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THE IDENTITY $L(E,F) = LB(E,F)$, TENSOR PRODUCTS AND INDUCTIVE LIMITS

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The purpose of this article is to extend the study of pairs of locally convex spaces (E,F) such that every continuous linear map from E into F is <u>bounded</u> (i.e. maps a 0-nghb in E in a bounded subset of F), and to give applications of our results in this direction to the interchangeability of inductive limits and projective or injective tensor products and to the projective description of weighted inductive limits of spaces of vector valued continuous functions. Our investigations are related to the study of certain classes of Fréchet spaces which have been relevant in several areas recently (see [2], [3], [13], [24], [25], [37], [45]).

Our notation for locally convex spaces (l.c.s.) is standard and we refer the reader to [33], [34], [40]. Our notation for Köthe echelon spaces is as in [11]. For a l.c.s. E, $cs(E)$ denotes the set of all continuous seminorms on E and $\mathcal{U}_0(E)$ is the basis of all absolutely convex 0-nghbs in E. If E and F are l.c.s., we let $L(E,F)$ (resp. $LB(E,F)$) denote the space of all continuous linear maps (resp. bounded linear maps) from E into F. According to [21], a pair (E,F) of l.c.s. is said to satisfy the <u>*localization property*</u> if, for every equicontinuous subset H of $L(E,F)$, there is a 0-nghb U in E such that $H(U) := \cup\{T(U) : T \in H\}$ is bounded in F (i.e., H is equibounded).

The study of the localization property was iniciated by Grothendieck [29]. He showed that if E is (DF) and F is a Fréchet space (or if E is Fréchet and F is (DF)), then (E,F) has the localization property. Several authors have extended the results of Grothendieck (e.g. Defant, Floret [21], S. Dierolf [23], Ruess [41]). In these extensions the «symmetric» case was not treated. Vogt [44] characterized the pairs (E,F) of Fréchet spaces such that $L(E,F) = LB(E,F)$ (or, in this case, equivalently satisfying the localization property). Recently Bonet [16] and Terzioglu [42] complemented the results of Vogt analysing the relevance in this setting of quojections and Fréchet spaces not satisfying property (*) of Bellenot and Dubinsky (c.f. [3]).

When one studies the maximal class of l.c.s. E (or F) such that $L(E,F) = LB(E,F)$ for certain fixed classes of spaces F (or E), one sees immediately that quojections (i.e.; surjective limits of Banach spaces, c.f. [3], [25]) and l.c.s. with the countable neighbourhood property, c.n.p. (i.e.; $\forall(p_n) \subset cs(E) \exists p \in cs(E), \lambda_n > 0, n \in N$, with $p_n \leq \lambda_n p, n \in N$; c.f. [14], [26]), appear very often.

In section 1, we introduce two classes of l.c.s. The first one (l.c.s. satisfying the quotient c.n.p.) containing quojections and l.c.s. satisfying the c.n.p. and the second one (called countable generalized prequojections) which contains the Fréchet spaces which do not satisfy the property (*) of Bellenot and Dubinsky and the l.c.s. with the c.n.p. Their relevance for the identity $L(E,F) = LB(E,F)$ is established in 1.4 and 1.11.

In section 2 we treat the incidence of the identity $L(E,F) = LB(E,F)$ in the interchangeability of projective and injective tensor products and countable (always assumed separated) inductive limits. Grothendieck proved that if E has the c.n.p. and $F = indF_n$, then $E \otimes_\pi F = ind(E \otimes_\pi F_n)$ holds topologically. The first author observed in [14] that this topological identity for a strict (LF)-space F already implies that E has the c.n.p. Several pairs (E,F) for which the topological identity holds without assuming the c.n.p. in E were given in [15], [19]. This problem is throughly studied in Section 2 (see 2.2, 2.5, 2.7). Concerning the injective topology we complement the study of Hollstein [31], [32] with several remarks.

Section 3 is devoted to the problem of algebraic and topological projective description of weighted inductive limits of spaces of vector valued continuous functions. The algebraic identity is completely characterized for Fréchet valued functions or sequences in terms of suitable modifications of a condition of Vogt [44]. A very general projective description result, which implies a result on commutativity for the injective topology, is obtained in 3.5. For a survey on the relevance of projective description of weighted inductive limits we refer to [9].

1. TWO CLASSES OF LOCALLY CONVEX SPACES AND THE IDENTITY $L(E,F) = LB(E,F)$

We start with the following definition.

Definition 1.1. (i) a l.c.s. E satisfies the *quotient countable neighbourhood property* (q.c.n.p.) if every quotient of E with a continuous norm satisfies the c.n.p. or equivalently, if $\forall U \in \mathscr{U}_0(E), \forall (U_n) \subset \mathscr{U}_0(E) \exists V \in \mathscr{U}_0(E) \exists \lambda_n > 0 (n \in N)$ with $V \subset \lambda_n U_n + \text{Ker } p_U, n \in N$ (here p_U is the Minkowski functional of U).

(ii) A l.c.s. E is said to be a *countable generalized prequojection* (countable g-prequojection) if $\forall U \in \mathscr{U}_0(E), \forall (U_n) \subset \mathscr{U}_0(E) \exists V \in \mathscr{U}_0(E) \exists \lambda_n > 0 (n \in N)$ with $E'_{U^0} \cap U_n^0 \subset \lambda_n V^0, n \in N$ (here E'_{U^0} is the span of U^0 in E').

Clearly a Fréchet space E is a quojection if and only if it satisfies the q.c.n.p., and every l.c.s. E with the c.n.p. (hence every (DF) and (gDF)-space) also has the q.c.n.p. Nachbin [39] in the study of holomorphic factorization introduced the following definition. A l.c.s. E has the *openess condition* if $\forall U \in \mathscr{U}_0(E) \exists V \in \mathscr{U}_0(E) \forall W \in \mathscr{U}_0(E)$ $\exists \rho > 0$ with $V \subset \rho W + \text{Ker } p_U$ (i.e.; $E/\text{Ker}p$ is normable for every $p \in cs(E)$). Certainly if E has the openess condition, then it has the q.c.n.p. We recall from [39] that the following spaces satisfy the openess condition. (i) l.c.s. endowed with the weak topology, (ii) $C(X,F)$ endowed with the compact open topology if X is completely regular and Hausdorff and F a normed space, (iii) $L^p_{loc}(\mu, F)$ for every positive Radon measure μ on a topological space X and every normed space $F (1 \leq p \leq \infty)$, (iv) $C^m(U,F)$ for every non-void open subset U of $\mathbb{R}^n$ and every normed space F. As in [15], if U is an open subset of $\mathbb{R}^n$, $\mathscr{D}'(U)$ satisfies the q.n.c.p. (it is isomorphic to a countable product of (DF)-spaces (c.f. [43], [46])) but it satisfies neither the openess condition nor the c.n.p.

According to Terzioglu [42], a l.c.s. E is said to satisfy condition (b) if $\forall U \in \mathscr{U}_0(E)$ $\exists V \in \mathscr{U}_0(E) \forall W \in \mathscr{U}_0(E) \exists \rho > 0$ with $E'_{U^0} \cap W^0 \subset \rho V^0$. A Fréchet space E satisfies (b) if and only if E does not satisfy condition (*) of Bellenot and Dubinsky, [3]. By [45; 0.3] this is, in turn, equivalent to E'' being a quojection. The l.c.s. with property (b) are called generalized prequojections in [35]. Clearly every generalized prequojection is a countable g-prequojection. Taking polars it is easy to see that if E has the q.c.n.p., then E is a countable g-prequojection, and that these two conditions coincide if E is reflexive. In general the converse is not tue. Fréchet countable g-prequojections are precisely the ones which do not satisfy condition (*) of Bellenot and Dubinsky. We refer to [2] or [38] for examples of non-normable countable g-prequojections with a continuous norm. We will mention later completely different examples of metrizable (non-complete) spaces which are countable g-prequojections but do not satisfy the q.n.c.p. Despite of that we have the following result (observe that it is not assumed that the l.c.s. has a continuous norm, and compare with [35; Theorem] and [42; Prop. 2]).

