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LOCALLY s-REGULAR MANIFOLDS AND SYMMETRIES

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Abstract. *We study properties of a field of special local diffeomorphisms on a Riemannian manifold and derive some new characterizations of locally s-regular manifolds and, as a special case, of locally 3-symmetric spaces.*

1. INTRODUCTION

As well-known, the local geodesic symmetries on a Riemannian locally symmetric space are local isometries. *Locally s-regular manifolds* and, in particular, *Riemannian k-symmetric spaces* are natural generalizations of locally symmetric spaces (see for example [1], [5]). For this class of manifolds the local geodesic symmetries are replaced by a special field of local isometries.

It is also possible to characterize other classes of Riemannian manifolds by special properties of the local geodesic symmetries. We refer to [9] for a survey. In this paper we continue the study of Riemannian manifolds (M,g) which are equipped with a field of special local diffeomorphisms $s_m, m \in M$, defined on a sufficiently small neighborhood of m by

$$s_m = exp_m \circ S_m \circ exp_m^{-1}$$

where S is a (1,1)-tensor field on M such that S preserves g and $I - S$ is invertible. (See [6], [7] for previous work.) In particular we concentrate on local diffeomorphisms s_m which preserve the (0,2)-tensor A given by $A(X,Y) = g(X,SY)$ for all tangent vector fields X,Y. This is similar to the study of s_m which preserve the Kähler form on an almost Hermitian manifold, that is, which are symplectic with respect to $\Omega(X,Y) = g(X,JY)$. In this way we obtain new characterizations of locally s-regular manifolds and, as a special case, of *locally 3-symmetric spaces* (see [2] for more details).

2. PRELIMINARIES

Let (M,g) be an n-dimensional smooth Riemannian manifold with Levi Civita connection ∇ and Riemannian curvature tensor R defined by

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all vector fields X,Y on M. A (1,1)-tensor field S is called a *symmetry tensor field* if $I - S$ is non-singular and g is S-invariant, that is $g(SX,SY) = g(X,Y)$ for all X,Y. In particular, if ∇S and $\nabla^2 S$ are S-invariant, then we say that S is *regular.*

Next, for any symmetry tensor field S on M we define on a sufficiently small neighborhood U_m of m a *local symmetry* s_m by

$$s_m = exp_m \circ S_m \circ exp_m^{-1}.$$

s_m is a local diffeomorphism on U_m. We denote by s the map $m \mapsto s_m$ so defined on M and we note that for each $m \in M$

$$s_{m*}|T_m M = S_m.$$

Finally, we recall from [1], [5] that (M, g) together with s is called a *Riemannian locally s-regular manifold* if each s_m is also a local *isometry* which preserves S, that is

$$s_{m*} \circ S = S \circ s_{m*},$$

for each $m \in M$. Then s is called a *local regular s-structure* on (M, g). Moreover, if s_m^k=identity for all $m \in M$, where $k(\geq 2)$ is the smallest integer with this property, then the s-structure is said to be of *order* k and (M, g) is called a Riemannian *k-symmetric space.* Note that for $k = 2$ (or $S = -I$) we obtain the locally symmetric spaces and for $k = 3$ we have a *locally 3-symmetric space.* Such a manifold is an almost Hermitian manifold (M, g, J) where the *canonical almost complex structure* J is defined by

$$S_m = -\frac{1}{2} I_m + \frac{\sqrt{3}}{2} J_m$$

(I_m denotes the identity on $T_m M$). We refer to [1], [5], [2] for more details about all these manifolds and for a lot of nice examples.

To finish this section we give two lemmas, contained in [1], which will be needed later.

Lemma 1. *Let S be a regular symmetry tensor field. Then R and ∇R are S-invariant if and only if (M, g) is a locally s-regular manifold with symmetry tensor field S.*

Lemma 2. *If S is a regular symmetry tensor field on (M, g) and the tensor fields P and ∇P are S-invariant, then $\nabla^2 P$ is S-invariant and hence all covariant derivatives are S-invariant.*

3. NEW CHARACTERIZATIONS OF LOCALLY s-REGULAR MANIFOLDS

Let (M, g) be a Riemannian manifold equipped with a symmetry tensor field S. We do not suppose that S is regular. Define the (0,2)-tensor field A by

$$A(X, Y) = g(X, SY)$$

for all vector fields X, Y on M. Now, we concentrate on *A-preserving local diffeomorphisms s_m* and prove the following results.

