Note di Matematica Vol. IX - n. 2, 249-261(1989)

MAXIMAL PARTIAL SPREADS AND CENTRAL GROUPS

N.L. JOHNSON [1]

1. INTRODUCTION

A *partial spread in* $PG(3,q)$ is a set of mutually skew lines. The cardinality of a partial spreads is less than or equal to $q^2 + 1$ and shall be called the *degree*. If the degree is t, the integer $g^2 + 1 - t$ is called the *deficiency*. Partial spreads of deficiency 0 are called spreads and are equivalent to translation planes of order q^2 and kernel containing $GF(q)$ obtained by considering the projective space as the lattice of subspaces of a 4-dimensional vector space V_4 over $GF(q)$. The $q^2 + 1$ lines of $PG(3,q)$ are realized as a set of 2-dimensional subspaces. The points of the plane are the 4-vectors and the lines are translate of the spread subspaces (components in the plane). Similarly, there is a correspondence between *translation nets of degree* $q^2 + 1 - t$ and partial spreads of deficiency t.

A partial spread P is *maximal* if and only if P cannot be extended to a partial spread of smaller deficiency. We shall adopt the terminology of Jungnickel [20] and say that a partial spread is *extendable* if can be imbedded into a partial spread of smaller deficiency.

If P is a partial spread, let T_P denote the corresponding translation net. If T_P can be embedded in an affine plane, we shall say that the net and the partial spread is imbeddable otherwise unimbeddable (Jungnickel [20] (1.2)). Note that if a maximal partial spread is imbeddable into a translation plane then the corresponding translation plane cannot have all of its components as 2-dimensional subspaces; the kernel of the corresponding translation plane is smaller than $GF(q)$.

In this article, we shall be concerned with the construction of maximal partial spreads of deficiency $q - 1$ or q and consider the question of imbeddability. By Jungnickel, if a partial spread of deficiency $q - 1$ or q can be embedded in some affine plane then the plane is a translation plane and we may use the terminology as above.

Jungnickel [20] shows that any maximal partial spread of deficiency q is unimbeddable. However, the known maximal partial spreads of deficiency $q - 1$ are imbeddable. In his concluding remarks of [20], Jungnickel lists several open problems which refer to maximal partial spreads of deficiency $q-1$. In particular, the following question is raised: *Are maximal partial spreads in* $PG(3,q)$ *and deficiency* $q - 1$ *always imbeddable?*.

We consider maximal partial spreads which may be constructed by collineation groups G acting on translation planes of order q^2 and kernel containing $K \cong GF(q)$ such that G fixes a 2-dimensional K-subspace pointwise and which leaves invariant a 2-dimensional K-subspace which is not a component. We shall call such a group a *central group*. We shall further say

[1] Partially supported by a grant from the National Science Foundation.

that a *central group is of type i* if and only if either the group is a central collineation group and the group leaves invariant exactly i Baer subplanes of a particular net of degree $q+1$ which are 2-dimensional K-subspaces or the group is a Baer group and the net defined by the parallel classes of the fixed point space of the group contains exactly i Baer subplanes which are K-subspaces. We note in section 2 that $i = 1, 2$ or $q+1$.

In section 2, we give our main result on central groups and maximal partial spreads:

Theorem 2.6. *Let π denote a translation plane of order q^2 and kernel containing $K \cong GF(q)$. Let G be a central group in the translation complement of type i. Let N_i denote the partial spread of degree $q+1$ which contains the i Baer subplanes determined by a G invariant 2-dimensional K-subspace π_O. Let M denote the set of components of π which do not belong to N_i. Then*

(1) *$M \cup \{$Baer subplanes of N_i which are K-subspaces$\}$ correspond to a maximal partial spread in $PG(3,K)$ of degree $q^2 - q + i$ (deficiency $q+1-i$).*

(2) *The maximal partial spread of (1) is imbeddable if and only if the net N_i is a derivable net.*

(3) *If $i = 1$ then the maximal partial spread of (1) is unimbeddable.*

In section 3, we list the known maximal partial spreads of deficiency $q-1$ and q constructed by maximal central groups. In particular, we give some examples of some unimbeddble maximal partial spreads of deficiency $q-1$ thus answering the question raised in Jungnickel and listed above. We shall see that there are a large number of examples of maximal partial spreads. However, isomorphisms results for partial spreads are fairly difficult to obtain. So, in section 4, we give some remarks on isomorphism of a particular type of maximal partial spread.