Proposition 1.2. *If a countable g-prequojection E has the bounded approximation property (b.a.p.), then E has the q.c.n.p.*

Proof. Let $(f_\alpha : \alpha \in D)$ be an equicontinuous net in $E' \otimes E \subset L(E,E)$, converging pointwisely to the identity on E. We fix $p \in cs(E)$. To show that $E/\mathrm{Ker}p$ has the c.n.p. we take $(A_n) \subset \mathscr{U}_0(E/\mathrm{Ker}p)$ and select, for each $n \in N, U_n \in \mathscr{U}_0(E)$ with $\pi_p(U_n) \subset A_n$, where $\pi_p : E \to E/\mathrm{Ker}p$ is the canonical surjection. We will find $V \in \mathscr{U}_0(E)$ and $\lambda_n > 0$ such that $\pi_p(V) \subset \lambda_n \pi_p(U_n)$ for all $n \in N$.

There is $q \in cs(E)$ with $p(f_\alpha(x)) \leq q(x)$ for all $x \in E, \alpha \in D$. Each f_α induces a continuous linear map $g_\alpha : E/\mathrm{Ker}q \to E/\mathrm{Ker}p, g_\alpha(\pi_q(x)) := \pi_p(f_\alpha(x))$, and $(g_\alpha(\pi_q(x) : \alpha \in D)$ converges to $\pi_p(x)$ for all $x \in E$. We put $U := \{x \in E : q(x) \leq 1\}$. For each $n \in N$ we find $V_n \in \mathscr{U}_0(E), V_n \subset U$, with $g_\alpha(\pi_q(V_n)) \subset \pi_p(U_n)$ for all $\alpha \in D$. By assumption, there are $V \in \mathscr{U}_0(E)$ and $\lambda_n > 0$ with $E'_{U^0} \cap V_n^0 \subset \lambda_n V^0, n \in N$.

Since each f_α has finite dimensional range, there is a linearly independent set $(\pi_p y_{n,\alpha} : 1 \leq n \leq N_\alpha) \subset E/\mathrm{Ker}p$ and $(y^*_{n,\alpha} : 1 \leq n \leq N_\alpha) \subset E'$ such that

$$g_\alpha(\pi_q(z)) = \sum_{n=1}^{N_\alpha} y^*_{n,\alpha}(z)\pi_p(y_{n,\alpha}) \quad \textit{for all} \quad z \in E.$$

Put $W_n := \pi_p(U_n)$. Given $\alpha \in D, x \in V, n \in N$ one has that

$$p_{W_n}(g_\alpha(\pi_q(x)) = sup(|\langle g_\alpha(\pi_q(x), v\rangle| : v \in W_n^0) =$$

$$\sup\left(\left|\sum_{n=1}^{N_\alpha} y_{n,\alpha}^*(x)v(\pi_p y_{n,\alpha})\right| : v \in W_n^0\right) =$$

$$\sup\left(\left|\left\langle\sum_{n=1}^{N_\alpha} v(\pi_p y_{n,\alpha})y_{n,\alpha}^*, x\right\rangle\right| : v \in W_n^0\right) \leq$$

$$\sup\left(p_{V^0}\left(\sum_{n=1}^{N_\alpha} v(\pi_p y_{n,\alpha})y_{n,\alpha}^*\right) : v \in W_n^0\right).$$

Now $(y_{n\alpha}^* : \alpha \in D, 1 \leq n \leq N_\alpha) \subset E'_{U^0}$. Indeed, the norm induced by p on E/Kerp and $\sum(\beta_n \pi_p y_{n\alpha} : 1 \leq n \leq N_\alpha) \to sup(|\beta_n| : 1 \leq n \leq N_\alpha)$ are equivalent norms on Img_α. Hence there is $C_\alpha > 0$ with $|y_{n\alpha}^*(x)| \leq C_\alpha p(g_\alpha(\pi_q x)) = C_\alpha p(f_\alpha(x)) \leq C_\alpha q(x)$ for all $x \in E$. This implies $y_{n\alpha}^* \in E'_{U^0}$.

Finally we have

$$p_{W_n}(g_\alpha(\pi_q x)) \leq sup\left(p_{V^0}\left(\sum_{n=1}^{N_\alpha} v(\pi_p y_{n,\alpha})y_{n,\alpha}^*\right) : v \in W_n^0\right) \leq$$

$$\lambda_n \sup\left(p_{V_n^0}\left(\sum_{n=1}^{N_\alpha} v(\pi_p y_{n,\alpha})y_{n,\alpha}^*\right) : v \in W_n^0\right) =$$

$$\lambda_n \sup\left(\left|\left\langle\sum_{n=1}^{N_\alpha} v(\pi_p y_{n,\alpha})y_{n,\alpha}^*, z\right\rangle\right| : v \in W_n^0, z \in V_n\right) =$$

$$\lambda_n \sup\left(\left|\left\langle v, \sum_{n=1}^{N_\alpha} y_{n,\alpha}^*(z)\pi_p y_{n,\alpha}\right\rangle\right| : v \in W_n^0, z \in V_n\right) =$$

$$\lambda_n \sup(|\langle v, g_\alpha(\pi_q z)\rangle| : v \in W_n^0, z \in V_n) \leq \lambda_n$$

since

$$g_\alpha(\pi_q(V_n)) \subset \pi_q(U_n) := W_n$$

Consequently,

$$p_{W_n}(\pi_p x) \leq \lambda_n$$

for every $n \in N, x \in V$, which implies $\pi_p(V) \subset \overline{\lambda_n \pi_p(U_n)} \subset \lambda_n A_n$ and E/Kerp has the c.n.p. ∎

Remark 1.3. As a consequence we obtain that a Fréchet space with the b.a.p. is a quojection if and only if it does not satisfy property (*). This is also a consequence of [35; Theorem] or [42; Prop. 2], if E has a continuous norm.

The next result should be compared with the characterization of the openess condition given by Terziogly [42; Prop. 1].

Proposition 1.4. *For a l.c.s. E the following conditions are equivalent (TFAE):*

(i) *(E,F) has the localization property for every metrizable space F with a continuous norm.*

(ii) *$L(E,F) = LB(E,F)$ for every metrizable space F with a continuous norm.*

(iii) *E satisfies the q.c.n.p.*

Proof. (ii) implies (iii). Given $U \in \mathcal{U}_0(E)$ and $(U_n) \subset \mathcal{U}_0(E)$ with $2U_{n+1} \subset U_n \subset U$ for all $n \in N$, we denote by $\pi : E \to E/\mathrm{Kerp}_U$ the canonical surjection. Clearly $(\pi(U_n))$ is a basis of 0-nghb of a metrizable l.c. topology t on E/Kerp_U. Put $F := (E/\mathrm{Kerp}_U, t)$. Clearly F has a continuous norm. By (ii) there is $V \in \mathcal{U}_0(E)$ such that $\pi(V)$ is bounded in F. This implies $V \subset \lambda_n U_n + \mathrm{Kerp}_U$ for all $n \in N$.

