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ISOPARAMETRIC SUBMANIFOLDS *

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In this talk we consider two classes of submanifolds in Euclidean spaces that are characterized by simple local invariants. The first class consists of *submanifolds with constant principal curvatures.* These are by definition submanifolds M^n such that for every parallel normal vector field $\xi(t)$ along a curve in M^n the eigenvalues of the shape operator $A_{\xi(t)}$ are constant, see [St] and [Ol 2]. The second class of submanifolds we will consider consists of *isoparametric submanifolds.* These are by definition submanifolds M^n such that the normal bundle is flat and the eigenvalues of the shape operator A_ξ are constant for every locally defined parallel normal vector field ξ, see [Ha], [CW], [Te] and [PT 2]. It is of course clear that isoparametric submanifolds have constant principal curvatures.

The following theorem was proved in [Th].

Theorem A. *Let M^n be an isoparametric submanifold of $\mathbf{R}^{n+r}$ that is compact, irreducible and full with codimension $r \geq 3$. Then there is a symmetric space $X = G/K$ of rank r and an isometry $A : \mathbf{R}^{n+r} \to T_{(K)}X$ that carries M^n onto a principal orbit of the isotropy representation of X.*

Conversely, it is shown in [PT 1] that every principal orbit of an isotropy representation of a symmetric space is isoparametric. It is also easy to show that all orbits have constant principal curvatures.

The compactness of M^n in theorem A is not a real restriction since noncompact complete isoparametric submanifolds are products of compact ones with Euclidean spaces, see [Te]. We recall that an isoparametric submanifold is said to be *irreducible* if it is not a product embedding of two isoparametric submanifolds. A submanifold of a Euclidean space is said to be *full* if it is not contained in a proper affine subspace.

Notice that a compact isoparametric submanifold with codimension one is a sphere. A large class of inhomogeneous examples of isoparametric submanifolds with codimension two was constructed by Ferus, Karcher and Münzner in [FKM] based on examples by Ozeki and Takeuchi [OT]. A compact isoparametric submanifold lies in a sphere. Hence isoparametric submanifolds with codimension two coincide with isoparametric hypersurfaces in spheres which have been extensively studied and are apparently still far from being classified.

It was proved by Terng [Te] that the normal bundle of an isoparametric submanifold M^n is globally flat. This means by definition that M^n has r linearly independent parallel globally

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defined normal vector fields. Terng proves furthermore that if ξ is a globally defined normal vector field over M^n, then the set $\{p + \xi(p) | p \in M^n\}$ is a submanifold, called a *parallel submanifold*, that is isoparametric if its dimension is n. If its dimension is less than n, then it is called a *focal manifold of* M^n. One sees easily that the set of focal points of M^n is a union over the focal manifolds of M^n. Focal manifolds of isoparametric submanifolds have constant principal curvatures. The following theorem of Olmos [Ol 2] shows that the converse is true.

Theorem B (Olmos). *Let M^n be a compact submanifold with constant principal curvatures. Then M^n is either isoparametric or a focal manifold of an isoparametric submanifold.*

E. Heintze and the author proved independently [HT] that a submanifold with constant principal curvatures and polar normal holonomy satisfies the conclusion in theorem B. Olmos proves [Ol 1] that the normal holonomy is always polar. The rest of his proof is similar to ours. One proves that if M^n has constant principal curvatures (and polar normal holonomy), then the image of a principal orbit of the normal holonomy group under the normal exponential map is isoparametric.

The following is a corollary of theorem A and B.

Corollary. *Let M^n be an irreducible compact submanifold with constant principal curvatures and assume that the normal holonomy of M^n is not trivial and does not have factors that act transitively on spheres. Then there is a symmetric space $X = G/K$ and an isometry $A : \mathbb{R}^{n+r} \to T_{(K)}X$ that carries M^n onto an orbit of the isotropy representation of X.*

Notice that M^n in the corollary is the focal manifold of an isoparametric submanifold with codimension two or is an isoparametric submanifold with codimension two if the normal holonomy is trivial or is transitive on spheres. Hence we do not have a complete classification of constant curvature submanifolds since isoparametric submanifolds of codimension two have not yet been classified.

In the rest of this talk we will give a sketch of the proof of theorem A. We divide the proof into three steps and first explain what is done in each of these steps.

In the first step we associate a compact topological Tits geometry $\Delta(M^n)$ of dimension r, also called a building of rank r, to an isoparametric submanifold M^n as in theorem A. Such geometries can also be associated to all known examples with codimension two. In fact, the Tits geometry can be seen as an explanation for the inhomogeneous examples with codimension $r = 2$ as will be explained below. The geometry $\Delta(M^n)$ is an incidence geometry which generalizes projective geometry. Its objects are points and lines if $r = 2$ and also 2-planes and so on if $r \geq 3$. In our situation, objects of a given type are compact manifolds that can be identified with certain of the focal submanifolds. One can associate to it the full flag space $Flag(\Delta(M^n))$ whose elements are r-tuples $(p, l, \pi, \ldots)$ where p is a point incident to the line l, l is incident to the 2-plane π etc. (Of course there are no 2-planes if $r = 2$). The space $Flag(\Delta(M^n))$ inherits a natural topology and it follows from the construction of the building that it is homeomorphic to M^n.

