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# ON SEMIDIRECT PRODUCT OF SEMIGROUPS *

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## 1. INTRODUCTION

Let $X$ be a subset of a semigroup $S$. We denote by $E(X)$ the set of idempotent elements of $X$.

An element $a$ of a semigroup $S$ is called *$E$-inversive* if there exists $x \in E(S)$ such that $ax \in E(S)$. We note that the definition is not one-sided. Indeed, an a element of a semigroup $S$ is $E$-inversive iff there exists $y \in S$ such that $ay, ya \in E(S)$ (see [7], [1] p. 98).

A semigroup $S$ is called *$E$-inversive* if all its elements are $E$-inversive. This class of semigroups is extensive. All semigroups with a zero and all eventually regular semigroups [2] are $E$-inversive semigroups.

Recently $E$-inversive semigroups reappeared in a paper by Hall and Munn [3] and in a paper by Mitsch [5]. The special case of $E$-inversive semigroups with pairwise commuting idempotents, called *$E$-dense*, was considered by Margolis and Pin [4].

Let $S$ and $T$ be semigroups, and let $\alpha : S \to End(T)$ be a homomorphism of $S$ into the endomorphism semigroup of $T$. If $s \in S$ and $t \in T$, denote $t(s\alpha)$ by $t^s$. Thus, if $s, s' \in S$ and $t \in T$ then $(t^s)^{s'} = t^{ss'}$. The *semidirect product* of $S$ and $T$, in that order, with structure map $\alpha$, consists of the set $S \times T$ equipped with the product

$$(s,t)(s',t') = (ss', t^{s'}t')$$

This product will be denoted by $S \times_\alpha T$.

In this note we determine which semidirect products of semigroups are $E$-inversive semigroups and $E$-dense semigroups, respectively. It turns out that the case in which $S$ induces only automorphism on $T$ allows a particularly simple description.

In [6], Preston has answered the analous question for regular semigroups and for inverse semigroups.

For the terminology and for the definitions of the algebraic theory of semigroups, we refer to [1].

## 2. $E$-INVERSIVE SEMIDIRECT PRODUCTS

**Lemma 2.1.** *An element $a$ of a semigroup $S$ is $E$-inversive if and only if there exists $y \in S$ such that $y = yay$.*

*Proof.* Let $x \in S$ and $ax \in E(S)$. Then, setting $y = xax$, we have $yay = (xax)a(xax) = x(ax)(ax)(ax) = xax = y$. The converse is clear.

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Consequently, we can say that $S$ is $E$-inversive iff $W_S(a) = \{t \in S : yay = y \neq \emptyset$ for every $a \in S\}$.

**Theorem 2.2.** *Let $U = S \times_\alpha T$ be a semidirect product of semigroups. Then $U$ is $E$-inversive if and only if for all $s \in S$ and $t \in T$ there exists $x \in W_S(s)$ such that $t^{xs}$ is an $E$-inversive element of $T^{xs}$.*

*Proof.* Suppose first that $U$ is $E$-inversive. Let $(s,t) \in U$ and $(x,y) \in W_U(s,t)$. Then $(x,y) = (x,y)(s,t)(x,y) = (xsx, y^{sx}t^x y)$, hence

$$x = xsx, \tag{1}$$

$$y = y^{sx}t^x y. \tag{2}$$

Condition (1) gives that $x \in W_S(s)$. By (2)

$$(y^s)^{xs} = y^{sxs} = (y^{sx}t^x y)^{sxs} = y^{sxsxs}t^{xsxs}y^{sxs} = y^{sxs}t^{xs}y^{sxs} =$$

$$= (y^s)^{xs}t^{xs}(y^s)^{xs}.$$

Thus $t^{xs}$ is an $E$-inversive element of $T^{xs}$.

Conversely, let $(s,t) \in U$ and $x \in W_S(s)$ such that $t^{xs}$ is an $E$-inversive element of $T^{xs}$. Then there exists $v \in T$ such that $v^{xs} \in W_T(t^{xs})$. We set $y = v^x$. Then $(x,y)(s,t)(x,y) = (xsx, y^{sx}t^x y) = (x, y^{sx}t^x y)$ and $y^{sx}t^x y = (v^x)^{sx}t^x v^x = v^{xsx}t^x v^x = v^{xsx}t^{xsx}v^{xsx} = (v^{xs}t^{xs}v^{xs})^x = (v^{xs})^x = v^{xsx} = v^x = y$.