Acknowledgement: The author is indebted to Professor Mauro Biliotti for helpful suggestions in the preparation of this article.

2. THE CONSTRUCTION

Let π denote a translation plane of order q^2 and kernel K isomorphic to $GF(q)$. Let G denote a collineation group in the translation complement.

According to the definition given in section 1, there are exactly the following types of central groups G:

(2.1) *Elation type i*; the group G is an affine elation group and fixes a Baer subplane π_O which is a 2-dimensional K-subspace. There are exactly i Baer subplanes in the net N_{π_O} defined by the parallel classes of π_O which are G invariant.

(2.2) *Homology type i*; the group G is an affine homology group and fixes a Baer subplane π_O which is a 2-dimensional K-subspace. There are exactly i Baer subplanes in the net N_{π_O} defined by the parallel classes of π_O which are G invariant.

(2.3) *Baer-shear type i*; the group G is a Baer group of order dividing q and > 2. The net N_{π_O} defined by the parallel classes of the fixed point space of G (which is a K-space by Foulser [4]) contains exactly i Baer subplanes which are K-subspaces and are G invariant.

(2.4) *Baer-strain type i*; the group G is a Baer group of order dividing $q-1$ and > 2. The net N_{π_O} defined by the parallel classes of the fixed point space of G (again is a K=subspace by Foulser [4]) contains exactly i Baer subplanes which are K-subspaces and are G invariant.

Lemma 2.5. *If G is a central group of type i of a translation plane π of order q^2 and kernel K isomorphic to $GF(q)$ then $i = 1, 2,$ or $q + 1$.*

Proof. By definition, $i \geq 1$. If $i \geq 3$ then by Foulser [3], then are $1 + |Kernel\pi_O|$ Baer subplanes incident with the zero vector of the net N_i so the net must be derivable as π_O is Desarguesian. Now, we may use Biliotti-Lunardon [1] to observe that in this latter case, $i = q+1$. If G is an elation or homology collineation group then G leaves every Baer subplane of the net N_i invariant (see e.g. Lüneburg [22] (4.7)) and in particular, leaves invariant each such Baer subplane which is a K-subspace.

We are essentially uninterested in the situation $i = q+1$ since it will not produce nontrivial maximal partial spreads. We shall say that a central group of type i is nontrivial if and only if $i \neq q+1$.

Probably the central groups that are easiest to study are the *maximal central groups;* the order of an elation or Baer-shear group of type i is q and the order of a homology or Baer-strain group of type i is $q-1$.

There is a 1-1 correspondence between nontrivial maximal Baer-shear or Baer-strain groups and maximal partial flocks of quadratic cones or of hyperbolic quadrics of deficiency one respectively (missing one circle - see the main result of Johnson [11]). By utilizing this connection, it is shown in Johnson [12] (using a result of Payne and Thas) that there are no nontrivial maximal Baer-shear central groups of *even* order.

Furthermore, there are no known odd order examples. There are, however, two interesting examples of nontrivial maximal Baer-strain groups of orders 4 and 9 respectively. These lead to maximal partial spreads of order q^2 and degree $q^2 - q + 2$ (deficiency $q - 1$) where $q = 3$ or 4 and, as mentioned above, to maximal partial flocks of hyperbolic quadrics of deficiency one in $PG(3, q)$ for $q = 4$ or 9 (see Johnson and Pomareda [18]). In both cases, the maximal partial spreads are imbeddable.

We now give the proof of (2.6) stated in the introduction. Assume the hypothesis of (2.6).

