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KNOTS AND PHYSICS

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1. INTRODUCTION

This paper traces the construction of the bracket model of the Jones polynomial, and how this model can be naturally interpreted as a vacuum-vacuum expectation in a combinatorial version of physical theory. From this point of view certain structures such as solutions to the Yang-Baxter equation, and the quantum group for $SL(2)$ emerge naturally from topological considerations. We then see how quantum groups give rise to invariants of links via solutions to the Yang-Baxter equation. Section 5, is an original treatment of the construction of the universal R-matrix. All the other material has, or will appear elsewhere in similar form.

I am pleased to thank the origanizers of the Topology/Geometry conference (Giornate di studio di Geometria Differenziale e Topologia) held in Lecce, Italy, on the days of June 21-23, 1989 for their kind hospitality. This paper contalns the contents of the lecture that I gave there.

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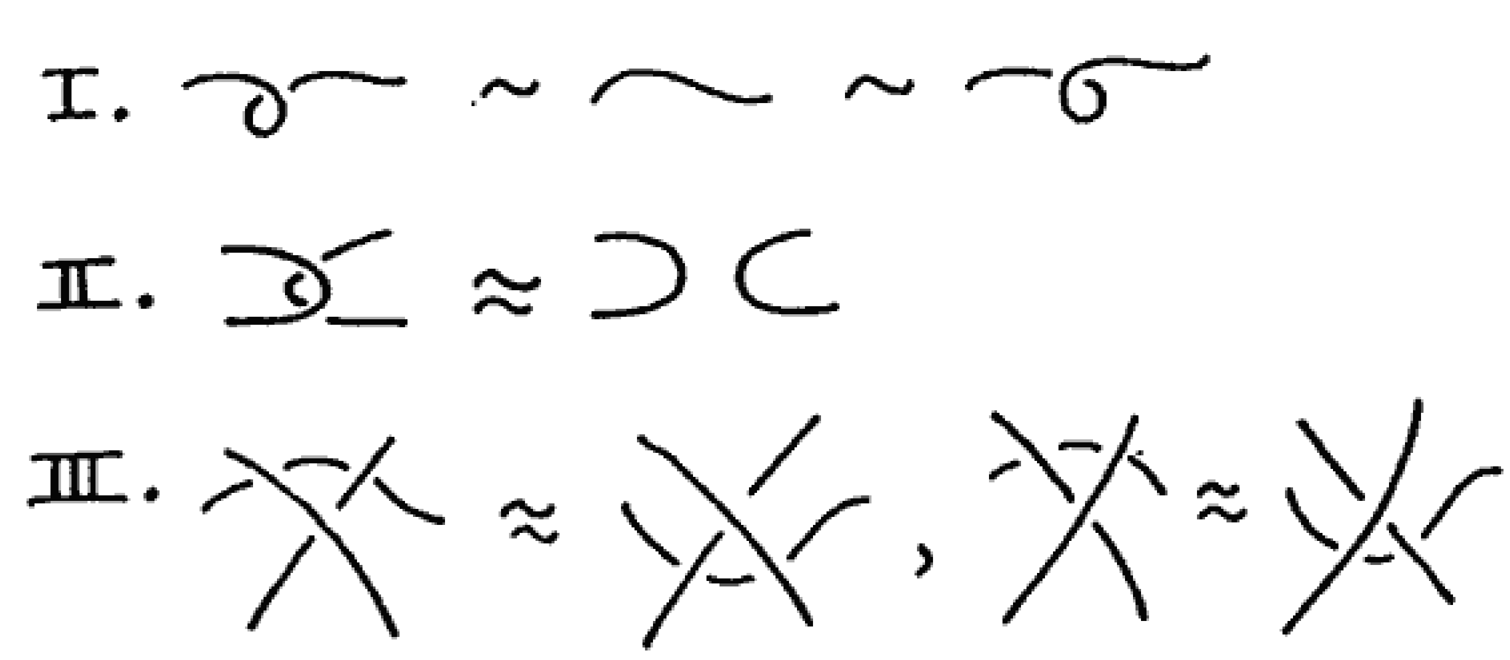
2. KNOT THEORY

Let's begin by recalling the Reidemeister moves:



These moves can be performed on a *link diagram.* A link diagram is a locally four-valent plane graph with extra structure at the vertices in the form of *crossings*

These crossings are taken to indicate the projection of arcs embedded in a three-space, and projected to the plane. The broken arc pair at a crossing indicates the arc that passes underneath the other arc in space. Any *link* (A link is a collection of circles imbedded in a three-sphere or Euclidean three space.) has a point of projection to the surface of a two dimensional sphere or to a plane, so that the projection (with under and over-crossing indications) becomes a diagram for that link.

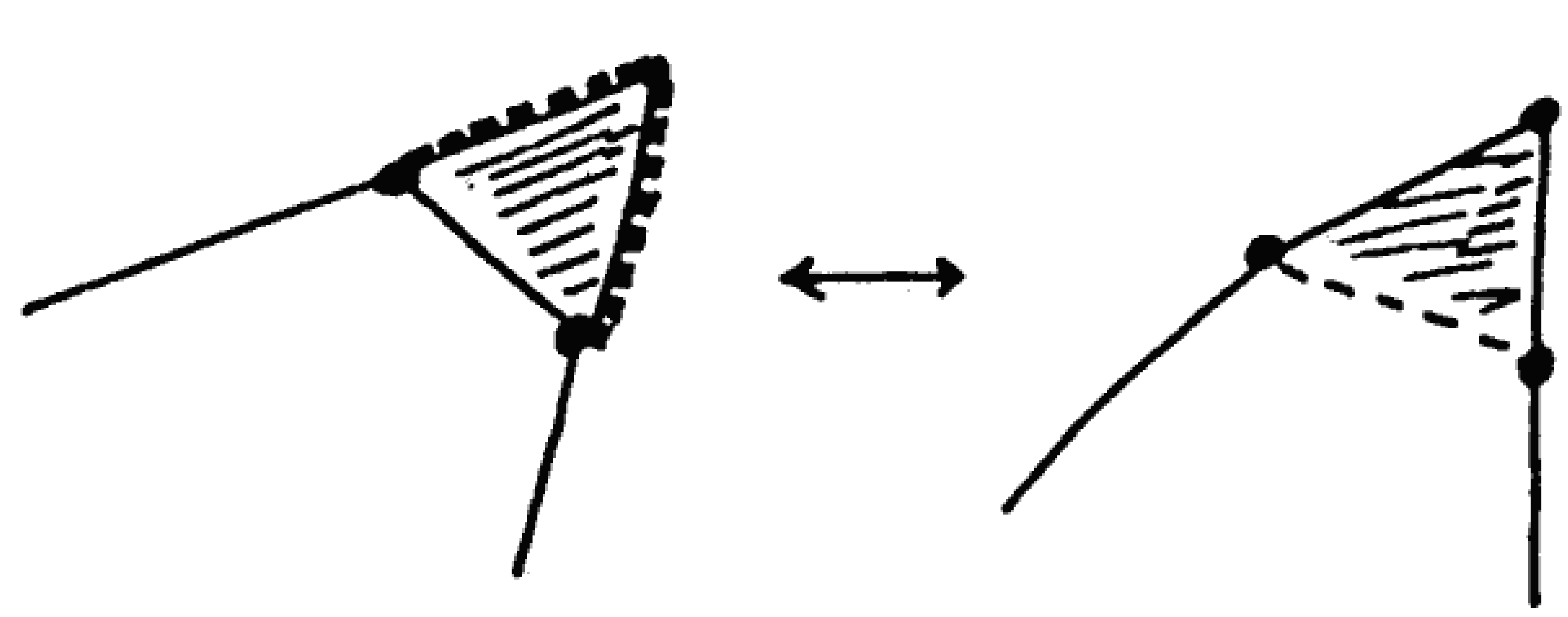
Two links are said to be *ambient isotopic* if there is a continuous time-parameter family of embeddings starting with one link and ending with the other one. The theory of knots and links is the theory of link embeddings under the equivalence relation of ambient isotopy. (A *knot* is a link with one component. That is, a knot is an embedding of a single circle into three-space).

It is assumed that all the embeddings are represented (up to ambient isotopy) by an embedding that is a differentiable curve(s) in the three-space. Links that do not admit such a representation are called *wild*, and must be treated separately.