Hence $U$ is $E$-inversive.

**Corollary 2.3.** *If $S$ and $T$ are $E$-inversive semigroups then every semidirect product $S \times_\alpha T$ is $E$-inversive.*

*Example 2.4.* Let $S = \{e, a\}$ be the semigroup with identity $e$ and $a = a^2$. Let $T$ be the multiplicative semigroup of positive integers. Let $\alpha : S \to End(T)$ defined by $t^e = t$ and $t^a = 1$ for every $t \in T$. Then $S$ and $S \times_\alpha T$ are $E$-inversive but $T$ is not $E$-inversive.

**Corollary 2.5.** *Let $S$ and $T$ semigroups and let $\alpha : S \to Aut(T)$ be a homomorphism of $S$ into the automorphism group of $T$. Then, $S \times_\alpha T$ is $E$-inversive if and only if $S$ and $T$ are $E$-inversive.*

*Proof.* Let $s \in S$ and $t \in T$. Then, by Theorem 2.2 there exists $x \in W_S(s)$ such that $t^{xs}$ is an $E$-inversive element of $T^{xs}$. Now, $xs \in E(S)$ and hence $(xs)\alpha$ is the identity of $Aut(T)$, thus $t$ is an $E$-inversive element of $T$.

The converse is immediate by Corollary 2.3.

## 3. $E$-DENSE SEMIDIRECT PRODUCTS

**Lemma 3.1.** *Let $S$ be a semigroup in which all idempotents commute. If $e \in E(S)$ and $e' \in W_S(e)$, then $e' = ee' = e'e$. In particular, $e' \in E(S)$.*

*Proof.* Since $e, ee', e'e \in E(S)$, we have $ee' = ee'ee' = e'eee' = e'$. Analogously $e'e = e'$.

**Lemma 3.2.** *Let $S$ be a $E$-dense semigroup and let $x, y \in S$. If $x' \in W_S(x) \cap W_S(y)$, then $xx'x = yx'y$.*

*Proof.* Since $x'y = x'xx'y = x'yx'x = x'x$ and, analogously $yx' = xx'$, we have $xx'x = yx'x = yx'y$.

Let $S$ be a semigroup. If $e, f \in E(S)$ we define $e \leq f$ if $e = ef = fe$. We call semilattice a commutative semigroup in which every element is idempotent.

**Proposition 3.3.** *Let $U = S \times_\alpha T$ be a semidirect product of semigroups and let $E(S)$ be a semilattice.*

*Then $E(U)$ is a semilattice if and only if*

(1) (i) *if $e \in E(S), t \in T$ such that $t = t^e t$, then $t = t^e$;*

(ii) *if $e, f \in E(S), f \leq e, t \in E(T^e)$, and $v \in E(T^f)$ then $t^f v = vt$.*

*Proof.* Suppose that $E(S)$ is a semilattice. Let $e \in E(S), t \in T$ such that $t = t^e t$. Then $(e,t)(e,t) = (e, t^e t) = (t,e)$ and $(e,t^e) = (e,t^e t^e) = (e,t)(e,t^e) = (e,t^e t)(e,t^e) = (e,t^e)(e,t)(e,t^e)$. Hence, by Lemma 3.1 $(e,t^e) = (e,t^e)(e,t) = (e,t^e t) = (e,t)$. Thus $t = t^e$.

How we prove part (ii) of (1). Let $e, f \in E(S), f \leq e, t \in E(T^e), v \in E(T^f)$ and $r \in T$ such that $v = r^f$. Evidently $(e,t),(f,v) \in E(U)$ and, since $E(U)$ is semilattice, $(e,t)(f,v) = (f,v)(e,t)$. Hence $t^f v = v^e t = (r^f)^e t = r^{ef} t = r^f t = vt$.

Conversely, suppose that (1) holds. Let $(e,t),(f,v) \in E(U)$. Then, by (i), $t = t^e$ and $v = v^f$. Thus

$$
(2)\qquad \begin{aligned}
t^f v &= t^{ef} v = t^{ef} v^f = (v^f)^{ef} t^{ef} && (by(i), since\ ef \leq f) \\
&= v^{ef} t^{ef} = (t^{ef})^{ef} v^{ef} && (by(i), since\ ef \leq ef) \\
&= t^{ef} v^{ef} = (t^e)^{ef} v^{ef} = v^{ef} t^e && (by(i), since\ ef \leq e) \\
&= (v^f)^e t^e = v^e t
\end{aligned}
$$

Hence, by (2), $(e,t)(f,v) = (ef, t^f v) = (fe, v^e t) = (f,v)(e,t)$.

