X}_i; i = 1, \dots, n\}$ *of common eigenvector fields which forms a holonomic frame. The coordinates* (q^i) *generated by such a holonomic frame are orthogonal separable coordinates.*

(ii) The diagonal components of the Killing tensors $\mathbf{K}_a$ *in the coordinates* (q^i) *form the inverse of a Stäckel matrix:* $\phi^i_{(a)} = K^{ii}_a$.

(iii) There exists an equivalent Stäckel system which contains the metric tensor $\mathbf{g}$.

Proof of (i). The first part is already proved. Let us write the commutation relation $[\mathbf{K}_a, \mathbf{K}_b] = 0$ in the coordinates (q^i) by using formulae (3) of Section 4. We get:

$$K^{ii}_a \partial_i K^{jj}_b - K^{ii}_b \partial_i K^{jj}_a = 0, \tag{9}$$

no sum with respect to the index i. This equation is equivalent to equation

$$\{E_{\mathbf{K}_a}, E_{\mathbf{K}_b}\}_i = 0$$

which shows that the first integrals $(E_{\mathbf{K}_a})$ are in separable involution with respect to the coordinates (q^i) (see Definition 2, Section 2). Theorem 1 of Section 2 shows that (q^i) are separable coordinates. ■

Proof of (ii) - We prove the following general property concerning Stäckel matrices:

Proposition 1. *Let* $(\mathbf{K}_a)$ *be* n *independent contravariant symmetric tensors of order 2. Assume that they are diagonalized in a coordinate system* (q^i) *, i.e. that* $K^{ij}_a = 0$ *for* $i \neq j$. *Then they commute if and only if the matrix of elements*

$$\phi^i_{(a)} = K^{ii}_a$$

is the inverse of a Stäckel matrix $[\phi^{(a)}_i]$.

Proof. The commutation relations are equivalent to equations (9), which can be written

$$\phi^i_{(a)} \partial_i \phi^j_{(b)} - \phi^i_{(b)} \partial_i \phi^j_{(a)} = 0, \tag{9'}$$

no sum w.r. to the index i. The independence of the tensors is equivalent to $det[\phi^i_{(a)}] \neq 0$. Let us take the inverse matrix $[\phi^{(a)}_i]$. Let us multiply equation (9') by $\phi^{(c)}_j \phi^{(a)}_h \phi^{(b)}_k$ and take the sum over the indices (j, a, b). We get, after a straightforward process:

$$\delta^i_h \partial_i \phi^{(c)}_k = \delta^i_k \partial_i \phi^{(c)}_h. \tag{10}$$

If $h = i$, then

$$\partial_i \phi^{(c)}_k = \delta^i_k \partial_i \phi^{(c)}_i,$$

and we see that $\partial_i \phi^{(c)}_k = 0$ for $k \neq i$. Hence, $[\phi^{(a)}_i]$ is a Stäckel matrix. Conversely, if $[\phi^{(a)}_i]$ is a Stäckel matrix, then (10) hold and we pass from (10) to (9) by reversing the process. ■

Proof of (iii) - We use a symplectic property of cotangent bundles: on T^*Q the maximum number of independent functions in involution is n. The functions $E_{\mathbf{K}_a}$ are independent and in involution. They are in involution also with H (they are first integrals). Hence, there is a C^∞ function $F : \mathbf{R}^n \to \mathbf{R}$ such that $H = F(E_{\mathbf{K}_a})$. We differentiate this function twice with respect to the momenta and we get the metric tensor as a linear combination with constant coefficients of the Killing tensors $\mathbf{K}_a$. ∎