Theorem 1. *(M, g, s) is a locally s-regular manifold if and only if R and ∇R are S-invariant and each s_m preserves A.*

Theorem 2. *Let s_m^k = identity for all $m \in M$ where $k(> 2)$ is odd. Then (M, g, s) is a locally s-regular manifold if and only if R is S-invariant and each s_m preserves A.*

Theorem 3. *Theorem 2 remains true when «R is S-invariant» is replaced by «$\nabla^2 S$ is S-invariant».*

Proof of the theorems. First, let (M, g, s) be a locally s-regular manifold. Then, from the definition and Lemma 1 it follows that $\nabla^2 S, R$ and ∇R are S-invariant. Moreover, since each s_m preserves g and S, A is also preserved.

To prove the converse results we will use a normal coordinate system $\{x^i, i = 1, \dots, n\}$ centered at m where $\left\{e_i = \frac{\partial}{\partial x^i}(m)\right\}$ is an orthonormal basis of $T_m M$. Let

$$A_{ij} = A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

and let $p = exp_m(ru)$ where $u \in T_m M$ is a unit vector. Then we have the following power series expansion (see for example [3])

$$\begin{aligned} A_{ij}(p) &= A_{ij}(m) + r(\nabla_u A)_{ij}(m) \\ &+ \frac{1}{2}r^2\left\{(\nabla^2_{uu}A)_{ij} - \frac{1}{3}\sum_t R_{uiut}A_{tj} - \frac{1}{3}\sum_t R_{ujut}A_{it}\right\}(m) \\ &+ \frac{1}{6}r^3\left\{(\nabla^3_{uuu}A)_{ij} - \sum_t R_{uiut}(\nabla_u A)_{tj} - \sum_t R_{ujut}(\nabla_u A)_{it}\right. \\ &\left. -\frac{1}{2}\sum_t (\nabla_u R)_{uiut}A_{tj} - \frac{1}{2}\sum_t (\nabla_u R)_{ujut}A_{it}\right\}(m) \\ &+ 0(r^4). \end{aligned} \tag{1}$$

Note that A is automatically S-invariant.

Now we express that s_m is A-preserving. First we note that $x^i \circ s_m = S^i_j(m)x^j$ and hence we see that s_m preserves A if and only if

$$A_{ij}(exp_m ru) = S^a_i(m)S^b_j(m)A_{ab}(exp_m rS_m u) \tag{2}$$

for all $u \in T_mM$ and all sufficiently small r.

So, from (1) and (2) we get as first necessary condition

$$(3) \qquad (\nabla_u A)_{xy} = (\nabla_{Su} A)_{SxSy}$$

for all $u, x, y \in T_mM$. This means that ∇A is S-invariant or equivalently, that ∇S is S-invariant.

To obtain the next condition we note that the coefficient of $r^2/2$ may be written as

$$(4) \qquad (\nabla^2_{uu} A)_{ij} - \frac{1}{3} R_{ue_i uSe_j} - \frac{1}{3} R_{uS^{-1}e_i ue_j}$$

and so the next condition yields

$$(5) \qquad \begin{aligned} (\nabla^2_{uu} A)_{xy} - \frac{1}{3} R_{uxuSy} - \frac{1}{3} R_{uS^{-1}xuy} = \\ = (\nabla^2_{SuSu} A)_{SxSy} - \frac{1}{3} R_{SuSxSuS^2y} - \frac{1}{3} R_{SuxSuSy}. \end{aligned}$$

Now we replace x by Sx in (5) to get

$$(6) \qquad \begin{aligned} (\nabla^2_{uu} S)_{Sxy} - \frac{1}{3} R_{uSxuSy} - \frac{1}{3} R_{uxuy} = \\ = (\nabla^2_{SuSu} A)_{S^2xSy} - \frac{1}{3} R_{SuS^2xSuS^2y} - \frac{1}{3} R_{SuSxSuSy}. \end{aligned}$$

Next, put

$$(7) \qquad T_{uxvy} = R_{uxvy} - R_{SuSxSvSy}.$$

Then (6) becomes

$$(8) \qquad (\nabla^2_{uu} A)_{Sxy} - (\nabla^2_{SuSu} A)_{S^2xSy} = \frac{1}{3}(T_{uxuy} + T_{uSxuSy}).$$

Now we prove that $\nabla^2 A$, or equivalently, $\nabla^2 S$ is S-invariant. The converse also holds if $s_m^k = \text{id.}$ for k odd and $k > 2$.