**Theorem 3.4.** *Let $U = S \times_\alpha T$ be a semidirect product of semigroups. Then $U$ is $E$-dense if and only if $S$ is $E$-dense, $T$ is $E$-inversive and (1) holds.*

*Proof.* Let $U$ be $E$-dense semigroup. By Theorem 2.2, $S$ is $E$-inversive. We show that the idempotents of $S$ are pairwise commuting. Let $e, f \in E(S)$. Then, by Theorem 2.2, there exists $h \in E(T)$, hence $(e, h^e)(e, h^e) = (e, h^e h^e) = (e, h^e)$. Analogously $(f, h^f)(f, h^f) = (f, h^f)$. Therefore, since $U$ is $E$-dense, we have $(f, h^f)(e, h^e) = (e, h^e)(f, h^f)$, hence $ef = fe$.

Now we prove that $T$ is $E$-inversive. Let $t \in T$. We choose $e \in E(S)$ and $(x, y) \in U$ such that $(x, y) = (x, y)(e, t)(x, y)$. Then (3) $x \in W_S(e)$, (4) $y = y^{ex} t^x y$.

It follows, by Lemma 3.1, that (5) $y^e = (y^{ex} t^x y) = y^x t^x y^e$. By Lemma 3.1, $(x, y)(x, t^x)(x, y) = (xx, y^x t^x)(x, y) = (x, y^x t^x y) = (x, y^{ex} t^x y) = (x, y)$.

Now, by Lemma 3.2, $(e, t)(x, y)(e, t) = (x, t^x)(x, y)(x, t^x)$, hence, by Lemma 3.1, (6) $t^x y^e t = t^x y^x t^x$. Then

$$\begin{aligned} y^e t y^e &= (y^x t^x y^e)^e t y^e && \text{by}(5) \\ &= y^x t^x y^e t y^e && \text{By Lemma 3.1} \\ &= y^x t^x y^x t^x y^e && \text{by}(6) \\ &= y^e && \text{by}(5). \end{aligned}$$

Thus $T$ is $E$-inversive.

By Prop. 3.3 the condition (1) holds.

Conversely, since $S$ ant $T$ are $E$-inversive semigroups, from Corollary 2.3, $U$ is $E$-inversive.

By Prop. 3.3, since $SE$-dense and (1) holds, it follow that $U$ is $E$-dense.

*Example* 3.5. Let $S = \{e, a\}$ be the semigroup with identity $e$ and $a = a^2$. Let $T$ be a semigroup with 0 and with two idempotent elements which do not commute. Let $\alpha : S \to End(T)$ defined by $t^e = t^a = 0$ for all $t \in T$. Then $S \times_a T$ is $E$-dense but $T$ is not $E$-dense.

*Example* 3.6. Let $S = \{e, a\}$ be the semigroup with identity $e$ and $a = a^2$. Let $T$ be a $E$-dense semigroup with identity 1. Let $\alpha : S \to End(T)$ defined by $t^a = t^e = 1$ for all $t \in T$. Then $S, T$ are $E$-dense, but $S \times_\alpha T$ is not $E$-dense.

**Corollary 3.7.** *Let $S$ and $T$ be semigroups and $\alpha : S \to Aut(T)$ be a homomorphism of $S$ into the automorphism group of $T$. Then $S \times_\alpha T$ is $E$-dense if and only if $S$ and $T$ are $E$-dense.*

*Proof.* If $S \times_a T$ is $E$-dense, by Theorem 3.4, $S$ is $E$-dense and $T$ is $E$-inversive. Moreover, since $S\alpha \subseteq Aut(T), T^e = T$ for all $e \in E(S)$. Thus, by (1) (ii) $T$ is $E$-dense.

Conversely, let $S, T$ be $E$-dense. Then, by Corollary 2.3, $S \times_\alpha T$ is $E$-inversive. Moreover, if $(s,t),(r,v) \in E(S \times_\alpha T)$ then $r,s \in E(S), t,v \in E(T)$ and $(s,t)(r,v) = (sr, t^r v) = (sr, tv) = (rs, vt) = (rs, v^s t) = (r,v)(s,t)$.

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