Remark. Let us consider a linear connection on the manifold Q. Let Γ_{ij}^h be the coefficients of the connection in a local coordinate system (q^i). We can write two differential systems:

$$\partial_i \phi_j - \Gamma_{ij}^h \phi_h = 0, \tag{11}$$

and

$$\partial_i v^j + \Gamma_{ih}^j v^h = 0. \tag{12}$$

If ϕ_i and v^i are two solutions then

$$v^i \phi_i = const. \tag{13}$$

Both systems (11) and (12) are completely integrable if and only if the Riemann tensor is zero. Then the general solution of system (11) has the form

$$\phi_i = c_k \phi_i^{(k)}, \quad c_k \in R, \tag{14}$$

where $\{\phi_i^{(k)}; k = 1, \dots, n\}$ are independent solutions:

$$\det[\phi_i^{(k)}] \neq 0.$$

The inverse matrix $[\phi_{(k)}^i]$ gives the general solution of system (12):

$$v^i = c^k \phi_{(k)}^i. \tag{15}$$

If in addition the connection is *symmetric*, then two solutions of system (12), interpreted as vector fields, commute; that is:

$$\phi_{(h)}^i \partial_i \phi_{(k)}^j = \phi_{(k)}^i \partial_i \phi_{(h)}^j. \tag{16}$$

We say that the connection is *separable* in the coordinates (q^i) if

$$\Gamma^h_{ij} = 0 \quad for \quad i \neq j. \tag{17}$$

Then the two differential systems become

$$\partial_i \phi_j = 0 \quad (i \neq j), \quad \partial_i \phi_i = B^h_i \phi_h, \tag{11'}$$

and

$$\partial_i v^j = -B^j_i v^i \tag{12'}$$

(no sum with respect to the index i), where $B^j_i = \Gamma^j_{ii}$. Integrability conditions (of both systems) are:

$$\partial_i B^h_j = B^i_j B^h_i \quad (i \neq j) \tag{18}$$

(no sum w.r. to the index i). From (12') we get the coefficients of the connection (we exclude $v^i = 0$):

$$B^j_i = -\frac{1}{v^i}\, \partial_i v^j. \tag{19}$$

As a consequence, the integrability conditions (18) become:

$$v^i v^j \partial_i \partial_j v^h - v^i \partial_i v^j \partial_j v^h - v^j \partial_j v^i \partial_i v^h = 0, \quad i \neq j. \tag{20}$$

These equations are similar to *Lamé equations* [4]. On the other hand, the first set of equations (11'), shows that any set of independent solutions form a Stäckel matrix $[\phi^{(k)}_i]$. Hence, the most general function satisfying equations (20) is of the kind (15) where $[\phi^i_{(k)}]$ is the inverse of a Stäckel matrix, or equivalently of the kind

$$v^i = \phi^i_{(n)} \tag{21}$$

where $\phi^i_{(n)}$ is a line of the inverse of a Stäckel matrix.

Furthermore, let v^i and u^i be two solutions of system (12'). Let us set

$$\rho_i = \frac{u^i}{v^i}. \tag{22}$$

From equations (12') and (18) it follows that

$$\partial_i \rho_j = (\rho_i - \rho_j)\partial_i \log |v^j|. \tag{23}$$

Equations (20) are still the integrability conditions of this differential system.

This is a general framework in which we can find some of the preceding results on Stäckel systems. Indeed if we set

$$v^i = g^{ii}, \quad \phi_i = (\partial_i W)^2$$

then in equation (13) we recognize the Hamilton-Jacobi equation. In the first set of equations (11') we recognize the assumption of separability. The existence of a complete integral fo the form (2) is then equivalent to the complete integrability of system (11') with the functions B_i^j given by

$$B_i^j = -\frac{1}{g^{ii}}\,\partial_i g^{jj}$$

The integrability conditions are the equations

$$g^{ii}g^{jj}\partial_i\partial_j g^{hh} - g^{ii}\partial_i g^{jj}\partial_j g^{hh} - g^{jj}\partial_j g^{ii}\partial_i g^{hh} = 0\,, \quad i \neq j, \tag{20'}$$

which are recurrent equations in the theory of separation of variables (see [6], [7]). In fact these equations directly follow from the separability conditions of Levi-Civita, (18), Section 2, when H is the geodesic Hamiltonian in orthogonal coordinates; hence they represent necessary and sufficient conditions for the separability of the orthogonal coordinates (q^i). As we have seen, the most general solution of equations (20') is of the form (21), i.e. $g^{ii} = \phi^i_{(n)}$, where $\phi^i_{(n)}$ is a line of the inverse of a Stäckel matrix. This proves the first part of Theorem 2. Furthermore, equations (16) show that the tensors $\mathbf{K}_a$ defined by equations

$$K_a^{ii} = \phi^i_{(a)}, \quad K_a^{ij} = 0 \quad (i \neq j),$$

are in involution, and the second part of Theorem 2 is also proved.