Assume that the partial spread P defined in the statement of (2.6) is not maximal. Then let L denote a 2-dimensional K-subspace which is disjoint from the component subspaces of P. Then L is either a component of π or a Baer subplane of π since π has dimension 4 over K. Thus, L must define a Baer subplane of the net N_i. If G is a central collineation group then G must leave L invariant. Thus, in any situation, then L must be one of the original i Baer subplanes of N_i which are 2-dimensional K-subspaces and thus we obtain a contradiction. This proves (1) of (2.6). Note that there is a similar construction in Jungnickel [20] (6.1).

Now by Jungnickel [20] (3.2), if the net P is imbeddable into an affine plane Σ then the plane Σ is a translation plane. Now Σ and π both share the net M. But, the net M has critical deficiency (Ostrom [23]). Hence, by Ostrom [23], Σ and π are either equal or Σ is derived from π. Thus, Σ must be derived from π which forces the net N_i to be derivable. This proves part (2).

Actually, the proof of part (3) may be obtained from Jungnickel [20] (3.4). But, note that if $i = 1$ and the net P is imbeddable then the net N_i is derivable by part (2), but then by Biliotti-Lunardon there must be at least 2 Baer subplanes of N_i which are K-subspaces. Thus, we have the proof to part (3) and thus to our construction result (2.6).

3. THE KNOWN EXAMPLES OF MAXIMAL CENTRAL GROUPS

In this section, we list the known maximal central groups. Note that by (2.6), for each example, we obtain a maximal partial spread in $PG(3,q)$ of deficiency q or $q-1$. In particular, we give some nonimbeddable maximal partial spreads of deficiency $q-1$. We shall organize the examples in the order given for the types of groups listed in section 2.

Elation type 1

We first consider the elation type 1 groups. This information is also given in Johnson [13] but here the emphasis was not on maximal partial spreads.

In Johnson [13] (4.4), one of Kantor's planes obtained by ovoids in an $\Omega^+(8,q)$ space may be represented as follows: Let $\{(x_1,x_2,y_1,y_2)|x_i,y_i \in K \cong GF(q) \text{ for } i=1,2\}$ denote the 4-dimensional vector space V_4 over K. Let $x=(x_1,x_2), y=(y_1,y_2), O=(0,0)$. If $q = 3^{2e-1}$ then the following equations define a translation plane π of order $3^{2(2e-1)}$:

$x = O, y = x\begin{bmatrix} u & u^\sigma + t^{2\sigma+3} \\ -t & u \end{bmatrix}$ for all u,t in K and where $\sigma = 3^e$. Furthermore, this translation plane admits the elation group $E = \left\langle \begin{bmatrix} I & \begin{bmatrix} b & b^\sigma \\ 0 & b \end{bmatrix} \\ O & I \end{bmatrix} \middle| b \in K^* \right\rangle$. Note that E is a central group of type 1 as it leaves the 2-dimensional K-space $\{(0,x_2,0,y_2)|x_2,y_2 \in K\}$ invariant and does not fix any further Baer subplane whose parallel classes agree with those of the previous K-space considered as a Baer subplane.

The complete list of maximal central groups of elation type 1 is as follows:

Elation type 1

Description	*Group*
(E1)(1) Lüneburg-Tits	$\begin{bmatrix} I & b & b^{\sigma} \\ & 0 & b \\ O & & I \end{bmatrix}$
order $2^{2(2e-1)}$	$b \in GF(2^{2e-1}), \sigma = 2^{e} (see[13])$
(E1)(2) Kantor	$\begin{bmatrix} I & b & b^{\sigma} \\ & 0 & b \\ O & & I \end{bmatrix}$
order $3^{2(2e-1)}$	$b \in GF(3^{2e-1}), \sigma = 3^{e} (see[13])$
(E1)(3) Biliotti-Menichetti	$\begin{bmatrix} I & u & u^4 + u^2 \\ & 0 & u \\ O & & I \end{bmatrix}$
order 64	$u \in GF(8) (see[2])$
(E1)(4) Jha-Johnson	$\begin{bmatrix} I & u & u + u^2 \\ & 0 & u \\ O & & I \end{bmatrix}$
order 64	$u \in GF(8) (see[2])$

Elation type 2

In order to describe the examples of elation type 2, we first recall the construction of *lifting* (see Johnson [10] or [14]).