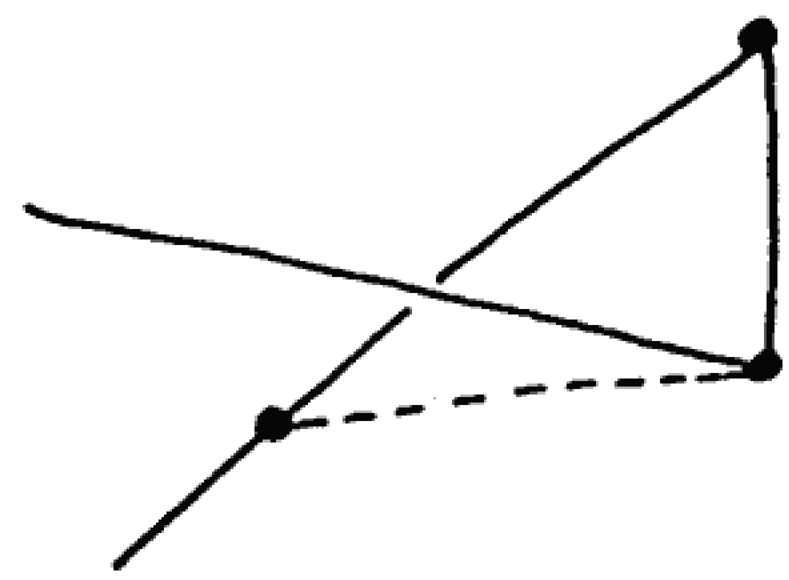
The Reidemeister moves generate the theory of knots and links in three-dimensional space in the sense of the following theorem:

Theorem. *(Reidemeister [18]). Let K and K be two links embedded in three-dimensional space (either the three-dimensional sphere, or the Euclidean space R^3). Then K and K' are ambient isotopic if and only if diagrams for K and K' are related by a finite sequence of the moves I, II, III.*

Remark. In Reidemeister's day the notion of ambient isotopy was also combinatorial. Let R denote Reidemeister. For R, ambient isotopy was generated by a *single* move type called an *elementary combinatorial isotopy* (or elementary isotopy for short). The knots and links for R are *piecewise linear* - meaning that they consist of interconnections of straight line segments embedded in Euclidean space. Vertices are regarded as the endpoints of these segments, and any straight segment can be regarded as the connection of two segments, by adding a vertex at an interior point. The elementary combinatorial isotopy has two directions: *expansion*, and *contraction*. In an expansion, one takes two vertices on the link, and a new vertex in the complement such that the (two dimensional) triangle spanned by these vertices intersects the link only at one of its three edges. Expansion consists in replacing this edge in the link by the two remaining edges in the triangle. Contraction is the opposite of expansion - three points on the link span a triangle intersecting the link only along two edges; these edges are replaced by the third edge of the triangle.

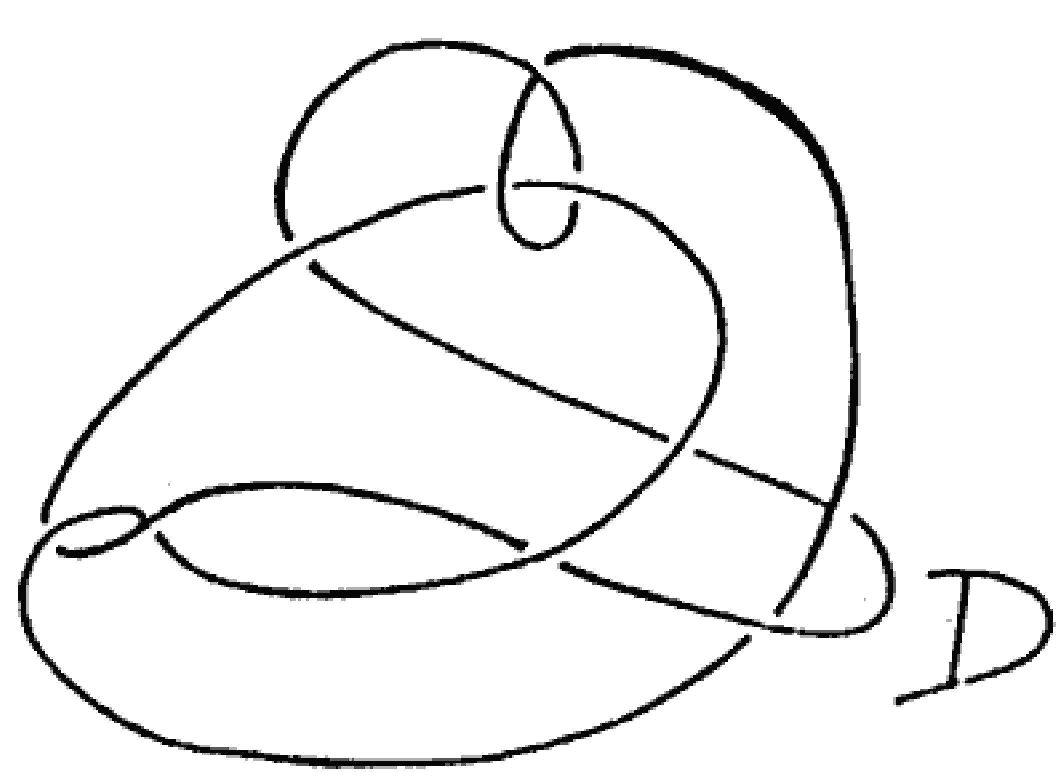


The Reidemeister moves come about via examination of the forms of planar projections of the elementary isotopies. For example, the diagram below shows how a type I Reidemeister move is the shadow of an elementary isotopy.



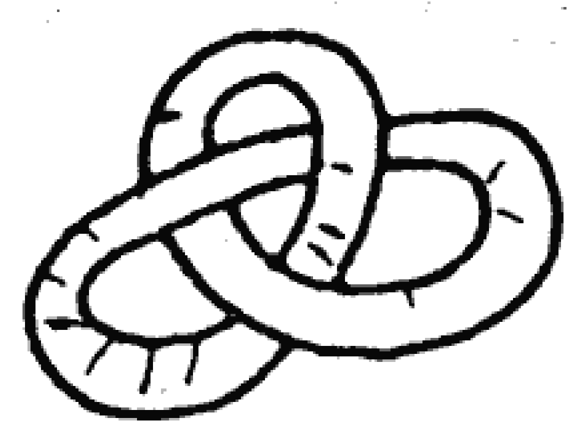
Reidemeister's approach to his theorem is a good way to get a geometric feel for the situation. For a modern, treatment of the Theorem, using the continuous (or differentiable) notion of isotopy, see [5].

While Reidemeister's Theorem is an excellent starting point for a combinatorial theory of knots and links, it does not make life easy. The easiest way to illustrate this is to exhibit a demon (This demon - shown to the author by Ken Millett - improves over previous culprits, and is the smallest possible for projections on a sphere.) such as the one shown below:



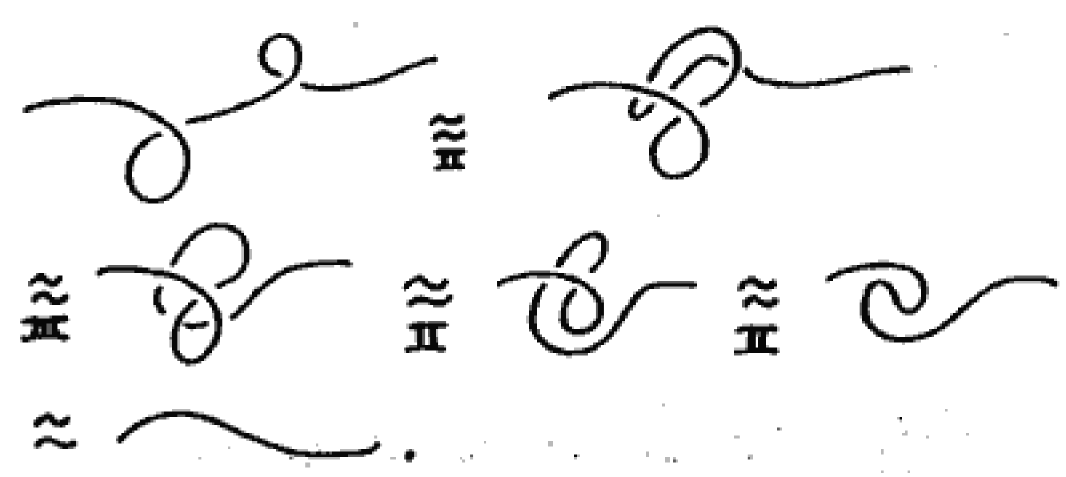
This demon D is unknotted, but does not admit any simplifying Reidemeister moves, nor does it admit any type three moves. (A Reidemeister move is said to be *simplifying* if it reduces the number of crossing in the diagram). In order to unknot D it is necessry to first make the diagram more complex before it can become simpler. Examples of this sort show that the equivalence relation generated by the Reidemeister moves is subtle, and that the matter of constructing invariants is non-trivial.

There are many accounts of the classical construction of knot and link invariants([1], [8], [4], [11], [21]). In the next section I shall go directly to a model for the Jones polynomial and discuss its physical interpretations. For these purposes it does make sense to make one remark about the process of abstraction leading to mathematical knots. If we were to make a knot or link from rope or other material, then the amount of twisting on the rope would make a difference in the behaviour of the resulting knotted form. Such twisting has been abstracted when we go to the diagram or to the mathematical curve embedded in space. We can recover some of this structure by considering *framed links*. A framed link is a link such that each component has a continuous normal vector field. This is equivalent to thinking about embeddings of *bands* rather than circles. Thus the figure below indicates a framed trefoil, with standard framing inherited from its planar embedding.