(iii) implies (i). Let H be an equicontinuous subset of $L(E,F)$, (W_n) a basis of 0-nghbs in F and $||.||$ a continuous norm in F. For each $n \in N$ select $U_n \in \mathcal{U}_0(E)$ with $T(U_n) \subset W_n$ for all $T \in H$ and find $U \in \mathcal{U}_0(E)$ with $||T(x)|| \leq 1$ for all $x \in U, T \in H$. By (iii) we find $V \in \mathcal{U}_0(E), \lambda_n > 0$ with $V \subset \lambda_n U_n + \mathrm{Kerp}_U$ for all $n \in N$. Therefore $T(V) \subset \lambda_n T(U_n) \subset \lambda_n W_n$ for all $n \in N, T \in H$ and H is equibounded. ■

The former proposition and some standard arguments are useful to prove the following hereditary properties of the classes considered above. For the class of countable g-prequojections direct arguments are needed.

Proposition 1.5. *(i) The class of spaces satisfying the q.c.n.p. is stable with respect to the formation of*

(1) *separated quotients*

(2) *countable direct sums*

(3) *arbitrary products*

(4) *finite codimensional subspaces* (here it is useful to have in mind [40; 2.6.18]).

(5) *if F has the q.c.n.p. and F is dense in E, then E has the q.c.n.p.*

(ii) *The class of countable g-prequojections is stable with respect to the formation of*

(1) *separated quotients*

(2) *countable direct sums*

(3) *arbitrary products*

(4) *if F is dense in E, then F is a* countable g-prequojection *if and only if E is a* countable g-prequojection.

None of the classes mentioned above is stable under the formation of uncountable direct sums, since $\oplus(\mathbb{K}, r \in \mathbb{R})$ has a continuous norm and the b.a.p. but it does not satisfy the c.n.p. ([23; p. 16]).

Since every Fréchet space is a (closed) subspace of a countable product of Banach spaces, the classes mentioned above are not stable by passing to a closed subspace.

Not every dense subspace of a space satisfying the q.c.n.p. satisfies this condition. Metafune and Moscatelli ([36]) characterize the Fréchet spaces without continuous norms having a dense subspace with a continuous norm. There exist quojections with a dense subspace with a continuous norm. Any such a subspace is a metrizable countable g-prequojection which does not satisfy the q.c.n.p.

Remark 1.6. If $(E_i : i \in I)$ is a non-void family of l.c.s. with the c.n.p. and F is a subspace of $E := \Pi(E_i : i \in I)$ with $\oplus(E_i : i \in I) \subset F$, then F satisfies the q.c.n.p.

Our next results complement the ones obtained in [16].

Proposition 1.7. *For a l.c.s. E, TFAE:*

(i) *(E, F) has the localization property for every metrizable l.c.s. F.*

(ii) *$L(E,F) = LB(E,F)$ for every metrizable l.c.s. F.*

(iii) *E satisfies the c.n.p.*

Proposition 1.8. *For a l.c.s. F, TFAE:*

(i) *(E, F) has the localization property for every quasibarrelled (DF)-space E.*

(ii) *$L(E,F) = LB(E,F)$ for every quasibarrelled (DF)-space E.*

(iii) *F has the countable boundedness condition (c.b.c., i.e.; for every sequence of bounded subsets (B_n) in F there are $\lambda_n > 0, n \in N$, such that $\cup(\lambda_n B_n : n \in N)$ is bounded in F, see [23]).*

Proposition 1.9. *For a l.c.s. F, TFAE:*

(i) *(E, F) has the localization property for every l.c.s. E with a total bounded set.*

(ii) *$L(E,F) = LB(E,F)$ for every l.c.s. E with a total bounded set.*

(iii) *the linear span of every bounded set of F is normable.*

The relevance of countable g-prequojections in this context is clarified by the next proposition.

Proposition 1.10. *For a l.c.s. F, TFAE:*

(i) *(E, F) has the localization property for every inductive limit of normed spaces $E := \text{ind}\, E_n$ with E_n dense in E_{n+1} for all $n \in N$.*

(ii) *$L(E,F) = LB(E,F)$ for every inductive limit of normed spaces $E := \text{ind}\, E_n$ with E_n dense in E_{n+1} for all $n \in N$.*

(iii) *the linear span of every bounded subset of F satisfies the c.b.c.*

If F is the strong dual of a quasibarrelled space G, then the conditions above are equivalent to

(iv) *G is a* countable g-prequojection.

Proof. The equivalence of (iii) and (iv) if there is a quasibarrelled space G with $F = G'_b$, follows from the following remark.

Remark 1.11. A quasibarrelled l.c.s. G is a countable g-prequojection if and only if the linear span of every absolutely convex bounded subset of G'_b satisfies the c.b.c.

(ii) implies (iii). Suppose that there is a bounded subset B of F such that its linear span H does not satisfy the c.b.c. We can find a sequence (B_n) of absolutely convex bounded subsets of H with $B \subset B_n \subset B_{n+1}$ for all $n \in N$ such that $\cup(c_n B_n : n \in N)$ is not bounded for every sequence (c_n) of positive real numbers. We denote by E_n the space H endowed with the norm defined by the Minkowsky functional of B_n. Then $E := ind\ E_n$ is an inductive limit of normed spaces with E_n dense in E_{n+1} for all $n \in N$. The inclusion $i : E \to F$ satisfies $i \in L(E,F)\backslash LB(E,F)$, which contradicts (ii).

(iii) implies (i). Let $E := ind\ E_n$ be an inductive limit of normed spaces with E_n dense in E_{n+1} for all $n \in N$. Let A_n be the unit ball of E_n and t_n the topology of E_n. If H is an equicontinuous subset of $L(E,F)$ then $\{T|E_1 : T \in H\}$ is an equicontinuous in $L(ind(E_1, t_n), F)$ and $B := H(A_1)$ is bounded in F. By assumption the linear span L of B has the c.b.c. and $(H(A_n \cap E_1); n \in N)$ is a sequence of bounded subsets in L. Hence there are $c_n > 0, n \in N$, with $\cup(c_n H(A_n \cap E_1) : n \in N)$ bounded in L (hence in F). Now $U := acx(\cup c_n(A_n \cap E_1) : n \in N)$ is a 0-nghb in $ind(E_1, t_n)$ and $H(U)$ is bounded in F. Since $ind(E_1, t_n)$ is a dense topological subspace of $ind\ E_n$, H is equibounded in $L(E,F)$. ∎

There are many examples of inductive limits satisfying the assumptions of 1.11(i). See [19; Examples 13].

Remark 1.12. If E and F are quasibarrelled l.c.s. then $L(E, F'_b) = LB(E, F'_b)$ is equivalent to $L(F, E'_b) = LB(F, E'_b)$. In general $L(E,F) = LB(E,F)$ does not imply $L(F'_b, E'_b) = LB(F'_b, E'_b)$. Indeed, take $E = \omega$, F a Fréchet space, non-Banach, with a continuous norm such that F'_b is a strict (LB)-space (c.f. [2], [38]).

2. COMMUTATIVITY OF INDUCTIVE LIMITS AND TENSOR PRODUCTS

Our first result is an extension of [17; Proposition].

Proposition 2.1. *Let $E = ind\ E_n$ be an inductive limit of normed spaces and F a quasibarrelled space TFAE:*

(i) $E \otimes_\pi F = ind(E_n \otimes_\pi F)$ *holds topologically.*

(ii) (E, F'_b) *has the localization property.*

Proof. Let B_n be the unit ball of E_n. Given $T \in L(E, F'_b)$ we denote by $u_T : E \otimes F \to \mathbb{K}$ the linear form $u_T(x \otimes y) := \langle T(x), y \rangle, x \in E, y \in F$.

(i) implies (ii). If H is equicontinuous in $L(E, F'_b)$, the set $(u_T | E_n \otimes_\pi F : u \in H)$ is equicontinuous in $(E_n \otimes_\pi F)'$ for all $n \in N$. By (i), $(u_T : u \in H)$ is equicontinuous in $(E \otimes_\pi F)'$, hence there are $U \in \mathscr{U}_0(E), V \in \mathscr{U}_0(E)$ with $|u_T(x \otimes y)| \leq 1$ for all $x \in U, y \in V, T \in H$. This implies $H(U) \subset V^0$ and H is equibounded.