Let π be a translation plane of order q^2 and kernel containing $K \cong GF(q)$. Let F contain K and be isomorphic to $GF(q^2)$ where $F = K[t]$ such that $t^2 = t\omega + \rho$ for ω, ρ in $GF(q)$. Represent the spread for π by $x = O, y = x \begin{bmatrix} \alpha & \beta \\ g(\alpha,\beta) - \omega h(\alpha,\beta) & h(\alpha,\beta) \end{bmatrix}$ (as above in the example of the Kantor planes) where x, y are 2-vectors over K and for all α, β in $GF(q)$ and g, h are functions from $GF(q) \times GF(q)$ into $GF(q)$.

Define a function f on F by $f(\alpha,\beta) = g(\alpha,\beta) - h(\alpha,\beta)t$.

Then there exists a translation plane π^L called the plane lifted from π by t of order q^4 and kernel isomorphic to $GF(q^2)$ with spread set defined by $x = O, y = x \begin{bmatrix} u & v \\ f(v) & u^q \end{bmatrix}$ where x, y are 2-vectors over F and for all u, v in F.

Note that $\left\langle \begin{bmatrix} I & \begin{matrix} u & 0 \\ 0 & u^q \end{matrix} \\ O & I \end{bmatrix} \middle| u \in F \right\rangle = E$ is a maximal central group of elation type 2.

Note that in this case, the maximal partial spreads of order q^4 and degree $q^4 - q^2 + 2$ (deficiency $q^2 - 1$) are always imbeddable as the corresponding net (see (2.6)) is derivable (see e.g. Johnson [8] to check that this statement is valid).

The examples given in Jungnickel [20] (section 6) may be obtained using the construction method above.

There are a vast number of maximal partial spreads of order q^4 and deficiency $q^2 - 1$ which may be obtained by elation groups of type 2 by the construction method of lifting. For example, every translation plane of order q^2 and kernel containing $GF(q)$ produces at least one and usually many. The isomorphism questions are completely open (but see section 4).

Homology type i

Lemma 3.1. *Every nontrivial maximal central group of homology type* i *is of type* 2.

Proof. Let G be the homology group of type i. Then $|G| = q - 1$ and hence the group can be diagonalized on the coaxis. It now follows that if the group is of type $i = 1$ or 2 then $\{(x_1, 0, y_1, 0)\}$ and $\{(0, x_2, 0, y_2)\}$ (see notation and representation of the Kantor planes) represent Baer subplanes which are left invariant by a homology group G represented in the form $\left\langle \begin{bmatrix} I & & \\ & u & 0 \\ & 0 & m(u) \end{bmatrix} \middle| u \text{ is in } GF(q)^* \right\rangle$ where m is a function on $GF(q)$. The sets of components of the two Baer subplanes above are $\{x = 0, y = 0, y = x \begin{bmatrix} u & 0 \\ n(u) & l(u) \end{bmatrix}$ where n, l are functions on $GF(q)$ such that $n(1) = 0\}$ and $\{x = 0, y = 0, y = x \begin{bmatrix} u & s(u) \\ 0 & r(u) \end{bmatrix}$ where s, r are functions on $GF(q)$ such that $s(1) = 0\}$ respectively. Since the two Baer subplanes share the line $y = x$ and are left invariant under G, it follows that the components distinct from $x = 0, y = 0$ are defined as an orbit of G. Hence, $s(u) = n(u) = 0$ which implies that the two Baer subplanes share all of their parallel classes. Thus, the group is of homology type 2 or $q + 1$. But, G is assumed nontrivial so the type is 2.

j-planes

In Johnson [13], a class of translation planes called $(q-1, q+1)$-planes of order q^2 and kernel containing $GF(q)$ is considered. In Johnson, Pomareda, and Wilke [19] this class is extended to what there is called the class of j-planes. The former set of planes are $j = 1$-planes.