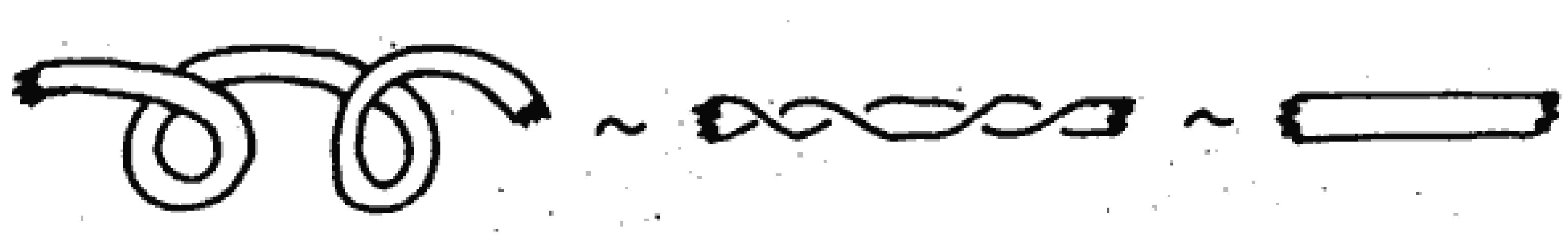


If we keep track of the framing then one no longer has invariance under the type I move:

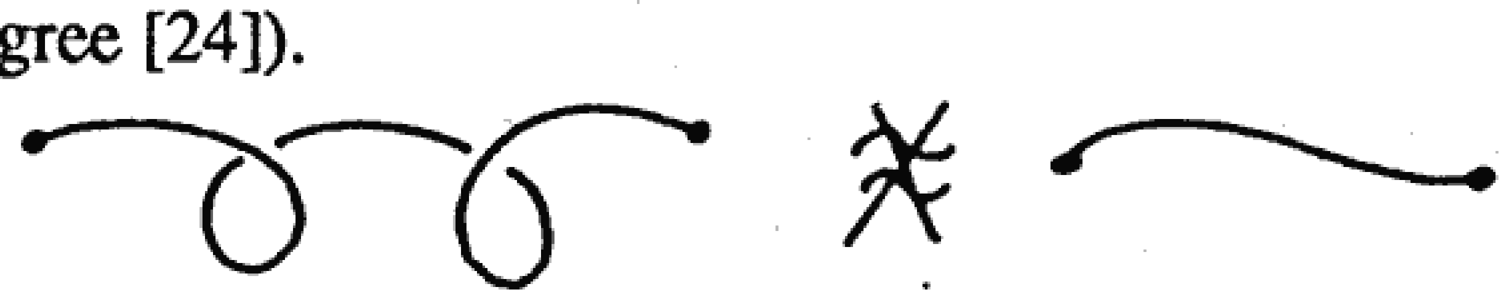
For this reason it is useful to have the concept of *regular isotopy.* Two links are said to be regularly isotopic if one can be obtained from the other by a sequence of type II and type III moves only. Regular isotopy is the equivalence relation generated by the type II and type III moves. Note:



Opposite curls cancel. This regular isotopy is the knot theoretic version of the Whitney trick [24]. Actually, regular isotopy is a bit subtler than simple framing. The bands shown below

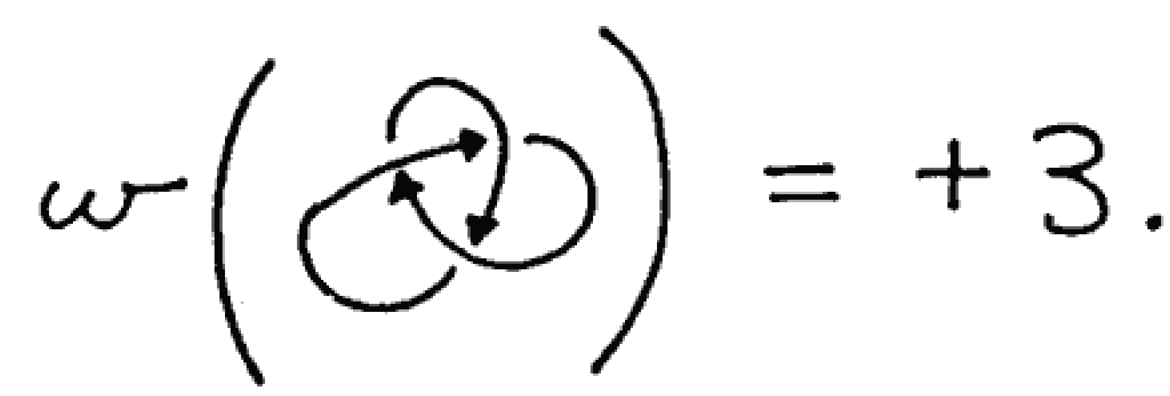


are isotopic, but the corresponding string diagrams are not regularly isotopic (They have different Whitney degree [24]).



A useful invariant of regular isotopy is the *writhe,* $w(K)$. The writhe is the sum of the crossing signs

in a given diagram. Thus



It is easy to see that the writhe is a regular isotopy invariant for diagrams. It is very useful for normalizing other invariants of regular isotopy. It turns out that most of the invariants we shall discuss need such a normalization.

3. LINK INVARIANTS AS VACUUM-VACUUM AMPLITUDES

First, a quick description of the bracket model [12] of the Jones polynomial [9]: We give a method of associating a well-defined polynomial in three variables, $\langle K\rangle(A, B, d)$ to an unoriented link K. This polynomial is defined recursively by the formulas:

$$2.\ \langle O K\rangle = d\langle K\rangle$$

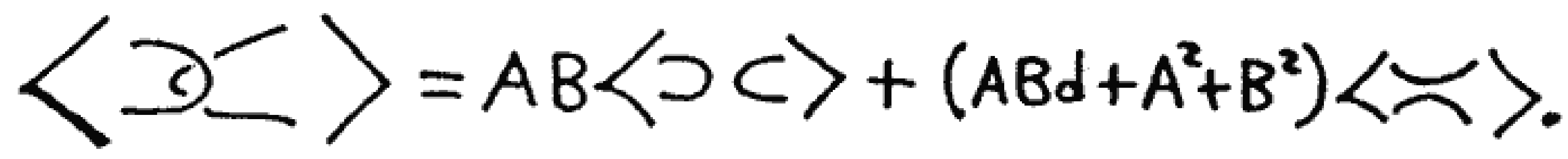
$$\langle O\rangle = d.$$

The first formula asserts that the polynomial for a given diagram is obtained as an additive combination of the polynomials for the diagrams obtained by splicing away the given crossing in two possible ways. Thus the small diagrams indicate larger diagrams that differ only as indicated. The second formula says that the value of a loop (simple closed curve in the plane) is d, and that if a loop occurs (isolated) inside a larger diagram, then the value of the polynomial acquires a factor of d from this loop. In particular, the value of a disjoint union of N simple closed curves is d^N.

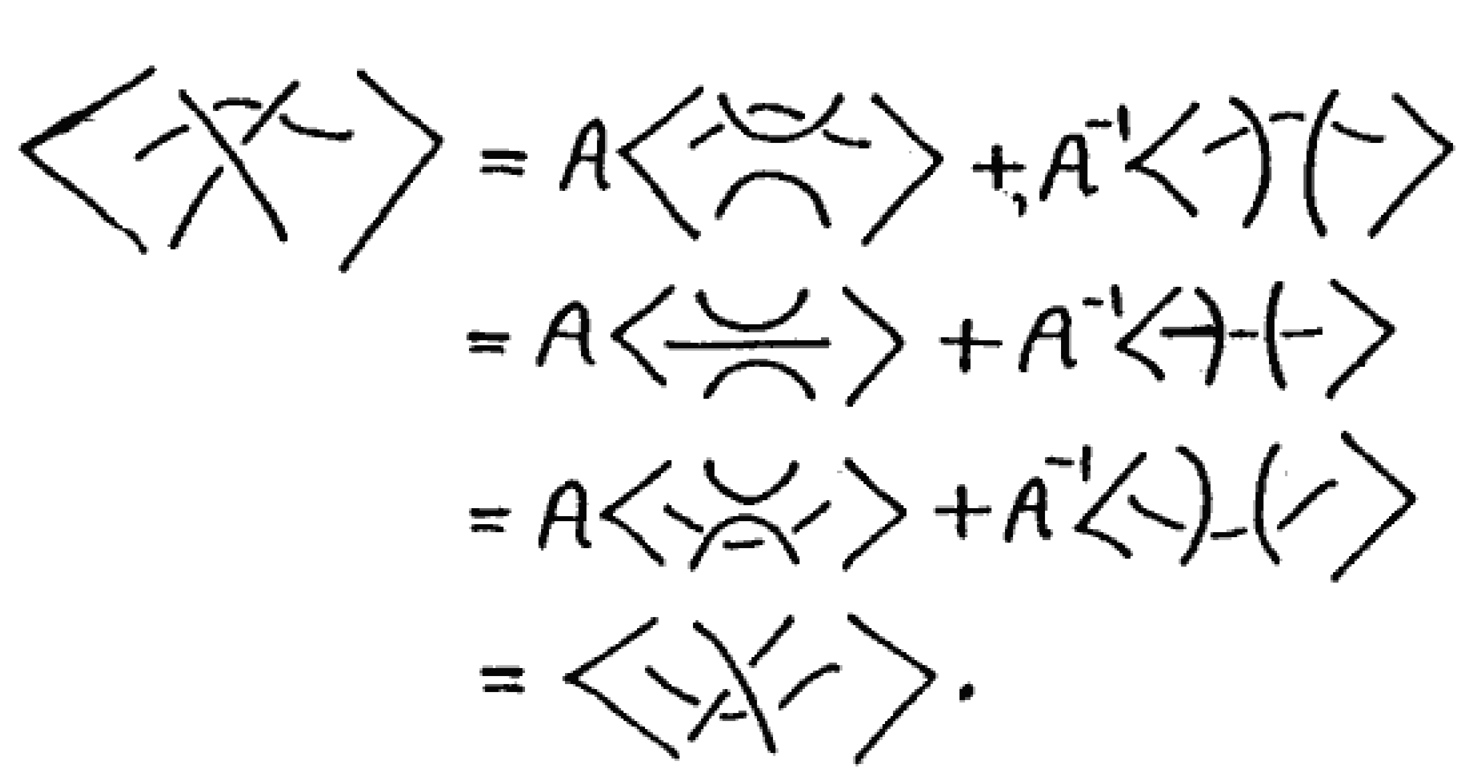
Together, the two formulas completely determine $\langle K\rangle$, and $\langle K\rangle$, is well-defined just so long as A, B and d commute with one another. Thus this polynomials takes values in the ring $Z[A, B, d]$ of polynomials in three variables with integer coefficients.