Equations (20') are also the integrability equations of the differential system

$$\partial_i \rho^j = (\rho_i - \rho_j)\partial_i \log |g^{jj}|.$$

These are the characteristic equations of the eigenvalues of a Killing tensors with respect to a holonomic frame of eigenvectors (see Remark 2, Section 4). This fact provides another proof of the point (iv) of Theorem 3; indeed, at the point (ii) we have proved that a Stäckel system admits a holonomic frame of eigenvectors; this means that in the corresponding coordinates (q^i) system (23') must be completely integrable; but the integrability conditions are just equations (20'), which characterize the separability of coordinates (q^i); thus (q^i) are separable. In this way we can avoid using Theorem 1 of Section 1.

REFERENCES

[1] S. Benenti, *Separable dynamical systems: characterization of separability structures on Riemannian manifolds*, Rep. Math. Phys., **12** (1977), 311-316.

[2] S. Benenti, *Separability structures on Riemannian manifolds*, L. N. Math., **863** (1980), 512-538.

[3] S. Benenti, M. Francaviglia, *The theory of separability of the Hamilton-Jacobi equation and its applications to General Relativity*, in "General Relativity and Gravitation", A. Held Ed., Plenum (1980).

[4] L. Bianchi, *Lezioni di geometria differenziale*, Spoerri (Pisa, 1902-1903).

[5] B. Carter, *Hamilton-Jacobi and Schrödinger separable solutions of Einstein equations*, Comm. Math. Phys. **10** (1968), 280-310.

[6] L.P. Eisenhart, *Separable systems of Stäckel*, Ann. Math., **35** (1934), 284-305.

[7] L.P. Eisenhart, *Riemannian Geometry*, Princeton Univ. Press (1950).

[8] E.G. Kalnins, *Separation of Variables for Riemannian Spaces of Constant Curvature*, Pitman Monographs **28** (1986).

[9] E.G. Kalnins, W. Miller Jr., *Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations*, SIAM J. Math. Anal., **11** (1980), 1011-1020.

[10] E.G. Kalnins, W. Miller Jr., *Killing tensors and nonorthogonal variable separation for Hamilton-Jacobi equations*, SIAM J. Math. Anal., **12** (1981), 617-629.

[11] E.G. Kalnins, W. Miller Jr., *Intrinsic characterization of variable separation for the partial differential equations of mechanics*, Proceeding IUTAM-ISIMM Symposium on Modem Developments in Analytical Mechanics, Torino, June 7-11, 1982, Acta Academiae Scientiarum Taurinensis **117** (1983), 511-533.

[12] H. Katzin, J. Levine, *Quadratic first integrals of the geodesics in spaces of constant curvature*, Tensor, **10** (1965), 97-104.

[13] T. Levi-Civita, *Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili*, Math. Ann., **59** (1904), 383-397.

[14] J.A. Schouten, *Ricci calculus*, Springer (1954).

[15] P. Stackel, *Über die Integration der Hamilton-Jacobischer Differentialgleichung mittelst Separation der Variablen*, Habilitationschrift, Halle (1891).

[16] P. Stackel, *Sur une classe de problème de dynamique*, C.R. Acad. Sci. Paris **116**, 485-487 (1893).

[17] N.M.J. Woodhouse, *Killing tensors and the separation of the Hamilton-Jacobi equation*, Commun. Math. Phys., **44** (1975), 39-44.

[18] K. Yano, S. Bochner, *Curvature and Betti numbers*, Princeton (1953).

S. Benenti
Istituto di Fisica Matematica
"J. Louis Lagrange"
Via Carlo Alberto 10
10123 Torino