Let $x^2 + xg - f$ be irreducible over $K \cong GF(q)$ where $f, g \in K$. Then $\left\{ \begin{bmatrix} u & t \\ tf & u + tg \end{bmatrix} \middle| u, t \in K \right\}$ is a field isomorphic to $GF(q^2)$. Let $\delta_{u,t}$ denote the determinant of $\begin{bmatrix} u & t \\ tf & u + tg \end{bmatrix}$ for $(u, t) \neq (0, 0)$. Let

$$\Gamma = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (\delta_{u,t}^{-j}) & 0 & 0 \\ 0 & 0 & u & t \\ 0 & 0 & tf & u + tg \end{bmatrix} \middle| u, t \in GF(q)^* \right\}.$$

Let $V = \{(x_1, x_2, y_1, y_2) | x_i, y_i \in GF(q) \text{ for } i = 1, 2\}$. Let $x = (x_1, x_2), y = (y_1, y_2), O = (0, 0)$. Then $x = O, y = O$ and the Γ images of $y = x$ define a spread if and only if $\delta_{u,t}^{j+1} - \delta_{u,t}^{j}(u + tg) + (1 - u) \neq 0$ for all u, t in $GF(q), (u, t) \notin \{(1, 0), (0, 0)\}$. If a spread is defined, the corresponding translation plane is called a j**-plane**.

Note that letting $t = 0$ and multiplying the corresponding element of Γ by the kernel homology $u^{-1} I$, we obtain the element $\begin{bmatrix} u^{-1} & 0 & 0 & 0 \\ 0 & u^{-2j-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Hence, there is a corresponding homology group with axis $y = 0$ and coaxis $x = 0$ which has an orbit of components represented by $y = x \begin{bmatrix} u & 0 \\ 0 & u^{2j+1} \end{bmatrix}$ for all nonzero u in $GF(q)$.

Theorem 3.2. *Any j-plane for j nonzero and order q^2 admits a maximal homology group of type 2. There is a corresponding maximal partial spread of order q^2 and deficiency $q - 1$. This maximal partial spread is imbeddable if and only if $2j + 1$ is a power of p where $p^r = q$ for p a prime.*

Proof. By (2.6)(2), it suffices to determine necessary and sufficient conditions on j so that the indicated net is derivable but is not a K-regulus. But, by Johnson [8], it must be that $2j + 1$ is a power of a prime in order that the indicated net is derivable. If $j \neq 0$ then the corresponding net cannot correspond to a regulus over the field in question (see also Johnson [7]).

There are a variety of examples of j-planes given in [19]. Each gives rise to a maximal partial spread in $PG(3, q)$ of order q^2 and deficiency $q - 1$. In this section, we shall list only the $j = 1$ planes of Johnson [13] which arise from certain ovoids in 8-dimensional hyperbolic space and which correspond to certain planes of Kantor [21].

Homology type 2

Translation plane of order q^2 (j =1-plane)	(f, g) for the polynomial $x^2 + xg - f$
(H2)(1) Kantor order q^2 even and $q \equiv 2 \bmod 3$	(1, 1) (see (3.9) Johnson [13])
(H2)(2) Kantor	$(\alpha^3, \alpha^2(-3/\alpha)^{1/2})$, for α a nonsquare (see [13] (3.9)

order q^2 even and $q \equiv 2 \bmod 3$

Note that the above two examples give rise to non-imbeddable maximal partial spreads of degree q^2 and deficiency $q-1$ (see 3.2).

(H2)(3) Kantor order 3^{2r}	(α, α), for α a nonsquare (see [13] (3.21))

Note that the above example gives rise to an imbeddable maximal partial spread of order q^2 and deficiency $q-1$ as $2j+1=3$ for $j=1$ (see (3.2)).

Baer-Shear types

We have mentioned above that there no known examples of maximal Baer-Shear type 1 and there can be no nontrivial examples of even order.

Baer-Strain types

There are exactly two examples of maximal Baer-strain types one of order 4 and one of order 9 producing maximal partial spreads of order 4^2 and deficiency 3 and order 9^2 and deficiency 8 respectively. Both of these exampels are imbeddable. See Johnson and Pomareda [19].