As it stands, $\langle K\rangle$, is not an invariant of any of the Reidemeister moves. However, the following formula is an easy consequence of 1 and 2 above.

Proposition.



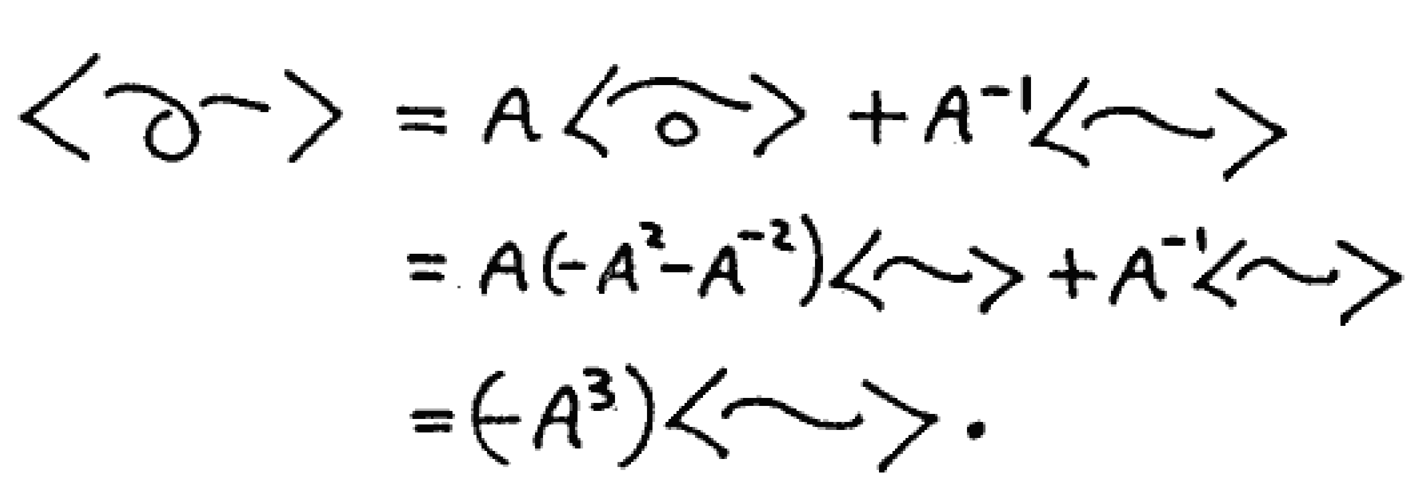
As a result we see that $\langle K\rangle$, is invariant under the type II move if we select $B = A^{-1}$ and $d = -A^2 - A^2$. Furthermore, it now follows directly that $\langle K\rangle$, is invariant under the move III:



Thus, with $B = A^{-1}$ and $d = -A^2 - A^{-2}$, we have that $\langle K \rangle$, is an invariant of regular isotopy. To obtain an invariant of ambient isotopy for oriented links, we form

$$f_K(A) = (-A^3)^{-w(K)} \langle K \rangle / \langle 0 \rangle$$

where K is oriented, $w(K)$ is the writhe of K as defined in the previous section, and $\langle K \rangle$, is the bracket evaluated on the unoriented link underlying K. The reason for this factor of $-A^3$ is that



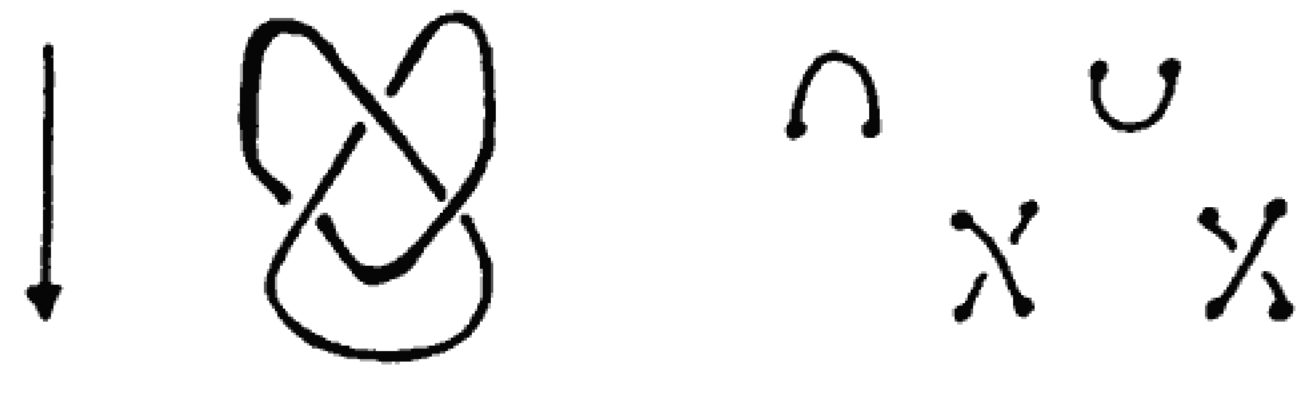
One then has the

Theorem 12. *For any oriented link* $k, V_K(t) = f_K(t^{-1/4})$ *where* V_K *denotes the original one-variable Jones polynomial.*

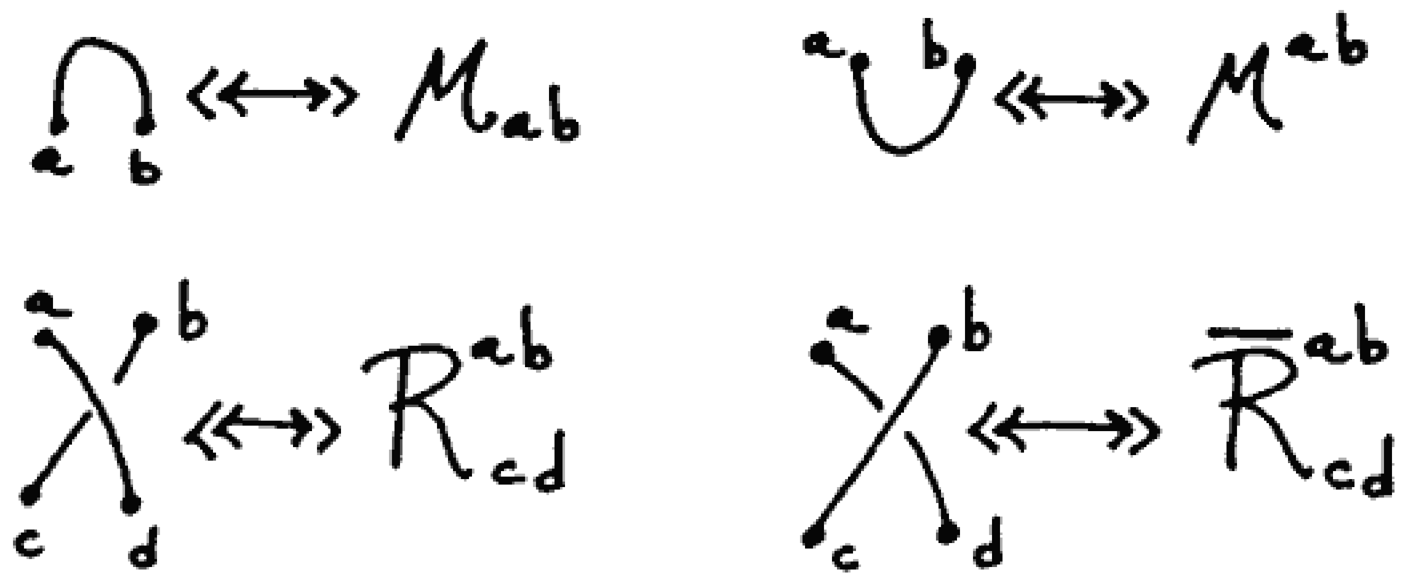
Thus, the bracket, suitably normalized, gives a direct model for the Jones polynomial.

THE VACUUM-VACUUM AMPLITUDE

In the rest of this section I shall stick to the bracket, and show how it can be seen as a "vacuum-vacuum amplitude" in a combinatorial version of topological quantum field theory [25]. More generally, we can consider an amplitude associated to a given diagram by regarding the plane as 1+1 spacetime. By convention, let time run vertically down the page, and space proceed from left to right (This is the convention of the reader of English). Position the link diagram so that it is transversal to the space levels except at critical points corresponding to maxima, minima and crossings.



Each maximum can be regarded as a creation of two particles from the vacuum, each minimum an annihilation, and each crossing is an interaction (thought of as involving braiding in the extra spatial dimension orthogonal to the page). To each of these events we associate a matrix whose indices go over (say) the spins of the particles, and whose values are the amplitudes for each of these processes.



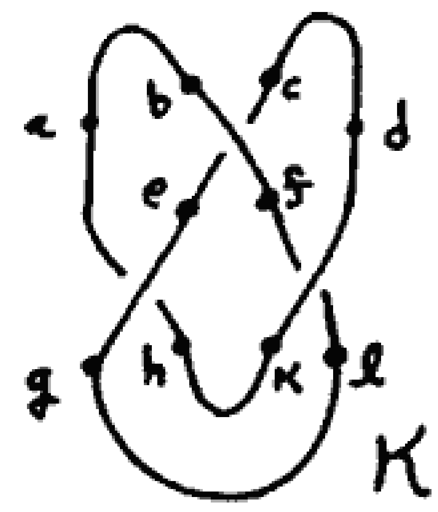
The amplitudes used here are a generalization of amplitudes in quantum mechanics, suitably generalized for the purposes of topology (In the process we take leave of the usual interpretations of observation in quantum mechanics. In the topology the amplitude itself is a real property of the system. There is no "collapse of the wave function"). Therefore the amplitudes will take values in a commutative ring (e.g. in $Z[A, A^{-1}]$), and the spins will run over an arbitrary finite index set (e.g. $\{-1, +1\}$). Amplitudes are calculated according to the *principles of quantum mechanics* [7]:

1. If an event occurs in a way that can be decomposed into a set of individual steps (e.g. creations, annihilations, interactions), then the amplitude of the given event is the product of the amplitudes of the individual steps.