(ii) implies (i). Let H be an equicontinuous subset of $(ind(E_n \otimes_\pi F))'$. If $u \in H$, define $T_u : E \to F'_b$ by $\langle T_u(x), y \rangle := u(x \otimes y), x \in E, y \in F$. Since $(u | E_n \otimes_\pi F : u \in H)$ is equicontinuous in $E_n \otimes_\pi F$ for all $n \in N$, it follows that $(T_u : u \in H)$ is equicontinuous in $L(E, F'_b)$. By (ii), it is even equibounded, but this implies that H is equicontinuous in $(E \otimes_\pi F)'$. ■

Applying Proposition 2.1 and 1.11 we obtain

Proposition 2.2. *Let E be a quasibarrelled l.c.s. TFAE:*

(i) *E is a* countable g-prequojection.

(ii) *$E \otimes_\pi F = ind(E \otimes_\pi F_n)$ holds topologically for all inductive limit $F = indF_n$ of normed spaces such that F_n is dense in F_{n+1} for all $n \in N$.*

In fact (i) implies (ii) holds without any quasibarrelledness assumption on E, using a direct argument.

Proposition 2.3. *Let E be a Fréchet space. TFAE:*

(i) *E does not satisfy condition (*) of Bellenot and Dubinsky (or E'' is a quojection, or E is a* countable g-prequojection).

(ii) *$E \otimes_\pi F = ind(E \otimes_\pi F_n)$ holds topologically for all inductive limit $F = indF_n$ of normed spaces such that F_n is dense in F_{n+1} for all $n \in N$.*

(iii) *$E \otimes_\pi (ind\ c_0(v_n)) = ind(E \otimes_\pi c_0(v_n))$ holds topologically for any decreasing sequence $\vartheta = (v_n)$ of strictly positive weights on N such that $(ind\ c_0(v_n))'_b$ is nuclear.*

Proof. We only have to prove that *(iii) implies (i).* By [16; 8] or [42; (3)], if the Fréchet space E satisfies property (*) (or equivalently is not a countable g-prequojection), there is a Köthe matrix $A = (a_n)$ such that $\lambda^1(A)$ is nuclear with a continuous norm and $L(E, \lambda^1(A)) \neq LB(E, \lambda^1(A))$. If $v_n(i) := a_n(i)^{-1}, i \in N, n \in N$, then $(ind\ c_0(v_n))'_b = \lambda^1(A)$ ([11; 2, 7]) and $E \otimes_\pi ind\ c_0(v_n)$ does not coincide topologycally with $ind(E \otimes_\pi c_0(v_n))$ by [17; Proposition]. This complete the proof. ■

Now we turn to the commutativity of inductive limits and injective tensor products. Hollstein introduced in [32] the inductive limits with *local partition of unity* and proved that $E \otimes_\varepsilon (ind\ F_n) = ind(E \otimes_\varepsilon F_n)$ holds topologycally for every l.c.s. E with the c.n.p. if and only if $ind\ F_n$ admits a local partition of unity. We also recall from [31] that a l.c.s.

E is an ε-space if and only if for every quotient map $Q : G \to G/H$, the canonical map $id \otimes Q : E \otimes_\varepsilon G \to E \otimes_\varepsilon (G/H)$ is open. We present some partial results for spaces which do not satisfy the c.n.p.

Lemma 2.4. *Let $E := \Pi(E_j : j \in N)$ be a countable product of Banach spaces. Let $F := ind F_n$ be an inductive limit with local partition of unity such that F_n is dense in F_{n+1} $(n \in N)$. Then $E \otimes_\varepsilon F = ind(E \otimes_\varepsilon F_n)$ holds topologically.*

Proof. Let W be a closed absolutely convex 0-nghb in $ind(E \otimes_\varepsilon F_n)$. There are $U \in \mathscr{U}_0(E)$ and $q \in cs(F_1)$ with $\{T \in E \otimes F_1 : q(Tu) \leq 1 \forall u \in U^0\} \subset 2^{-1}W$. If $m \in N$ satisfies $\Pi(E_j : j > m) \subset U$ then $\Pi(E_j : j > m) \otimes F_1 \subset 2^{-1}W$. Proceeding by recurrence, using the density of F_n in F_{n+1} for all $n \in N$, we obtain $\Pi(E_j : j > m) \otimes F_n \subset 2^{-1}W$ for all $n \in N$. Now, since $\Pi(E_j : j \leq m)$ is a Banach space and F admits local partition of unity we obtain $V \in \mathscr{U}_0(\Pi(E_j : j \leq m))$ and $p \in cs(F)$ such that

$$\{T \in (\Pi E_j : j \leq m) \otimes F : p(Tu) \leq 1 \forall u \in V^0\} \subset 2^{-1}W.$$

This implies

$$\{T \in E \otimes F : p(Tu) \leq 1 \forall u \in (V \times \Pi(E_j : j > m))^0\} \subset W$$

and the proof is complete. ∎

Proposition 2.5. *Let E be a quojection. Let $F := ind F_n$ be an inductive limit with local partition of unity such that every F_n is an ε-space dense in F_{n+1}. Then $E \otimes_\varepsilon F = ind(E \otimes_\varepsilon F_n)$ holds topologically.*

Proof. First we observe that F is an ε-space. Indeed, given a quotient map $f : G \to H$ between Banach spaces, then for each $n \in N$

$$f \otimes id_n : G \otimes_\varepsilon F_n \to H \otimes_\varepsilon F_n$$

is open, therefore

$$f \otimes id : ind(G \otimes_\varepsilon F_n) \to ind(H \otimes_\varepsilon F_n)$$

is also open, and, as F admits a local partition of unity this implies that

$$f \otimes id : G \otimes_\varepsilon F \to H \otimes_\varepsilon F$$

is open. Now, according to [18] there is a countable product of Banach spaces X and a quotient map $Q : X \to E$. The diagram

$$
\begin{array}{ccc}
ind(X \otimes_\varepsilon F_n) & \overset{i_1}{\longrightarrow} & X \otimes_\varepsilon F \\
g \Big\downarrow & & \Big\downarrow Q \otimes id \\
ind(E \otimes_\varepsilon F_n) & \overset{i_2}{\longrightarrow} & E \otimes_\varepsilon F
\end{array}
$$

is commutative. The canonical maps $g, Q \otimes id$ are open and i_1 is a topological isomorphism by lemma 2.4. Thus i_2 is also a topological isomorphism. ∎

We now treat another application of the identity $L(E,F) = LB(E,F)$. We study the algebraic identity $E\varepsilon indF_n = ind(E\varepsilon F_n)$. For the ε-product of Schwartz we refer the reader to [34; §44].

Our first result provides a certain converse to a result of Bierstedt and Meise [7; 1.4].