4. ISOMORPHISM RESULTS

We have constructed a large variety of maximal partial spreads in $PG(3,q)$ of deficiency q or $q-1$. However, there are essentially no isomorphism results on partial spreads so as to determine if the partial spreads are different. In this section, we offer a few remarks on the isomorphism problem.

First we note:

Theorem 4.0. *Let P_1 and P_2 be imbeddable partial spreads of order q^2 and deficiency q or $q-1$. Let π_1, π_2 denote the (unique) translation planes containing the partial spreads P_1, P_2 respectively. Then P_1 is isomorphic to P_2 only if π_1 is isomorphic to π_2.*

Proof. Let f be an isomorphism of P_1 onto P_2. Define $\pi_1 f$ to be the affine plane whose lines are defined by the f-images of lines of π_1. Now $P_1 f$ contains P_2 so that $\pi_1 f = \pi_2$ by uniqueness.

Also, we have the following fundamental result:

Theorem 4.1. *Let π denote a translation plane of order q^2 and arbitrary kernel K. Let M denote a partial subspread of degree $q^2 - q$ and let N denotes the complementary partial subspread relative to the spread for π. Let B denote a set of Baer subplanes of N which are (K-subspaces and let $P = B \cup M$. If a collineation g of the partial spread P leaves M (or B) invariant then g is a collineation of the translation plane π.*

Proof. First note that since the partial spread admits the full translation group of π, we may assume that g fixes the point O corresponding to the zero vector. Consider the affine plane πg whose points are those of π and whose lines are defined as the g-images of the lines of π. πg and π share the net M and the deficiency of M is what Ostrom [23] terms critical. Now since π and πg both extend M, it follows from Ostrom [23] that either π and πg are the same plane and hence g is a collineation group of π or πg is derived from π. However, it would then follow that N is the corresponding derivable net. Thus, the net P is extendable to the translation plane πg and by Jungnickel [20] (3.1), g must be a collineation group of the derived plane which fixes the derived net so that g is, in turn, a collineation of π.

Now let π_1 and π_2 be two nonisomorphic translation planes of order q^2 and kernel $K \cong GF(q)$ which admit central groups G_j for $j = 1, 2$ respectively. Construct the two maximal partial spreads P_j for $j = 1, 2$ as in (2.6). Then, by (4.1), P_1 is not isomorphic to P_2 unless perhaps an isomorphism does not map one Baer subplane set B_1 onto the other Baer subplane set B_2 (see (4.1)). If an isomorphism does not map the Baer subplane sets together and if we have a maximal central group then the situation becomes quite pathological. It may be possible to use group theory to weed out some of these cases.

To illustrate this, we consider the maximal elation type 1 situation for even order.

Let π denote a translation plane of order q^2 and kernel $K \cong GF(q)$ that admits an elation group E of order q and which leaves a Baer subplane π_O invariant such that the net N_{π_O} defined by the parallel classes of π_O contains exactly one Baer subplane which is a K-subspace. Let P denote the partial spread of components of π which are not in N_{π_O} union π_O. Then, by (2.6), P is a maximal partial spread in $PG(3, K)$. Further, the normalizer of E in the collineation group of P must leave π_O invariant and thus induce, by (4.1), a collineation group of the translation plane π. E is the maximal elation group of π with axis equal to the axis of E which can act as a collineation group on P. Thus, if a collineation g of P which fixes the zero vector O does not normalize E then π_O must be moved by g. Note that E^g can fix exactly one of the $q^2 - q + 1$(2-dimensional subspaces) elements of P. Let F denote the full group of P which fixes the zero vector O. Then $N_F(E) \cap E^g = \langle 1 \rangle$ for all $g \in F - N_F(E)$.