2. If an event may occur in several disjoint alternative ways, then the amplitude of this event is the sum of the amplitudes of the ways.

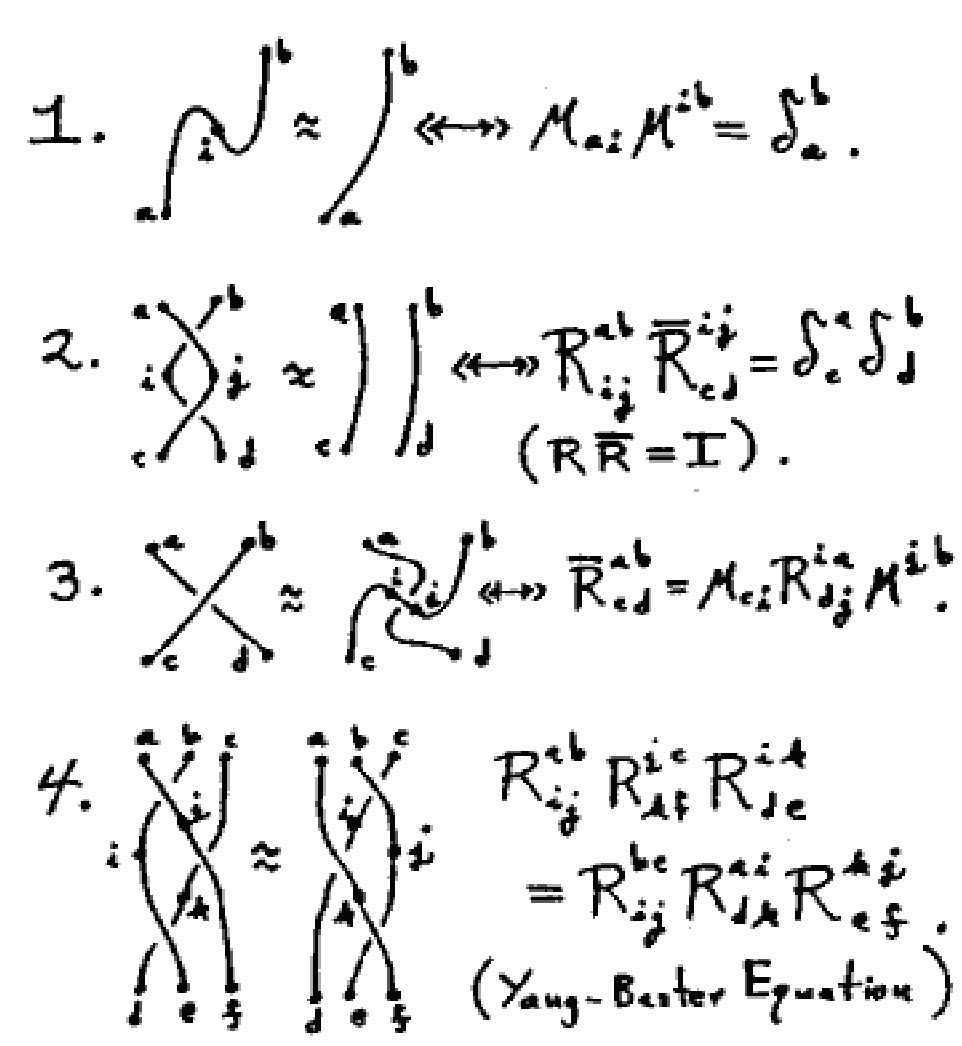
Given a diagram K, and a set of matrices as above, we can calculate the amplitude for particles to be created from the vacuum, interact in the pattern of the link diagram, and return to the vacuum. This amplitude decomposes as a sum of the amplitudes for *states* of the diagram. Each state σ is an assignment of spins to the *nodes* of the diagram. (The nodes are the input and output nodes of the small diagrams corresponding to the matrices). Given a state, each matrix has a well-defined value, and the amplitude of this state is the product of these values. Thus the vacuum-vacuum amplitude, $T(K)$ for a given diagram K is the sum over the states of the product of the matrix values for each state.

Symbolically, this works out in accord with the usual Einstein convention for repeated indices: Write down a product of all the matrices for the given diagram, in indices, with one index for each node. The amplitude is then the value of this expression interpreted as a sum over all cases of repetitions of an index in lower and upper positions.



$$T(K) = M_{ab} M_{cd} R^{bc}_{ef} \bar{R}^{ae}_{gh} \bar{R}^{fd}_{kl} M^{gl} M^{hk}.$$

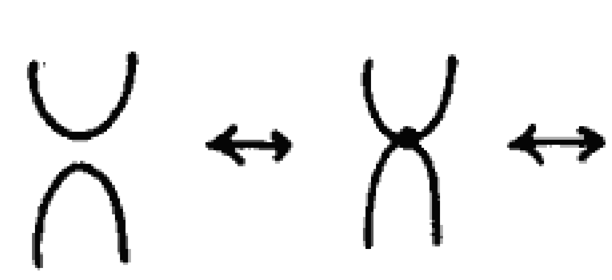
Having defined the vacuum-vacuum amplitude $T(K)$, we must see when it will be an invariant of regular isotopy, and when it will model the bracket. In order for $T(K)$ to be an invariant of regular isotopy, we need the following restrictions on the matrices [15]:



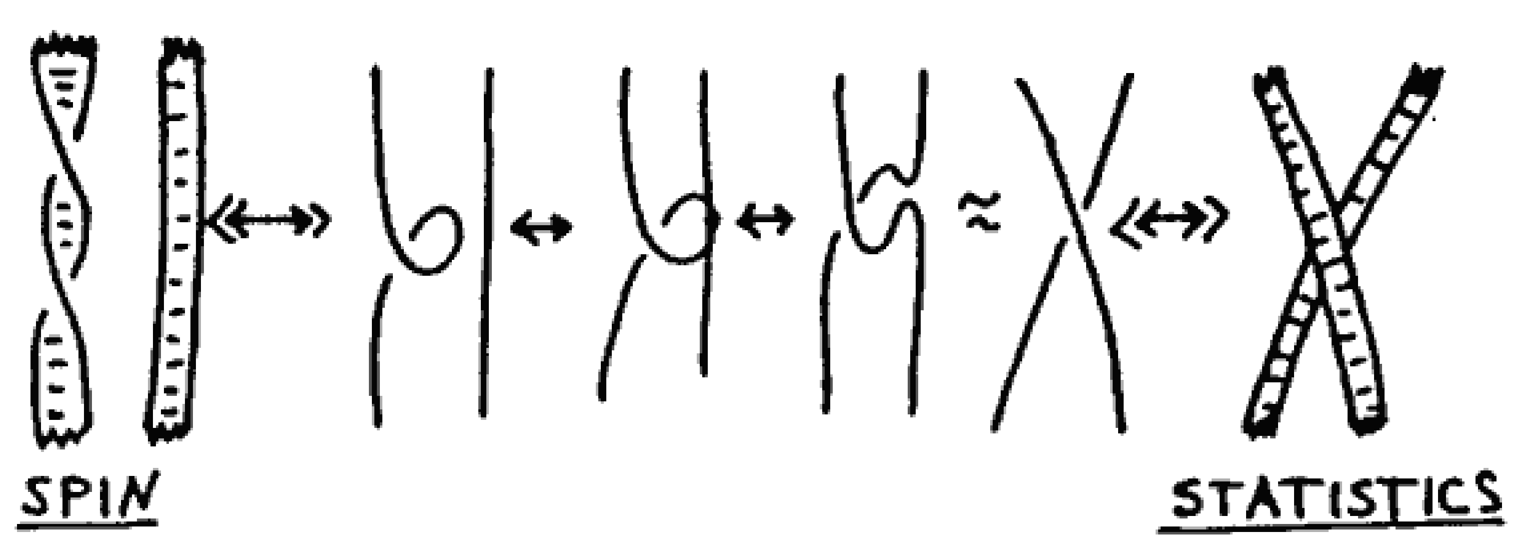
1. $M_{ai} M^{ib} = \delta_a^b$.
2. $R^{ab}_{ij} \bar{R}^{ij}_{cd} = \delta^a_c \delta^b_d$ $(R\bar{R} = I)$.
3. $\bar{R}^{ab}_{cd} = M_{ci} R^{ia}_{dj} M^{jb}$.
4. $R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef}$. (Yang-Baxter Equation)

(In 3 there is a corresponding left-hand twist, and in 4, there is also the same equation for all crossing reversed). Equation 4 is called the Yang-Baxter Equation [3] (here given without rapidity parameter).

Remark. It is interesting to speculate about the physical meaning of these restrictions. The twist condition 3 is the most mysterious since it relates R and R^{-1} via creations and annihilations. A simpler physical situation may lend some insight here. In the simplified scenario, we assume that 1 holds, and that parallel identity lines are interchangeable with pairs of creations and annihilations:



Then one has the sequence of identifications



This has been interpreted as a depiction of *the equivalence of spin and statistics* (see e.g. [22] and references therein) where spin is regarded as catalogued by the twist of framing (become curl of diagram) and statistics corresponds to the braiding of the two lines. This shows that part of our diagrammatics correspond to ordinary physical interpretations, and that *where the topology begins the equivalence of spin and statistics leaves off.* In this sense, the topology is an index of the non-standard statistics.