In the second step we assume that $r \geq 3$. It was shown by Tits [Ti 1] that a Tits geometry $\Delta(M^n)$ of dimension at least three is Moufang and in [BS] this was extended to the topological case by Burns and Spatzier. The Moufang condition is a strong homogeneity assumption. In particular, the topological automorphisms of the geometry are transitive on $Flag(\Delta(M^n))$. It was proved by Burns and Spatzier [BS] that the connected component of the topological automorphism group G of such a topological Tits geometry is a noncompact simple Lie group. It follows that $Flag(\Delta(M^n))$ is homeomorphic to the coset space G/P where P is the isotropy group of a flag in $\Delta(M^n)$. One proves that P is a minimal parabolic subgroup of G. It is then not difficult to prove that G/P is homeomorphic to a principal orbit of the symmetric space $X = G/K$ where K is a maximal compact subgroup of G. This step can certainly not be carried through when $r = 2$. There are for example many examples of topological projective planes that are not homogeneous under their projective transformations. The inhomogeneours isoparametric submanifolds of [FKM] do not correspond to projective planes, but to so-called polar planes.

In the third step one notices that G acts on $\mathbb{R}^{n+r}$ with M^n and its parallel submanifolds as orbits. The action can of course not be isometric since G is a noncompact group with a fixed point. The geometry $\Delta(M^n)$ has a certain involutive automorphism σ. Now let K_σ be the maximal compact subgroup of G that commutes with σ. It is then proved that K_σ acts isometrically on $\mathbb{R}^{n+r}$ with M^n as an orbit and that the action of K_σ is up to an isometry of ambient spaces an isotropy action of the symmetric space $X = G/K_\sigma$. In an essential step in this argument we use the normal holonomy of M^n.

We now explain the first two steps in special cases. Let us first explain Tits geometries in dimension two. A *generalized n-gon* or a *Tits building of rank two*, see [Ti 2], is a graph Δ with the following properties:

(i) The diameter of Δ is equal to n, i.e., any two vertices in the graph can be joined by a path consisting of less than or equal to n edges and for some two vertices one needs n edges to join them.

(ii) There are no cycles in Δ of length less than $2n$.

(iii) Any vertex is contained in at least three edges.

An example for $n = 3$, i.e. of a generalized triangle, can be obtained as follows. Let P be an incidence geometry that satisfies the axioms for projective planes. Let $\Delta(P)$ be the graph whose vertices are the points and lines of P and whose edges are flags (p, l), i.e., $p \in l$. Then p and l are vertices of the edge (p, l). In other words, we do not join two points by an edge nor do we join two lines by an edge, but we join a point and a line if and only if the point lies on the line. We can call $\Delta(P)$ the *flag complex of P*. To see that condition (i) is satisfied, let p be a point and l a line and assume that p does not lie on l. Let l_1 be a line that passes through p. Then l_1 intersects l in a point p_1. A path between p and l then goes over the edges $(p, l_1), (p_1, l_1)$ and (p_1, l). Hence the distance between p and l is three. It is as obvious to see that the distance between two points or two lines is two, and that the distance between a point and a line containing the point is one. Condition (ii) is also easy to prove and (iii) is equivalent to the property of projective planes that a point lies on at least three lines and a line contains at least three points.

What is remarkable is that the converse can be proved: Every generalized triangle is isomorphic to the flag complex of a projective plane.

Examples for $n = 4$, i.e. of generalized quadrangles, are flag complexes of so-called polar planes, see [Ti 1]. Again it turns out that any generalized quadrangle is isomorphic to the flag complex of a polar plane.