Now assume that q is even. Then $|E|$ is even and we may apply Hering's results on trivial normalizer intersection [6]. It follows direclty from Hering that the normal closure of E is either $SL(2, q), S_z(q), PSU(3, q)$, or $SU(3, q)$. In each of these latter three cases, the number of Sylow 2-subgroups is larger than $q^2 - q + 1$. Let S denote a Sylow 2-subgroup which contains E. Then S fixes the zero vector O and has order 2^{2r} for some integer r, and the number of elements of P is $q^2 - q + 1$ so it follows that S must fix en element of P. It follows that S fixes exactly one element of P. But, this is a contradiction unless perhaps the normal closure is $SL(2,q)$. In this case, there are exactly $q + 1$ elements of P which are fixed by Sylow 2-groups of $SL(2,q)$ so tht the group must permute the remaining $q(q-2)$ elements of P. Let ρ denote an element of prime order dividing $q - 1$ in $SL(2,q)$. Then ρ must fix one of these remaining elements. Now since no 2-element can fix any element of this set R, we must have that the stabilizer of a point T in R is cyclic of order dividing $q - 1$. Hence, there are at least 1/2 $(q(q+1))$ points in R which are fixed by groups of order $|\rho|$. And, no two of these points can be equal. Let $R\_$ denote the remaining set of $q^2 - 2q - q(q+1)/2 = (q^2 - 1 + (1-q) - 4q)/2$ elements. Since $|\rho|$ cannot divide this integer, the above argument may be applied so that there is another set of $(q(q+1)/2)$ elements (i.e. ρ must fix at least two elements) within $R\_$ which cannot be the case. Hence, it must be that the normalizer of E is the full group F so that the full collineation group of P must leave invariant the Baer subplane π_O and thus induce a collineation group in π.

So, we have the following result:

Theorem 4.2. *Let E_1 and E_2 be central groups of elation type 1 acting on translation planes π_1 and π_2 respectively. Let P_1 and P_2 denote the maximal partial spreads of $PG(3,q)$ and deficiency q constructed respectively via E_i for $i = 1,2$ as in (2.6). Assume further that q is even. Then P_1 is isomorphic to P_2 only if the translation planes π_1 and π_2 are isomorphic.*

Proof. By the above argument, the full collineation group of P_2 must leave invariant the Ber subplane π_O^2 of π_2 used in the construction. Thus, clearly, any isomorphism from P_1 onto P_2 must map the Baer subplane π_O^1 of π_1 onto π_O^2. It follows that there is an induced isomorphism from π_1 onto π_2.

Corollary 4.3. *The three maximal partial spreads in $PG(3,8)$ of deficiency 8 of maximal elation type 1 arising fromthe Lüneburg-Tits plane, the Biliotti-Menichetti plane, and the Jha-Johnson plane each of order 64 are mutually nonisomorphic.*

5. MORE CENTRAL GROUPS

Actually, the construction (2.6) uses the central group to locate the particular Baer subplanes used in the construction and is not actually essential to the construction. The following result of the author shows where to look for such Baer subplanes.

Theorem 5.1. *(see Johnson [17]). Let π denote a translation plane of order q^4 admitting a collineation group isomorphic to $SL(2,q)$ where for $q = p^r$ and p a prime, the p-elements are shears. Then there exists a derivable net N containing the set of shears axes which is left invariant by $SL(2,q)$.*

Now, we may use the construction method noted in Jungnickel [20] (section 6) to construct maximal partial spreads in some projective geometry provided the derivable net contains Baer subplanes incident with the zero vector which are not kernel subspaces.

Note that in this situation, we would have a central group of order q acting on a translation plane of order q^4.

This actually occurs. That is, there are Foulser-Ostrom planes (see [5]) of odd order q^4 and kernel $GF(q^2)$ admitting $SL(2,q)$ where the p-elements are elations. And, there are derivable nets N containing the $q+1$ elation axes such that not all of the Baer subplanes incident with the zero vector are $GF(q^2)$-subspaces. Hence:

(5.2). There are Foulser-Ostrom planes of odd order q^4 that produce maximal partial spreads in $PG(3,q^2)$ of deficiency q^2-1.

In section 4, we mentioned the question of isomorphism of maximal partial spreads. In this above situation, this poses no problem by (4.0) since these maximal partial spreads are imbeddable.

There are two additional planes worth mentioning here.