MODELLING THE BRACKET

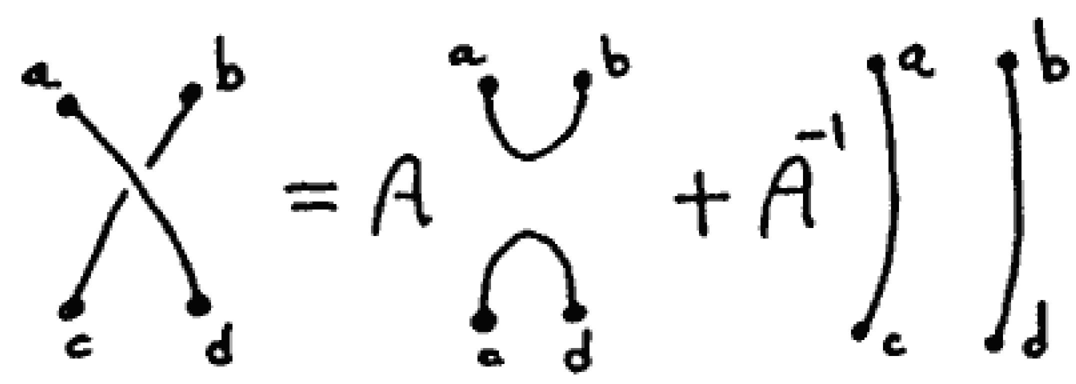
In order to model the bracket with a vacuum-vacuum amplitude we need to find creation and annihilation matrices that are inverse to one another, and that give a loop value of $-A^2 - A^{-2}$. Here is an answer to that puzzle:

$$M_{ab} = M^{ab}$$

$$M = \begin{bmatrix} 0 & \sqrt{-1}A \\ -\sqrt{-1}A^{-1} & 0 \end{bmatrix}.$$

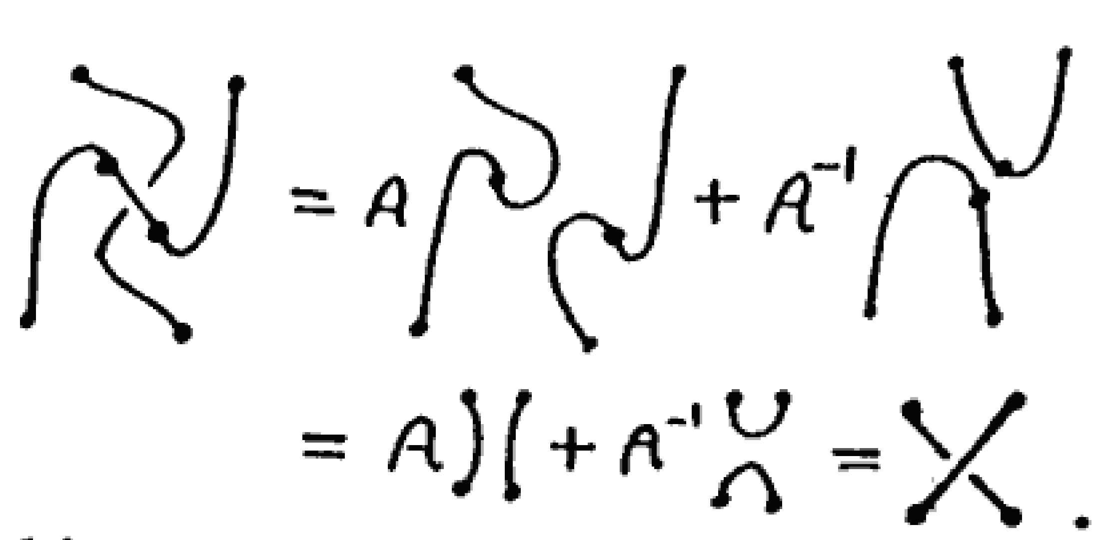
Note that the matrix M has square the identity, and that the loop value is therefore the sum of the squares of the entries of M. (See [13], [14], [15] for motivations for this construction).

With a given choice for the creations and annihilations, there is one choice for the R matrix to give the bracket:

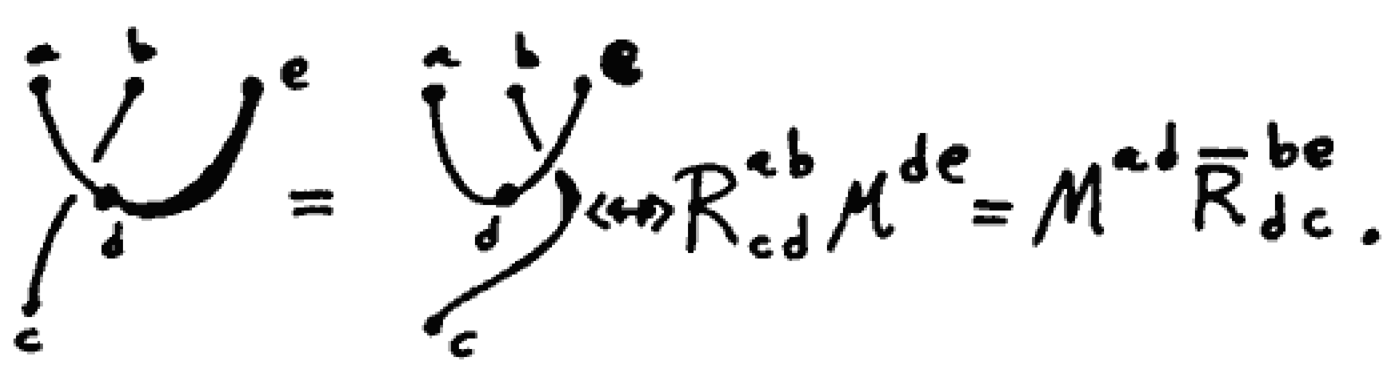


With this choice, $T(K)$ will satisfy the defining equations of the bracket, and therefore $\langle K \rangle = T(K)$ (since we have correctly adjusted the loop value).

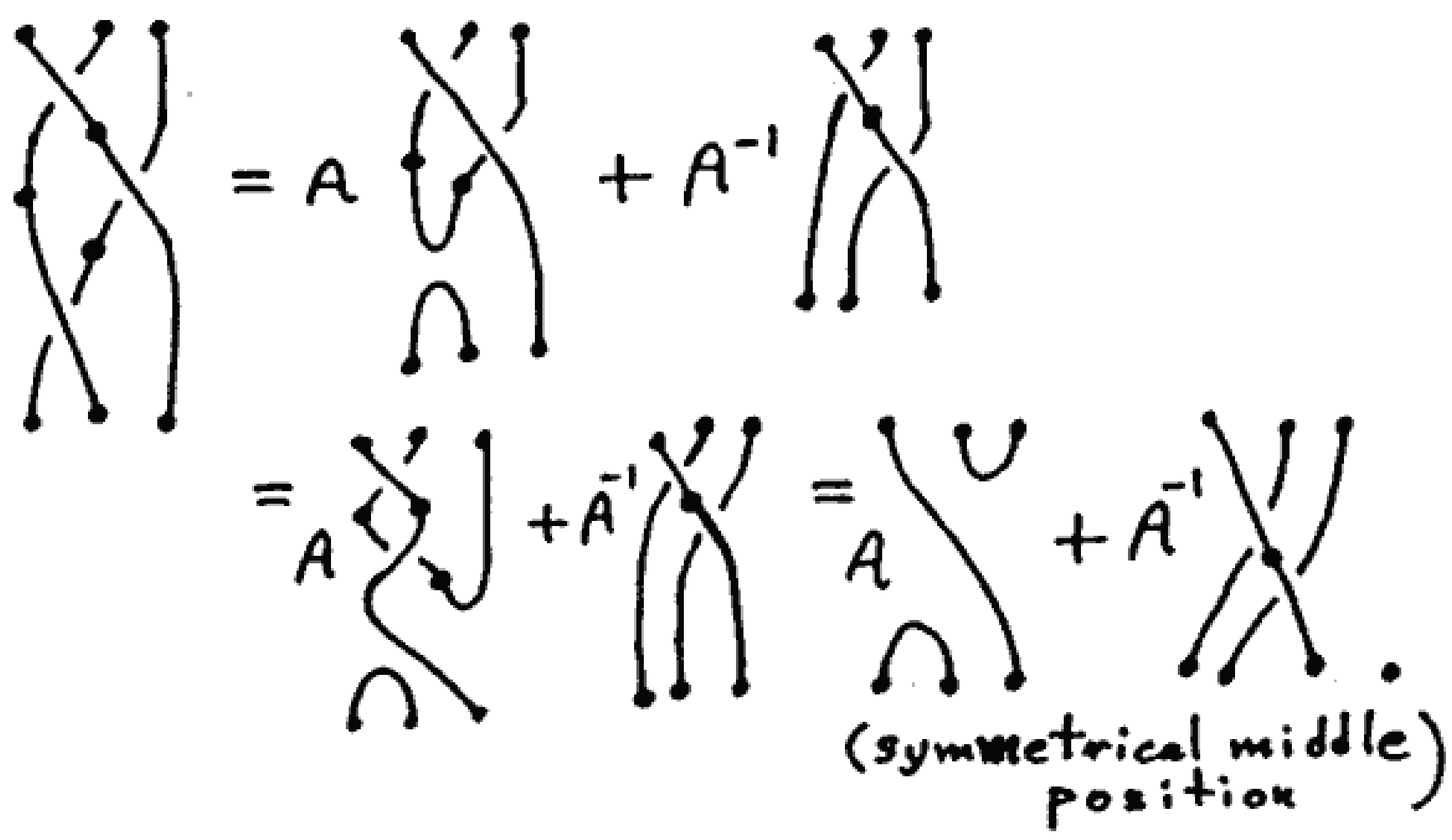
Remark. In fact it is interesting to note that if the creation and annihilation are inverse matrices, and R is defined as above, then 2 follows easily, while 3 goes as below



and 4 is proved by first checking



and then performing the following variation on our bracket derivation of the invariance under the III move



The upshot of this discussion is that by simply adjusting the creation and annihilation matrices correctly, we automatically produce a model of the bracket and a solution to the Yang-Baxter equation. This is the simplest instance of a solution to the Yang-Baxter equation appearing naturally from the knot theory. This is the well-known [19] R-matrix corresponding to the $SL(2)$ quantum group. In fact, the structure that we have created so far will now enable us to see one motivation for the construction of the quantum group.