Proposition 2.6. *For an inductive limit $F := indF_n$ of l.c.s., TFAE:*

(i) *every compact absolutely convex set in F is compact in some step F_n*

(ii) *$E\varepsilon F = ind(E\varepsilon F_n)$ holds algebraically for every Banach space E.*

Proof. *(i) implies (ii)* by [7; 1.4]. We prove that *(ii) implies (i)*. Let K be an absolutely convex compact subset of F. $U := K^0$ is a 0-nghb in $(F', \mu(F', F))$. We put E for the completion of the normed space $F'_{(U)}$ (notation as in [33; 8.3]). Then E' is canonically isomorphic to F_K and then E'_{co} coincides with $(F_K, co(F_K, F'_{(U)}))$, where $co(F_K, F'_{(U)})$ is the topology of the uniform convergence on the compact convex subsets of the second space. Since K is the unit ball of E' (cf. [33; proof of 8.3.4]), K is compact in E'_{co}. It is easy to see that the inclusion $E'_{co} \to F$ is continuous, hence it belongs to $E\varepsilon F$. By (ii), there is $n \in N$ such that E'_{co} is continuously included in F_n. Then K is compact in F_n and the proof is complete. ∎

An inductive limit $F := indF_n$ is called compactly regular (see e.g. [39; Chapter 8]) if every compact subset of F is compact in some step F_n. An (LF)-space $F := indF_n$ is compactly regular if and only if it is regular and satisfies 2.6 (i). Indeed, the necessity is obvious. To prove the sufficience, by [20; 3] it is enough to show that F is sequentially retractive. Given a sequence converging to the origin in F its closed absolutely convex hull K is compact, since F is regular. By the condition 2.6 (i), K is compact in some step F_n, hence the zero sequence also tends to the origin in F_n and we are done.

Proposition 2.7. *Let E be a Fréchet space and let $F := \mathrm{ind} F_n$ be a compactly regular (LF)-space. TFAE:*

(i) *$E\varepsilon F = \mathrm{ind}(E\varepsilon F_n)$ holds algebraically.*

(ii) $L(F'_{co}, E) = LB(F'_{co}, E)$

(iii) *For every compact subset H of $E\varepsilon F$ there is $m \in N$ such that H is contained and compact in $E\varepsilon F_m$.*

Proof. (ii) implies (i). Given $T \in L(F'_{co}, E) = E\varepsilon F$, we apply (ii) to obtain an absolutely convex bounded subset B of E such that $T \in L(F'_{co}, E_B) = E_B \varepsilon F$. By [7; 1.4] there is $n \in N$ with $T \in E_B \varepsilon F_n \subset E\varepsilon F_n$.

(i) implies (ii). Given $T \in L(F'_{co}, E)$, we can apply (i) to find $n \in N$ such that $T \in L((F_n)'_{co}, E)$. By [41; 1.7], there is a precompact subset K of F_n such that $T(K^0)$ is precompact in E. In particular $T \in LB(F'_{co}, E)$.

(i) implies (iii). We fix a basis (U_n) of absolutely convex 0-nghbs in E and a compact subset H of $E\varepsilon F$. For each $n \in N$, H is compact in $E_{(U_n)}\varepsilon F$, hence we can apply [22] to find a strictly increasing sequence $(k(n)) \subset N$ such that H is compact in $E_{(U_n)}\varepsilon F_{k(n)}$ for each $n \in N$. Since $E_{(U_n)}\varepsilon F_{k(n)}$ is metrizable we find a compact absolutely convex subset K_n such that H is compact in the Banach space $(E_{(U_n)}\varepsilon F_{k(n)})_{Kn}$. We set L_n for the compact absolutely convex subset $K_n(U_n^0)$ of $F_{k(n)}$. The map $\psi : (E_{(U_n)}\varepsilon F_{k(n)})_{Kn} \to l_\infty(U_n^0, (F_{k(n)})_{Ln}), \psi_n(T) := (T(u) : u \in U_n^0)$, is linear and continuous, consequently $\psi_n(H)$ is compact in $l_\infty(U_n^0, (F_{k(n)})_{L_n})$. We denote by q_n the Minkowski functional of L_n and we put

$$G := \{T \in L(E'_{co}, F) : T(U_n^0) \subset F_{Ln}, r_n(T) := \sup(q_n(Tu) : u \in U_n^0) < \infty \, \forall n \in N\}$$

endowed with the l.c. topology generated by the seminorms (r_n). G is a Fréchet space. Indeed, if (T_m) is a Cauchy sequence in G, on easily defines a linear map $T : E'_{co} \to F$ such that (T_m) converges to T uniformly on each U_n^0, hence $T|U_n^0$ is continuous for each $n \in N$. Since (U_n^0) is a fundamental sequence of bounded subsets of the (gDF)-space E'_{co} (cf. [33; Chapter 12] or [40; Chapter 8]), we conclude that $T \in L(E'_{co}, F), T \in G$ and (T_m) converges to T in G.

By (i), $G \subset \cup(E\varepsilon F_n : n \in N)$. Therefore we can apply Grothendieck's factorization theorem [30; p. 148] to obtain $m \in N$ such that G is continuously included in $E\varepsilon F_m$. We are done if we observe that H is compact in G. This is trivial since $r_n(T)$ coincides with the norm in $l_\infty(U_n^0, (F_{k(n)})_{L_n})$ on $\psi_n(H)$ and $\psi_n(H)$ is compact in this space. ■

We utilize our former proposition to derive the following unpublished result due to Floret. In fact 2.7 was invented as an extension of Floret's result.

Proposition 2.8. *(Floret). Let E be a Fréchet space and let $F := indF_n$ be a (DFN)-space. TFAE:*

(i) $E\hat{\otimes}_\varepsilon F = ind(E\hat{\otimes}_\varepsilon F_n)$ *algebraically and topologically.*

(ii) $L(E,F'_b) = LB(E,F'_b), L(F'_b,E) = LB(F'_b,E)$.

Proof. F has the approximation property and, passing to a suitable equivalent spectrum, we may suppose that every F_n has also the approximation property and that the injection $F_n \to F_{n+1}$ is nuclear for each $n \in N$. In this case $E\varepsilon F = E\hat{\otimes}_\varepsilon F$ and $E\varepsilon F_n = E\hat{\otimes}_\varepsilon F_n$, hence the algebraic identity in (i) is equivalent to $L(F'_b,E) = LB(F'_b,E)$ by 2.7.

On the other hand, since $E \otimes_\pi F = E \otimes_\varepsilon F$ and $ind(E \otimes_\varepsilon F_n) = ind(E \otimes_\pi F_n)$ we have that the topological identity in (i) is equivalent to the topological identity $E \otimes_\pi F = ind(E \otimes_\pi F_n)$, which in turn is equivalent to $L(E,F'_b) = LB(E,F'_b)$ by [17; Proposition]. ∎

3. PROJECTIVE DESCRIPTIONS OF WEIGHTED INDUCTIVE LIMITS OF SPACES OF CONTINUOUS FUNCTIONS AND SEQUENCES

First of all we recall the basic definitions. In this section X will denote a completely regular Hausdorff topological space, $\vartheta = (v_n)$ a decreasing sequence of strictly positive continuous functions (i.e., weights) on X. We denote by $\overline{V} = \overline{V}(\vartheta)$ the maximal Nachbin family associated to ϑ.

$$\overline{V} := \left\{\overline{v} : X \to R_+ \, upper\ semicontinuous : \ sup\left(\frac{\overline{v}(x)}{v_n(x)} : x \in X\right) < \infty \ \forall n \in N\right\}$$

For a l.c.s. E, we define

$$Cv_n(X,E) := \{f \in C(X,E) : (v_n f)(X) \, is\ bounded\ in\ E\}$$

$$C(v_n)_0(X,E) := \{f \in Cv_n(X,E) : p \circ (v_n f) \, vanishes\ at\ infinity\ on\ X \quad for\ every\ p \in cs(E)\}$$

both endowed with the l.c. topology generated by the system of seminorms

$$q_{v_n,p}(f) := \sup(v_n(x)p(f(x)) : x \in X), f \in Cv_n(X,E), p \in cs(E).$$

The weighted inductive limits are defined by (cf. [10])

$$\vartheta C(X,E) := ind\, Cv_n(X,E), \vartheta_0 C(X,E) := ind\, C(v_n)_0(X,E).$$

The projective hulls associated to these inductive limits are

$$C\overline{V}(X,E) := \{f \in C(X,E) : (\overline{v}f)(X) \, is \, bounded \, in \, E \; \forall \overline{v} \in \overline{V}\}$$

$$C\overline{V}_0(X,E) := \{f \in C(X,E) : p \circ (\overline{v}f) \; vanishes \; at \; infinity \; on \; X \; \forall \overline{v} \in \overline{V} \quad \forall p \in cs(E)\}$$

with the topology given by all the seminorms $q_{\overline{v},p}, \overline{v} \in \overline{V}, p \in cs(E)$.