In [15], the author shows that there are exactly three mutually nonisomorphic translation planes of order 16 which may be derived from the semifield plane of order 16 and kernel $GF(4)$ (including the so-called Lorimer-Rahilly, Johnson-Walker planes of order 16 admitting $PSL(2,7)$ (see [16])). Since each of these three planes have kernel $GF(2)$, it is clear that they may be constructed by derivable nets not all Baer subplanes of which are $GF(4)$-subspaces.

Thus:

Note 5.3. *There are three mutually nonisomorphic maximal partial spreads in $PG(3,4)$ of deficiency 3 constructed from the same semifield plane of order 16 and kernel $GF(4)$.*

These maximal partial spreads also appear in Bruen and Thas [3].

REFERENCES

[1] M. BILIOTTI, G. LUNARDON, *Insiemi di derivazione e sottopiani di Baer in un piano di traslazione*, Atti Accad. Naz. Lincei. Cl. Sci. Fis. Mat. Natur. (8) **69** (1980).

[2] M. BILIOTTI, V. JHA, N.L. JOHNSON, G. MENICHETTI, *A structure theory for translation planes of order q^2 that admit a collineation group of order q^2*, Geom. Dedicata. 29 (1989), 7-43.

[3] A. BRUEN, J.A. THAS, *Partial spreads, packings and Hermetian manifolds in $PG(3,q)$*,, Math. Z. **151** (1976) 207-214.

[4] D.A. FOULSER, *Subplanes of partial spreads in translation planes*, Bull. London Math. Soc.. **4** (1972) 32, 38.

[5] D.A. FOULSER, *A translation plane admitting Baer collineations of order p*, Arch. Math. **24** (1973) 323-326.

[6] Ch. HERING, *On subgroups with trivial normalizer intersection*, J. Algebra. **20** (1972) 622-629.

[7] N.L. JOHNSON, *A note on translation planes of large dimension*, J. Geometry. **33** (1988) 66-72.

[8] N.L. JOHNSON, *Lezioni sui piani di traslazione*, Quaderni di Matematica, Lecce,. **3** (1986) 1-121.

[9] N.L. JOHNSON, *Sequences of derivable translation planes*, Osaka J. Math. **25** (1988) 66-72.

[10] N.L. JOHNSON, *Translation planes admitting Baer groups and partial flocks of quadric sets*, Simon Stevin. Vol. 63, no. 3, (1989), 163-187.

[11] N.L. JOHNSON, *Flocks and partial flocls of quadric sets*, Contem. Math. III (1990), 111-116.

[12] N.L. JOHNSON, *Ovoids and translation planes revisited*, Geom. Dedicata (submitted).

[13] N.L. JOHNSON, *Translation planes of large dimension*, IX-Elam Conf. Santiago 88 (to appear).

[14] N.L. JOHNSON, *A note on the derived semifield planes of order 16*,, Aequationes Math. **18** (1978) 103-111.

[15] N.L. JOHNSON, *The translation planes of order 16 that admit nonsolvable collineation groups*, Math. Z. **185** (1984) 355-372.

[16] N.L. JOHNSON, *On Desarguesian extensions of elation nets*, J. Geometry 23 (1984) 72-77.

[17] N.L. JOHNSON, R. POMAREDA, *A maximal partial flock of deficiency one of the hyperbolic quadric in $PG(3,9)$*, Simon Stevin, vol. 64, no. 2 (1990), 169-177.

[18] N.L. JOHNSON, R. POMAREDA, F.W. WILKE, *j-planes*, J. Comb. Theory, vol. 56, no. 2 (1991), 271-284.

[19] D. JUNGNICKEL, *Maximal partial spreads and translation nets of small deficiency*, J. Algebra **90** (1984) 119-132.

[20] W.M. KANTOR, *Ovoids and translation planes*, Canad. J. Math. **34** (1982) 1195-1207

[21] H. LÜNEBURG, *Translation planes*, Springer, 1980.

[22] T.G. OSTROM, *Nets with critical deficiency*, Pacific J. Math. **14** (1964) 1381-1387.

Received March 21, 1989.
Norman L. Johnson
Mathematics Dept.
University of Iowa
Iowa City, Iowa 52242
USA