4. THE $SL(2)$ QUANTUM GROUP

Note that we can write

$$M = \begin{bmatrix} 0 & \sqrt{-1}A \\ -\sqrt{-1}A^{-1} & 0 \end{bmatrix} = \sqrt{-1}\widetilde{\varepsilon},$$

$$\widetilde{\varepsilon} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix}.$$

and that, as A approaches 1, the matrix $\widetilde{\varepsilon}$ approaches

$$\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix ε is significant in linear algebra because it expresses the determinant of a 2×2 matrix. That is, let P be a 2×2 matrix with commuting entries. Then

$$P\varepsilon P^T = Det(P)\varepsilon.$$

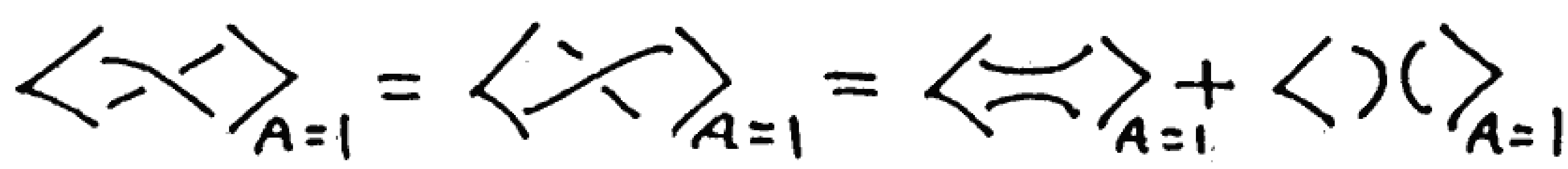
(T denotes transpose).

Now $SL(2)$ (over a commutative ring) is the set of matrices of determinant one, and can therefore be characterized as the set of matrices leaving the epsilon invariant:

$$SL(2) = \{P | P\varepsilon P^T = \varepsilon\}$$

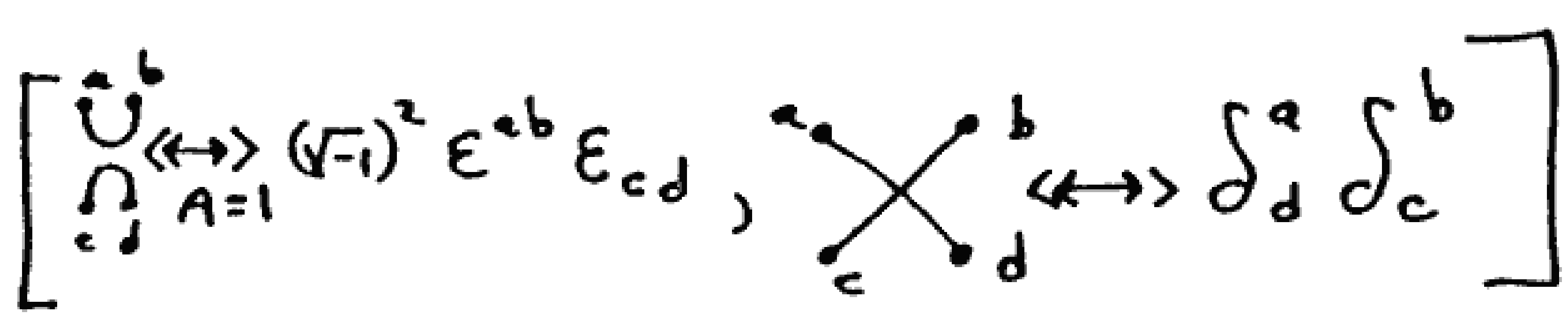
Here T denotes matrix transpose.

At $A = 1$ the bracket does not discriminate between under and overcrossings, and the identity



corresponds directly to the Fierz identity

$$\varepsilon^{ab}\varepsilon_{cd} = \delta^{ac}\delta_{bd} - \delta^{ad}\delta_{bc}$$



Thus at $A = 1$ (and also at $A = -1$) the diagrams become interpreted as tensor diagrams for $SL(2)$ invariant expressions.

It is then natural to ask whether there is a generalization of this symmetry for the topology and link diagrams. Specifically, we ask whether $\tilde{\varepsilon}$ has a symmetry group analogous to $SL(2)$. Some experimentation shows that the way to ask this question is to consider

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

with associative, possibly non-commutative entries, and ask for the invariances:

$$P\tilde{\varepsilon}P^T = \tilde{\varepsilon}$$

and

$$P^T\tilde{\varepsilon}P = \tilde{\varepsilon}.$$

It is then an exercise in elementary algebra to see that these conditions are equivalent to the equations:

$$(q = \sqrt{A})$$

$$ca = qac \quad db = qbd$$

$$ba = qab \quad dc = qcd$$

$$bc = cb$$

$$ad - da = (q^{-1} - q)bc$$

$$ad - q^{-1}bc = 1$$

These are the defining relations for the algebra $U^* = SL(2)q$ ([6], [17]), sometimes called the $SL(2)$ quantum group. It is not a group, but rather a Hopf algebra. The co-algebra structure is given by the map

$$\Delta : U^* \longrightarrow U^* \otimes U^*$$

where

$$\Delta(P^i_j) = \sum_k P^i_k \otimes P^k_j.$$

Thus

$$\Delta(a) = a \otimes a + b \otimes c$$

$$\Delta(b) = a \otimes b + b \otimes d$$

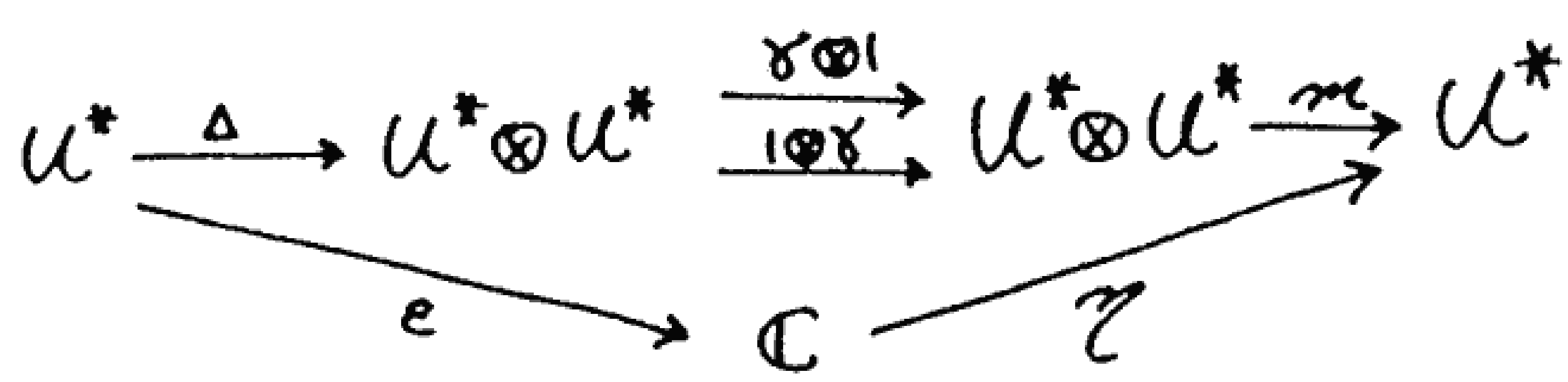
$$\Delta(c) = c \otimes a + d \otimes b$$

$$\Delta(d) = c \otimes b + d \otimes d.$$

In this case the Hopf algebra has an antipode and this is directly related to the fact that the matrix P has an inverse $\gamma(P)$:

$$\gamma(P) = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

Recall that the antipode is a map $\gamma : U^* \to U^*$ such that the following diagram commutes



where ε and η are the co-unit respectively and m denotes the multiplication in the algebra. Here

$$\varepsilon(P^i_j) = \delta^i_j \quad and \quad \eta(\delta^i_j) = \delta^i_j.$$

Thus the condition that γ be an antipode is just that

$$\sum_k \gamma(P_k^i)P_j^k = \delta_j^i$$

$$\sum_k P_k^i \gamma(P_j^k) = \delta_j^i.$$

And this is the same as saying that P and $\gamma(P)$ are inverse matrices.

We could now and discuss a number of things about the relationship of this quantum group to solutions to the Yang-Baxter equation, and to its dual form as a deformation of the Lie Algebra for $SL(2)$. (See [6], [19]). But here there is not space for this. The purpose of this section has been to show how the quantum group arises naturally from a combination of the topology and a desire to extend the algebraic symmetry inherent in a significant special case of the vacuum-vacuum expectation model.