If $E = \mathbb{K}(\mathbb{R} \, or \, \mathbb{C})$, we drop it from our notations and we write $Cv_n(X), \vartheta C(X)$, etc.

The basic problem of projective description as stated in [10; 0.5] is to determine when (i) $\vartheta C(X,E) = C\overline{V}(X,E)$, (ii) $\vartheta_0 C(X,E) = C\overline{V}_0(X,E)$ hold algebraically or topologically. We will concentrate in the case of a Fréchet space E. The scalar cases were treated in [10]. Extensions to spaces E with the c.n.p. can be found in [12]. In [15; 2.2] it is proved that if X is locally compact (ii) holds if and only if

$$E \otimes_\varepsilon (ind \, C(v_n)_0(X)) = ind(E \otimes_\varepsilon C(v_n)_0(X))$$

holds topologically, which gives the relation of this question with our study in Section 2. The topological identity in problem (i) turned out to be much more complicated. In the scalar case the complete solution is X is discrete can be seen in [5]. More information is given in [6] and [1].

If X is discrete, i.e., and index set $X = I$, our notation is as in [5], [6] and [11], $\vartheta C(X,E) = k_\infty(E), \vartheta_0 C(X,E) = k_0(E), C\overline{V}(X,E) = K_\infty(E), C\overline{V}_0(X,E) = K_0(E)$. If $1 < p < \infty$, the spaces $k_p(E) = ind \; l_p(v_n, E)$ and $K_p(E) = K_p(\overline{V}, E)$ are defined in a canonical way (see [5] and [6]).

We start with the following observation which gives the key of our results on the algebraic identity in (i) and (ii). Let E be a Fréchet space with an increasing fundamental sequence of seminorms $(||.||_k)$. Let $A = (a_n)$ be a Köthe matrix on N. We define $\vartheta = (v_n), v_n(i) := a_n(i)^{-1}$. By [6; 2.2], the mapping $L(\lambda^1(A),E) \to K_\infty(E)$, given by $T \to (Te_i)_i$, is an algebraic isomorphism, then the algebraic identity $K_\infty(E) = k_\infty(E)$ is equivalent to $L(\lambda^1(A),E) = LB(\lambda^1(A),E)$ and, by [44; 1.3] this is in turn equivalent to

$$\forall (k(l)) \uparrow \exists k \, \forall n \, \exists l_0 = l_0(n), C_n > 0 :$$

$$||e||_n v_k(i) \leq C_n \max(||e||_l v_{k(l)}(i) : 1 \leq l \leq l_0) \forall i \in N, e \in E.$$

Although the isomorphism $L(\lambda^1(A),E) \cong K_\infty(E)$ is not longer true for $K_p(E)$, $p \neq \infty$, we will see that adecuate reformulations of the above condition provide the right conditions.

Proposition 3.1. *Let E be a Fréchet space with a increasing fundamental sequence of seminorms $(\|.\|_l)$. Let $\vartheta = (v_k)$ be a decreasing sequence of strictly positive weights on an index set I, and let $1 \leq p \leq \infty$. TFAE:*

(i) *for every bounded subset B of $K_p(E)$ there is $n \in N$ such that B is bounded in $l_p(v_n, E)$.*

(ii) $K_p(E) = k_p(E)$ *algebraically.*

(iii) $\forall (k(l)) \uparrow \exists k \, \forall n \, \exists l_0 = l_0(n), C_n > 0 : \|e\|_n v_k(i) \leq C_n \max(\|e\|_l v_{k(l)}(i) : 1 \leq l \leq l_0) \forall i \in N, e \in E$.

Proof. (for $p \neq \infty$): *(ii) implies (iii).* Given the strictly increasing sequence $(k(l))$ in N, we set

$$G := \left\{ x = (x(i)) \in K_p(E) : \pi_l(x) := \left(\sum_i (v_{k(l)}(i) \|x(i)\|_l)^p \right)^{\frac{1}{p}} < \infty \quad \forall l \in N \right\}$$

endowed with the metrizable l.c. topology given by the seminorms (π_l). G is certainly a Fréchet space and, by (ii), $G \subset \cup(l_p(v_n, E) : n \in N)$. We apply Grothendieck's factorization theorem [30; p. 148] to obtain $k \in N$ such that G is continuously injected in $l_p(v_k, E)$. Hence for every $n \in N$ there are $C_n > 0$ and $l_0 \in N$ with

$$\left(\sum_i (v_k(i) \|x(i)\|_n)^p \right)^{\frac{1}{p}} \leq C_n \max \left(\left(\sum_i (v_{k(l)}(i) \|x(i)\|_l)^p \right)^{\frac{1}{p}} : 1 \leq l \leq l_0 \right)$$

Condition (iii) follows from this.

(iii) implies (i). Let B be a bounded subset of $K_p(E)$. For each $l \in N$ the set $\{(\|x(i)\|_l)_i : x \in B\}$ is bounded in $K_p(\overline{V})$. We can apply [11; 2.3] to obtain a strictly increasing sequence $(k(l))$ in N such that

$$M_l := \sup \left(\sum_i (v_{k(l)}(i) \|x(i)\|_l)^p : x \in B \right) < \infty.$$

Let $k, C_n > 0$ and $l_0(n) (n \in N)$ be given by condition (iii). Then for every $n \in N$ and $x \in B$ we have

$$\sum_i (v_k(i) \|x(i)\|_n)^p \leq C_n \sum_{l-i}^{l_0} M_l$$

This implies that B is contained and bounded in $l_p(v_k, E)$. ∎

Corollary 3.2. *(a) Let E be a Fréchet space without a continuous norm and $A = (a_n)$ a Köthe matrix such that $\lambda^1(A)$ is not normable. For $v = (v_n), v_n(i) := a_n(i)^{-1}$, we have $K_\infty(E) \neq k_\infty(E)$ algebraically.*

(b) *If a Fréchet space E satisfies $K_\infty(E) = k_\infty(E)$ for every decreasing sequence of strictly positive weights on N, then E is normable.*

Proof. (a) By a well-known result of Eidelheit, there is a closed subspace H of $\lambda^1(A)$ such that $\lambda^1(A)/H \cong \omega$. Let $q : \lambda^1(A) \to \omega$ be the quotient map, and $i : \omega \to E$ a topological isomorphism into, which exist because E does not have a continuous norm. Then $i \circ q \in L(\lambda^1(A), E)$ but is not bounded. Consequently $K_\infty(E) \neq k_\infty(E)$.

(b) By (a), E certainly admits a continuous norm, hence it is not isomorphic to $X \times \omega$, with X a Banach space. If E is not normable we can apply a result of Bessaga, Pelczynski, Rolewitz [4] to obtain a subspace F of E isomorphic to a Köthe nuclear space $\lambda^1(A)$ with a continuous norm. This implies $L(\lambda^1(A), E) \neq LB(\lambda^1(A), E)$. ∎

Theorem 3.3. *Let E be a Fréchet space with an increasing fundamental sequence of continuous seminorms $(\|.\|_l)$. Let X be a locally compact space and $\vartheta = (v_k)$ a decreasing sequence of strictly positive continuous weights on X. TFAE:*

(i) *every bounded subset of $C\overline{V}(X,E)$ is contained and bounded in some $C(v_k)(X,E)$.*

(ii) $\vartheta C(X,E) = C\overline{V}(X,E)$

(iii) $\forall (k(l)) \uparrow \; \exists k \, \forall n \, \exists l_0 = l_0(n), C_n > 0$:
$\|e\|_n v_k(x) \leq C_n \max(\|e\|_l v_{k(l)}(x) : 1 \leq l \leq l_0) \; \forall e \in E, x \in X$

Proof. The proof of *(iii) implies (i)* is analogous to the corresponding one in proposition 3.1, using that every bounded subset of $C\overline{V}(X)$ is contained and bounded in some $Cv_n(X)$ and that $C\overline{V}(X,E)$ is complete.