First, let R be S-invariant. Then, from (7) we get $T = 0$ and (8) becomes

$$(9) \qquad (\nabla^2_{uu} A)_{Sxy} = (\nabla^2_{SuSu} A)_{S^2xSy}$$

or equivalently,

$$(\nabla^2_{uu}A)_{xy} = (\nabla^2_{SuSu})_{SxSy}. \tag{10}$$

Then linearization of (10) yields

$$(\nabla^2_{uv}A)_{xy} + (\nabla^2_{vu}A)_{xy} = (\nabla^2_{SuSv}A)_{SxSy} + (\nabla^2_{SvSu}A)_{SxSy}. \tag{11}$$

By using the Ricci identity, we get

$$\begin{aligned}(\nabla^2_{vu}A)_{xy} &= (\nabla^2_{uv}A)_{xy} - A(R_{vu}x, y) - A(x, R_{vu}y) \\ &= (\nabla^2_{uv}A)_{xy} - R_{vuxSy} - R_{vuyS^{-1}x}.\end{aligned} \tag{12}$$

Finally, using (12) and the S-invariance of R, (11) becomes

$$(\nabla^2_{uv}A)_{xy} = (\nabla^2_{SuSv}A)_{SxSy},$$

which means that $\nabla^2 A$ is S-invariant.

Conversely, suppose that $\nabla^2 A$ (or equivalently, $\nabla^2 S$) is S-invariant. Then (8) gives

$$T_{uxuy} = -T_{uSxuSy}$$

and so

$$T_{uxuy} = (-1)^k T_{uS^kxuS^ky}. \tag{13}$$

If $s^k_m = \text{id.}$ for k odd, (13) yields

$$T_{uxuy} = 0 \tag{14}$$

and since T satisfies the same identities as a Riemann curvature tensor, (14) implies $T = 0$, which means that R is S-invariant.

Now, we note that when $R, \nabla S$ and $\nabla^2 S$ are S-invariant, Lemma 2 and (1), (2) yield as next condition

$$\nabla_u R_{uxuSy} + \nabla_u R_{uS^{-1}xuy} = \nabla_{Su} R_{SuSxSuS^2y} + \nabla_{Su} R_{SuxSuSy} \tag{15}$$

and hence

$$\nabla_u R_{uSxuSy} + \nabla_u R_{uxuy} = \nabla_{Su} R_{SuS^2xSuS^2y} + \nabla_{Su} R_{SuSxSuSy}. \tag{16}$$

Next, put

$$\overline{T}_{uxyzw} = \nabla_u R_{xyzw} - \nabla_{Su} R_{SxSySzSw}.$$

Then (16) may be written as

$$\overline{T}_{uuxuy} = -\overline{T}_{uuSxuSy} \tag{17}$$

which implies

$$\overline{T}_{uuxuy} = (-1)^k \overline{T}_{uuS^kxuS^ky}.$$

So, if $s_m^k = \text{id.}$ for k odd, we get

$$\overline{T}_{uuxuy} = 0.$$

Since $\overline{T}$ satisfies the same identities as the covariant derivative of a Riemannian curvature tensor, we obtain $\overline{T} = 0$ (see for example [4], [8]) and this means that ∇R is S-invariant.

The proof of the three theorems follows now easily from the results above and Lemma 1.

4. LOCALLY 3-SYMMETRIC SPACES

We first recall that an almost Hermitian manifold (M, g, J) belongs to the class $\mathcal{AH}_2$ if and only if

$$R_{XYZW} = R_{JXJYZW} + R_{JXYJZW} + R_{JXYZJW} \tag{18}$$

for all tangent vector fields X, Y, Z, W.

As a corollary of Theorem 2 we get the following characterization of locally 3-symmetric spaces.

Theorem 4. *Let (M, g, J) be an almost Hermitian manifold and define the symmetry tensor field S by*

$$S = -\frac{1}{2} I + \frac{\sqrt{3}}{2} J.$$

Then (M, g, s) is a locally 3-symmetric space if and only if each s_m is A-preserving and $(M, g, J) \in \mathcal{AH}_2$.

Proof. We have only to note that R is S-invariant if and only if the identity (18) holds (see for example [2]).

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