5. AND BACK

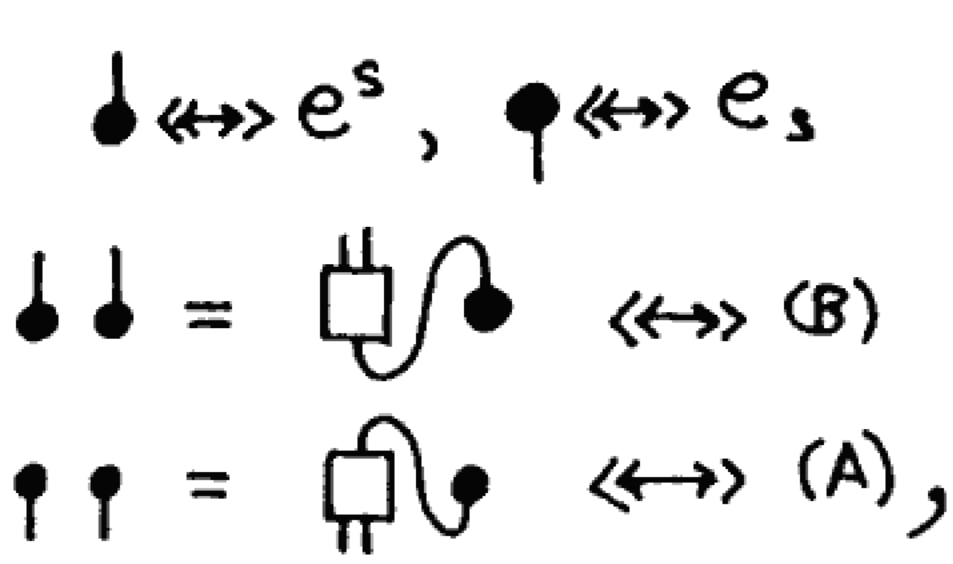
In order to indicate how the trail looks going back from quantum grops to link invariants I shall make a leap to the formalism behind the so-called quantum double construction of Drinfeld [6]. We shall then see how a Hopf algebra structure can give rise through its matrix representations, to invariants of links.

We begin with an algebra with generators $e_0, e_1, \ldots, e_n$ and $e^0, e^1, \ldots, e^n$ and the following relations describing multiplication in the algebra:

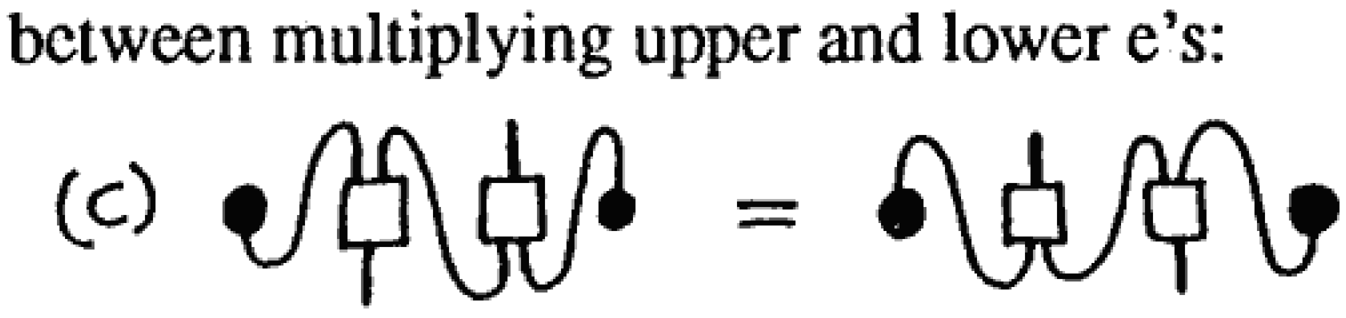
$$(A) \qquad e_s e_t = m_{st}^i e_i$$

$$(B) \qquad e^s e^t = \mu_i^{st} e^i$$

Diagrammatically, I shall write



where it is understood that the boxes denote the product expansion coefficients, and hence the boxes commute with the e-nodes (↓, ↑) and with each other. We further assume the following relationship between multiplying upper and lower e's:



$$c_i \mu_a^{ij} m_{jk}^b e^k n = e^k m_{kj}^b \mu_a^{ji} c_i.$$

Then we have

Theorem. *With the above assumptions, the element*

$$\rho = \sum_s e_s \otimes e^s =$$ 

satisfies the algebraic form of the Yang-Baxter equation:

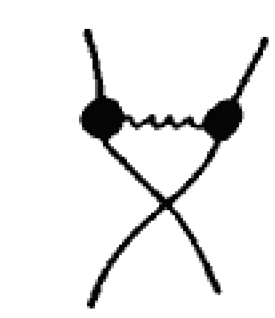
$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}.$$

$$\left[\rho_{12} = \sum_s e_s \otimes e^s \otimes 1, \quad \rho_{13} = \sum_s e_s \otimes 1 \otimes e^s, \ldots\right]$$

Proof. (See [15]). In a matrix representation the e-nodes sprout indices, and an algebraic solution to the Yang-Baxter Equation becomes and knot theorist's matrix solution via an added permutation. Thus if

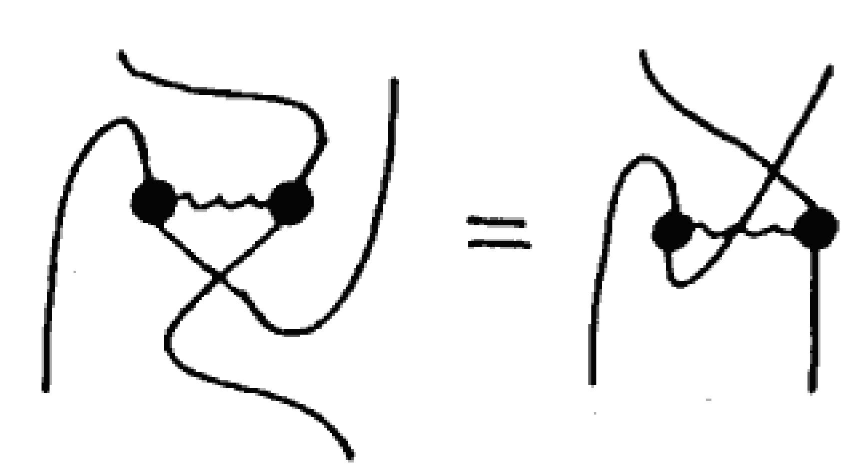
$\rho =$ 

denotes the algebraic solution, then

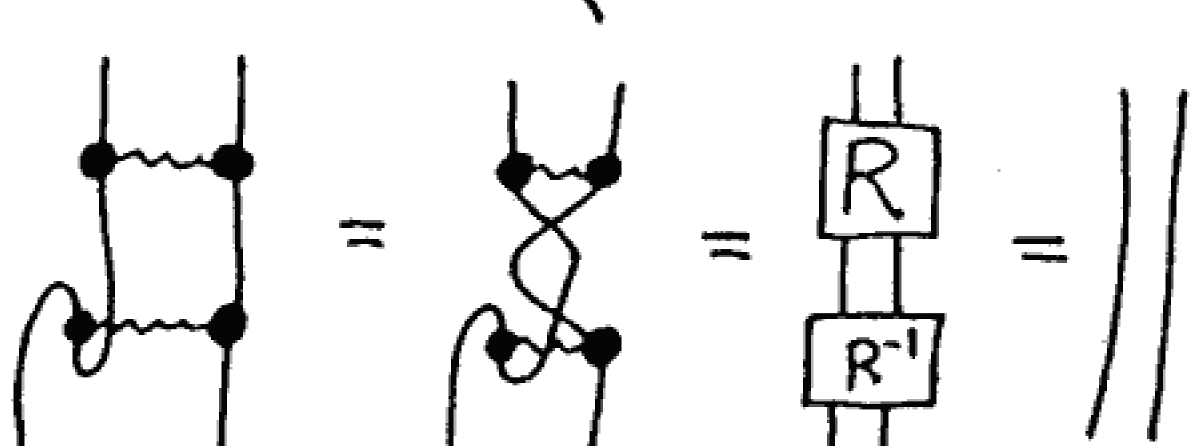
$R =$ 

denotes the corresponding knot theoretic R-matrix in some representation (the indices of this representation correspond to the new lines).

As we know from the previous section, the knot theory demands a relationship between the creation and annihilation matrices and the R-matrix. This "twist relation" is given diagrammatically as follows:

$R^{-1} =$ 

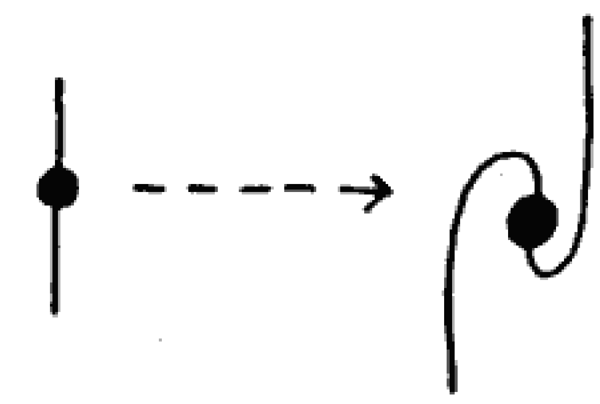
Thus



We conclude that there should be an antimorphism

$$x \longrightarrow \gamma(x)$$

of the abstract algebra that corresponds to the map in the representation



and we need that $\gamma(e_s) \otimes e^s$ is the inverse of $e_s \otimes e^s$ (with summation on repeated lower and upper indices). It is the not hard to see that if we were to make the algebra into a Hopf algebra such that the co-multiplication for the lower index e's is the multiplication for the upper index e's and vice versa (this is the double construction), then this inverse requirement is equivalent to γ being an antipode. Thus the twist conditions of the knot theory are intimately tied up with the Hopf algebra structure for the quantum group. This completes the journey back, albeit in an abstract mode.

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