(ii) implies (iii). Given the sequence $(k(l))$ in N we set

$$G := \{f \in C\overline{V}(X,E) : P_l(f) := \sup(v_{k(l)}(x)\|f(x)\|_l : x \in X) < \infty \; \forall l\}$$

endowed with the metrizable l.c. topology given by the sequence of seminorms (P_l). G is a Fréchet space. By (ii), $G \subset \cup(C(v_k)(X,E) : k \in N)$ and Grothendieck's factorization theorem yields $k \in N$ such that G is continuously included in $C(v_k)(X,E)$. Hence for each $n \in N$ there are $C_n > 0$ and $l_0(n) \in N$ with

$$(1) \; \sup(v_k(x)\|f(x)\|_n : x \in X) \leq C_n \max(\sup(v_{k(l)}(x)\|f(x)\|_l : x \in X) : 1 \leq l \leq l_0)$$

for all $f \in G$.

To prove (iii) we now fix $n \in N, x_0 \in X, e \in E$ with $\|e\|_n \neq 0$. Taking $l_0 = l_0(n) > n$ there is a least $1 \leq m \leq l_0(n)$ with $\|e\|_m > 0$. We set

$$W := \{x \in X : \sup(v_{k(l)}(x)\|e\|_l : 1 \leq l \leq l_0) < 2 \sup(v_{k(l)}(x_0)\|e\|_l : 1 \leq l \leq l_0)\}.$$

W is an open neighbourhood of x_0. There is $g \in C(X,[0,1])$ with $g(x_0) = 1, \operatorname{supp} g \subset W$. Since, for $z \in W$,

$$v_{k(m)}(z)g(z) \leq \frac{2}{\|e\|_m} \sup(v_{k(l)}(x_0)\|e\|_l : 1 \leq l \leq l_0),$$

we have $g \in Cv_{k(m)}(X)$, hence $f := g \otimes e \in G$.

Applying (1) to this f we obtain

$$v_k(x_0)\|e\|_n = v_k(x_0)\|f(x_0)\|_n \leq$$
$$C_n \max(\sup(v_{k(l)}(x)\|f(x)\|_l : x \in X) : 1 \leq l \leq l_0) \leq$$
$$2C_n \max(v_{k(l)}(x_0)\|e\|_l : 1 \leq l \leq l_0),$$

and condition (iii) is proved. ∎

Proposition 3.4. *Let E be a Fréchet space. Let $\vartheta = (v_n)$ be a decreasing sequence of strictly positive continuous weights on a locally compact space X. TFAE:*

(i) *every bounded subset of $C\overline{V}_0(X,E)$ is contained and bounded in some $C(v_n)_0(X,E)$.*

(ii) $\vartheta C_0(X,E) = C\overline{V}_0(X,E)$

(iii) (a) ϑ *is regularly decreasing* (i.e., $\forall n\ \exists m > n\ \forall a > 0\ \exists \overline{v} \in \overline{V} : v_m \leq \max(av_n, \overline{v})$ (cf. [10; Section 2])).

(b) $\vartheta C(X,E) = C\overline{V}(X,E)$ *algebraically (or equivalently 3.3 (iii)).*

Proof. (i) Clearly implies (ii) and (i) follows from (ii) because every closed absolutely convex bounded subset of $C\overline{V}_0(X,E)$ is a Banach disc, because $C\overline{V}_0(X,E)$ is complete, and Grothendieck's factorization theorem can be applied. To prove that (ii) implies (iii), first observe that (ii) really implies $\vartheta C_0(X) = C\overline{V}_0(X)$, which is equivalent to ϑ being regularly decreasing by [10; 2.6]. On the other hand, given $f \in C\overline{V}(X,E)$ we may form a bounded net in $C\overline{V}_0(X,E)$ multiplying f by continuous functions valued in [0, 1], equal to 1 on any compact subset of X and vanishing outside a compact neighbourhood of this compact set. This bounded net must be bounded in some step $C(v_n)_0(X,E)$ by (ii) (equivalent to (i)!), hence f belongs to $Cv_n(X,E)$. Finally we show that (iii) implies (ii). Given $f \in C\overline{V}_0(X,E)$, by (iii) (b), there is $n \in N$ with $f \in Cv_n(X,E)$. Select m as in the regularly decreasing condition (a) and suppose that $f \notin C(v_m)_0(X,E)$. Then there are $p \in cs(E)$ and $\varepsilon > 0$ such that for all compact subset K of X there is $x(K) \cdot \notin K$ with $v_m(x(K))p(f(x(K))) > \varepsilon$. We find $M > 0$ such that $v_n(x)p(f(x)) \leq M$ for all $x \in X$. Given $a = \varepsilon/M$ there is $\overline{v} \in \overline{V}$ with $v_m \leq \max(av_n, \overline{v})$. Since $f \in C\overline{V}_0(X,E)$, given $\varepsilon > 0$ there is a compact subset L of X with $\overline{v}(x)p(f(x)) \leq \varepsilon$ if $x \notin L$. For $x(L) \notin L$ we have $v_m(x(L))p(f(x(L))) \leq \max(av_n(x(L))p(f(x(L))), \overline{v}(x(L))p(f(x(L)))) \leq \varepsilon$.

This is a contradiction. ∎

We close this section with a very general universal answer to question (ii) of the basic problem of projective description in the context of arbitrary l.c.s. where the countable g-prequojections of Section 1 play an important role. Our results extends [15; 2.5].

Theorem 3.5. *Let $\vartheta = (v_n)$ be a decreasing sequence of strictly positive weights on a locally compact σ-compact space X. If a l.c.s. E is a countable g-prequojection, then $\vartheta_0 C(X,E)$ is a topological subspace of $C\overline{V}_0(X,E)$.*

Proof. We fix an equicontinuous subset H of $(\vartheta_0 C(X,E))'$. We will see that H is equicontinuous for the topology induced by $C\overline{V}_0(X,E)$. The canonical injection $C(v_n)_0(X) \otimes_\pi E \to C(v_n)_0(X,E)$ is continuous, hence we can apply proposition 2.2. to obtain that H induces an equicontinuous on $(\vartheta_0 C(X) \otimes_\pi E)'$. Therefore we can find $U \in U_0(E)$ such that the map $\phi(g,.)$ given by $x \in E \to \phi(g \otimes x)$ belongs to E'_{U^0} for all $g \in \vartheta_0 C(X), \phi \in H$.

For each $n \in N$ there is $p_n \in cs(E)$ with $|\phi(f)| \le \sup(v_n(x) p_n(f(x)) : x \in X) \forall f \in C(v_n)_0(X,E), \phi \in H$. Put $U_n := \{y \in E : p_n(y) \le 1\}, n \in N$. Since E is a countable g-prequojection we can find $r \in cs(E), \lambda_n > 0, n \in N$, with $U_n^0 \cap E'_{U^0} \subset \lambda_n V^0 \ \forall n \in N (V := \{y \in E : r(y) \le 1\})$. Since X is locally compact and σ-compact we can assume without loss of generality that $\overline{v} := \inf 2^n \lambda_n v_n$ is strictly positive and continuous (cf. [10; p. 112]). Now we proceed in 3 steps.