Now let us explain how a generalized g-gon can be associated to the known examples of compact isoparametric submanifolds M^n in $\mathbb{R}^{n+2}$. As we already mentioned, M^n lies in a round sphere that we can take to be the unit sphere S^{n+1}. It turns out that exactly two of the focal manifolds of M^n lie in S^{n+1} and that each connected component of $S^{n+1} - M^n$ contains one of them. We denote these focal manifolds by F_1 and F_2. Now we define $\Delta(M^n)$ as follows: the set of vertices is $F_1 \cup F_2$. The edges are the great circle arcs which start in F_1, meet M^n orthogonally, have their endpoints, on F_2 and do not have any point except their endpoints in $F_1 \cup F_2$. It follows that $\Delta(M^n)$ is a generalized g-gon where g is the number of different principal curvatures of M^n as a hypersurface of the sphere. Notice that it is a result of Münzner [Mü] that g has to be one of the numbers 1, 2, 3, 4 and 6. In the proof that $\Delta(M^n)$ is a generalized g-gon one uses topological arguments to verify condition (i) and special properties of the examples for (ii) that has not yet been proved in full generality. Condition (iii) follows rather trivially. If $g = 3$, then M^n has associated to it a projective plane by the above remarks. More precisely, it follows that the focal manifolds F_1 and F_2 carry the structure of projective planes. This is not a surprise since it is an old result of Cartan [Ca] that the focal manifolds F_1 and F_2 in this case are the standard embeddings of $P_2\mathbb{R}, P_2\mathbb{C}, P_2\mathbb{H}$ or $P_2\mathbb{O}$ and that actually all cases occur. It turns out that a generalized triangle coming from an isoparametric submanifold satisfies the Moufang condition and that M^n is a principal orbit of a symmetric space. On the other hand, the Clifford examples [FKM] give rise to generalized quadrangles that are not Moufang.

We would now like to explain how an isoparametric submanifold M^n in $\mathbb{R}^{n+r}, r \geq 3$, gives rise to a Tits geometry. For this we need the following result of Terng [Te], see also [CW]: The set $\mathcal{F}_p$ of focal points of M^n that are contained in the normal space $p + \nu(M)_p \subset \mathbb{R}^{n+\nu}$ at a point $p \in M$ is a union over finitely many hyperplanes in $p + \nu(M)_p$ and the reflections in these normal planes generate a finite Coxeter group W_p that leaves $\mathcal{F}_p$ invariant and is irreducible on $p + \nu(M)_p$. Actually, $\mathcal{F}_p$ is the union over the mirrors of all reflections in W_p. The hyperplanes in $\mathcal{F}_p$ all pass through a point (different from p) that we consider as the origin of $p + v(M^n)_p$.

The Coxeter groups at two different points of M^n are isomorphic. Coxeter groups have been classified, see e.g. [GB]. Let us assume that $r = 3$ to simplify the discussion. The Coxeter group W_p acts irreducibly on a three-dimensional Euclidean space. There are up to isomorphisms exactly three such groups: the symmetry group of the tetrahedron that is denoted by A_3, the symmetry group of the cube or the octahedron that is denoted by C_3 and the symmetry group of the dodecahedron or the icosahedron that is denoted by H_3. Now it follows from results of Münzner [Mü] and Hsiang, Palais and Terng [HPT] that the group H_3 cannot be the Coxeter group of an isoparametric submanifold. So we are left with the case A_3 that will lead to projective geometry and C_3 that leads to polar geometry.

Let C_p be the connected component of the complement of $\mathcal{F}_p$ in $p + v(M)_p$ that contains p. It follows that C_p is a cone over a triangle. In particular, the boundary of C_p lies in three 2-planes that intersect in three lines. We denote the three rays by R_1, R_2 and R_3 that lie in these lines and are contained in the boundary of C_p.

Now let us assume that the Coxeter group W_p is A_3. Then it follows that the angles between the 2-planes containing the boundary of C_p are $\pi/3$ along two of the lines of intersection and $\pi/2$ along the third. We assume that the angle is $\pi/3$ along R_1 and R_3 and $\pi/2$ along R_2. Now let $f_1(p), f_2(p)$ and $f_3(p)$ denote the unit vectors in R_1, R_2 and R_3 respectively. Let F_1, F_2 and F_3 be the set of $f_1(p), f_2(p)$ and $f_3(p)$ for all $p \in M$ respectively. One can show that F_1, F_2 and F_3 are focal manifolds of M^n. We call the elements of F_1 points, the elements of F_2 lines and the elements of F_3 2-planes. We say that elements x of F_i and y of F_j, $i \neq j$, are incident if there is a q in M^n such that $x = f_i(q)$ and $y = f_j(q)$. It follows that this incidence geometry satisfies the axioms for a three-dimensional projective space. Furthermore the spaces F_1, F_2 and F_3 have a topology induced from $\mathbb{R}^{n+v}$ with respect to which they are compact and connected and it is clear that the incidence axioms are continuous with respect to this topology. It is now a theorem of Kolmogorov [Ko] that a compact and connected topological projective space of dimension $r \geq 3$ is homeomorphic and projectively equivalent to $P_r\mathbb{F}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. It follows that F_1 is homeomorphic to a $P_3\mathbb{F}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and the same is true for F_3 by duality in projective geometry. From the construction of the projective geometry associated to M^n, it follows that M^n is homeomorphic to the full flag manifold of $P_3\mathbb{F}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We have thus finished step two in the sketch of the proof of theorem A for isoparametric submanifolds with A_3 as Coxeter group. The theorem of Kolmogorov was generalized to topological Tits geometries by Burns and Spatzier in [BS] and the proof of our theorem in the general case relies on their work.

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