1. Let K be a compact subset of $X, m \in N, \{\Omega_j : 1 \le j \le n\}$ an open cover of $K, \varphi_j \in C_c(X)$ satisfying $0 \le \varphi_j \le 1, \operatorname{supp}\varphi \subset \Omega_j, \Sigma(\varphi_j : 1 \le j \le n) \le 1$, and for each j there is $x_j \in \Omega_j$ such that $v_m(x)/v_m(x_j) < 2$ for all $x \in \Omega_j$. If $(z_j : 1 \le j \le n)$ is included in E and $\sup(\lambda_m v_m(x_j) r(z_j) : 1 \le j \le n) < 1$ then

$$\left|\phi\left(\sum_{j=1}^{n} \varphi_j \otimes z_j\right)\right| \le 2 \quad for\ all \quad \phi \in H.$$

Indeed,

$$\left|\phi\left(\sum_{j=1}^{n} \varphi_j \otimes z_j\right)\right| \le \sum_{j=1}^{n} \left|\phi\left(\frac{\varphi_j}{v_m(x_j)\lambda_m},.\right)(\lambda_m v_m(x_j) z_j)\right| \le$$

$$\sum_{j=1}^{n} P_{V^0}\left(\phi\left(\frac{\varphi_j}{\lambda_m v_m(x_j)},.\right)\right)$$

were P_{V^0} is the Minkowski functional of V^0 (observe that $\lambda_m v_m(x_j) z_j \in V$ for all $1 \le j \le n$). Since $U_m^0 \cap E'_{U^0} \subset \lambda_m V^0$ and

$$\phi\left(\frac{\varphi_j}{\lambda_m v_m(x_j)},.\right) \in E'_{U^0}$$

one has

$$P_{V^\circ}\left(\phi\left(\frac{\varphi_j}{\lambda_m v_m(x_j)},.\right)\right) \leq P_{U_m^\circ}\left(\phi\left(\frac{\varphi_j}{v_m(x_j)},.\right)\right)$$

Consequently

$$\left|\phi\left(\sum_{j=1}^{n} \varphi_j \otimes z_j\right)\right| \leq \sum_{j=1}^{n} P_{U_m^\circ}\left(\phi\left(\frac{\varphi_j}{v_m(x_j)},.\right)\right)$$

Now, if $y_j \in U_m (1 \leq j \leq n)$ one has

$$\sum_{j=1}^{n}\left|\phi\left(\frac{\varphi_j}{v_m(x_j)} \otimes y_j\right)\right| = \left|\phi\left(\sum_{j=1}^{n} \frac{\varphi_j}{v_m(x_j)} \otimes y_j \eta_j\right)\right|$$

for certain $\eta_j, |\eta_j| = 1$. If g denotes the function

$$\sum_{j=1}^{n} \frac{\varphi_j}{v_m(x_j)} \otimes y_j \eta_j$$

we obtain

$$v_m(x) p_m(g(x)) = v_m(x) p_m\left(\sum_{j=1}^{n} \frac{\varphi_j(x)}{v_m(x_j)} \eta_j y_j\right) \leq \sum_{j=1}^{n} \frac{v_m(x)}{v_m(x_j)} \varphi_j(x) p_m(y_j) \leq 2$$

for every $x \in X$, from where it follows $|\phi(g)| \leq 2$ for every $\phi \in H$.

2. Let $f \in C_c(X, E)$ be such that $\sup(\overline{v}(x) r(f(x)) : x \in X) < 1$. Given $x \in X$, we find $n(x) \in N$ such that $2^{n(x)} v_{n(x)}(x) \lambda_{n(x)} < 2\overline{v}(x)$, and we set

$$\Omega_x := \{y \in X : 2^{n(x)} v_{n(x)}(y) \lambda_{n(x)} < 2\overline{v}(y), v_{n(x)}(y) < 2 v_{n(x)}(x),$$

$$p_{n(x)}(f(y) - f(x)) v_{n(x)}(y) < 2^{-n(x)}\},$$

which is an open neighbourhood of x.

Since the supp f is compact we find $x_1, \ldots, x_n \in X$ such that supp $f \subset \cup(\Omega_{x_j} : 1 \leq j \leq n)$. Find $\varphi_j \in C_c(X), 1 \leq j \leq n$, such that

$$\operatorname{supp}\varphi_j \subset \Omega_{x_j}, 0 \leq \varphi_j \leq 1, \sum_{j=1}^{n} \varphi_j = 1 \textit{ on } \operatorname{supp} f, \sum_{j=1}^{n} \varphi_j \leq 1 \textit{ on } X.$$

We take

$$h := \sum_{j=1}^{n} \varphi_j \otimes f(x_j) \in C_c(X) \otimes E$$

$$I_k := \{j = 1, \ldots, n : n(x_j) = k\} (k \in N)$$

Then

$$h = \sum_k \left(\sum_{j \in I_k} \varphi_j \otimes f(x_j) \right) \text{(finite sum!)}$$

For each $j \in I_k$ we have

$$\lambda_k v_k(x_j) r(f(x_j)) = \lambda_{n(x_j)} v_{n(x_j)}(x_j) r(f(x_j)) \leq 2^{-k} 2\overline{v}(x_j) r(f(x_j)) \leq 2 2^{-k}$$

and, according to step 1, we obtain

$$\left| \phi \left(\sum_{j \in I_k} \varphi_j \otimes f(x_j) \right) \right| \leq 4.2^{-k}$$

for all $k \in N, \phi \in H$. Moreover

$$f = \sum_k \sum_{j \in I_k} \varphi_j f$$

and we have

$$p_k \left(\sum_{j \in I_k} \varphi_j(x)(f(x) - f(x_j)) \right) v_k(x) \leq \sum_{j \in I_k} \varphi_j(x) p_k(f(x) - f(x_j)) v_k(x) \leq 2^{-k}$$

since, for $x \in \Omega_{x_j}$

$$p_k(f(x) - f(x_j)) v_k(x) = p_{n(x_j)}(f(x) - f(x_j)) v_{n(x_j)}(x) < 2^{-n(x_j)} = 2^{-k}.$$

Consequently

$$\left| \phi \left(\sum_{j \in I_k} \varphi_j f \right) \right| \leq 2^{-k} + \left| \phi \left(\sum_{j \in I_k} \varphi_j \otimes f(x_j) \right) \right| \leq 5.2^{-k} \textit{ for all } k \in N; \phi \in H,$$

hence

$$|\phi(f)| \leq \sum_k \left| \phi\left(\sum_{j \in I_k} \varphi_j f \right) \right| \leq 5 \ for\ all\ \phi \in H.$$

3. We take now $f \in C(v_n)_0(X, E)$ for some $n \in N$, with sup $(\overline{v}(x)r(f(x)) : x \in X) < 1$. We can find $g \in C_c(X, E)$ with

(a) $\sup(\overline{v}(x)r(g(x)) : x \in X) < 1$

(b) $\sup(v_n(x)p_n(f(x) - g(x)) : x \in X) < 1$

(a), (b) and step 2 imply $|\phi(f)| \leq 6$ for all $\phi \in H$. This shows that H is equicontinuous on $\vartheta_0 C(X, E)$ endowed with the topology induced by $C\overline{V}_0(X, E)$. This completes the proof. ∎

In [28] the second author has proved that a Fréchet space E is a countable g-prequojection if and only if $\vartheta_0 C(X, E)$ is a topological subspace of $C\overline{V}_0(X, E)$ for all sequence ϑ on any locally compact and $\sigma-$compact space X, as a consequence of a complete characterization of the pairs (ϑ, E) of sequences on X and Fréchet spaces E such that $\vartheta_0 C(X, E)$ is a topological subspace of $C\overline{V}_0(X, E)$ in terms of a modification of the condition of Vogt in [44; 1.4].

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