Note di Matematica Vol. IX - n. 1, 77-87 (1989)

ON THE CHARACTERIZATION OF THE RANGE OF AN INTEGRAL FUNCTIONAL

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Abstract. *In this note we characterize the range of an integral functional on decomposable spaces and successively also on non-decomposable space.*

INTRODUCTION

In [1], [2] we studied an existence theorem for the following non convex problem of calculus of variations.

Let $f : [0,1] \times \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a normal proper, lower semicontinuous integrand in the sense of R.T. Rockafellar [4]; let moreover $L^1(0,1,\mathbf{R}^n)$ be the usual space of summable functions and let

$$L_0^1(0,1,\mathbf{R}^n) = \left\{ x \in L^1(0,1,\mathbf{R}^n) : \int_0^1 x(t)\mathrm{d}t = 0 \right\}.$$

Since f is a normal integrand, we can consider the integral functional I_f defined for every measurable function x as

$$I_f(x) = \int_0^1 f(t, x(t))\mathrm{d}t$$

and we can state the following problems:

$$(1) \qquad Minimize\{I_f(x) : x \in L^1(0,1,\mathbf{R}^n)\}$$

$$(2) \qquad Minimize\{I_f(x) : x \in L_0^1(0,1,\mathbf{R}^n)\}.$$

When a milder version of the classical «basic growth condition» is satisfied we prove that, [2], problem (1) has a solution, while to prove a similar result for problem (2) we need the «basic growth condition» together with an assumption which assures that f^{**} is a Caratheodory integrand.

Proving the above results we find some inclusions between the ranges of I_f and of $I_{f^{**}}$; here, using the same assumptions on f, we intend to study the ranges of I_f and $I_{f^{**}}$ on $L^1(0,1,\mathbf{R}^n)$ and on $L_0^1(0,1,\mathbf{R}^n)$.

We prove a sort of intermediate value theorems for I_f and successively we show that

$$R_1(I_f) = R_1(I_{f^{**}}) = [\lambda_1, +\infty)$$

$$R_2(I_f) = R_2(I_{f^{**}}) = [\lambda_2, +\infty)$$

where $R_1(I_f)$ and $R_2(I_f)$ stands for the range of I_f on $L^1(0,1,\mathbf{R}^n)$ and $L^1_0(0,1,\mathbf{R}^n)$, respectively, while λ_1 and λ_2 are the minimum values in problems (1) and (2), respectively.

An example shows that, when some of the assumptions we used are missing, equality of ranges is no longer true.

NOTATION AND ASSUMPTIONS

We collect in this section all notation we use in the sequel in order to make the following sections free of thecnical definitions.

$\mathbf{R}^n$ is the usual n-dimensional euclidean space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$; [0, 1], the unit interval of the real line is equipped with the Lebesge measure; $\mathscr{L}$ indicates the σ*-algebra* of all Lebesgue sets in [0, 1] while $\mathscr{B}$ is the σ*-algebra* of all Borel sets in $\mathbf{R}^n$.

$f : [0,1]\times\mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ is said to be a normal integrand when it is measurable with respect to the σ*-algebra* $\mathscr{L}\otimes\mathscr{B}$ in $[0,1]\times\mathbf{R}^n$; it is said proper when $f(t,\cdot)$ is a proper function (not identically $+\infty$), $a.e. - t \in [0,1]$; while it is said lower semicontinuous (l.s.c. in abreeged) when $f(t,\cdot)$ is a l.s.c. function $a.e. - t \in [0,1]$. By standard results [4], for every measurable function $x : [0,1] \to \mathbf{R}^n, f(\cdot, x(\cdot))$ is a measurable function and we can define

$$I_f(x) = \int_0^1 f(t, x(t))\,\mathrm{d}t.$$

We also consider the integrands defined as

$$f^*(t,y) = \sup\{\langle x,y\rangle - f(t,x) : x \in \mathbf{R}^n\}$$

and by

$$f^{**}(t,x) = (f^*(t,\cdot))^*(x).$$

By [4], f^* and f^{**} are normal proper l.s.c. integrands whenever f is so and we can also consider the corresponding integral functionals I_{f^*} and $I_{f^{**}}$.

$L^1(0,1,\mathbf{R}^n)$ stands for the usual space of summable functions from $[0,1]$ to $\mathbf{R}^n$ while

$$L^1_0(0,1,\mathbf{R}^n) = \left\{x \in L^1(0,1,\mathbf{R}^n) : \int_0^1 x(t)\,\mathrm{d}t = 0\right\}.$$

We define the range of I_f on $L^1(0,1,\mathbf{R}^n)$ as

$$R_1(I_f) = \{I_f(x) : x \in L^1(0,1,\mathbf{R}^n)\}$$

while $R_1(I_{f^{**}})$ has obvious meaning.

Moreover we set

$$R_2(I_f) = \{I_f(x) : x \in L_0^1(0,1,\mathbf{R}^n)\}$$

and $R_2(I_{f^{**}})$ is consequently defined.

In [2], we gave two existence theorems and some other related results, which need some assumption which we recall here for the use in the present work.

We say that f satisfies (M.B.G.C.) when

$$(M.B.G.C.) \qquad \begin{array}{l} \exists r \in \mathbf{R}_+ \quad , \quad \exists \gamma \in L^1(0,1,\mathbf{R}) \quad \text{such that} \\ f^*(t,p) \leq \gamma(t) \qquad \forall p \in \mathbf{R}^n \qquad \|p\| \leq r \end{array}$$

we say that f (B.G.C.) when

$$(B.G.C.) \qquad \begin{array}{l} \forall p \in \mathbf{R}^n \quad , \quad \exists \gamma_p \in L^1(0,1,\mathbf{R}) \quad \text{such that} \\ f^*(t,p) \leq \gamma_p(t). \end{array}$$

and we say that f satisfies $(C.C.^{**})$ when:

$$(C.C.^{**}) \qquad f^{**}(t,x) < +\infty \qquad \forall x \in \mathbf{R}^n \qquad a.e. - t \in [0,1]$$

We also always assume a condition of consistence for problems (1) and (2).

Working in $L^1(0,1,\mathbf{R}^n)$ we suppose that

$$(C1) \qquad \exists \hat{x} \in L^1(0,1,\mathbf{R}^n) : I_f(\hat{x}) \in \mathbf{R}$$

while, when we deal with $L_0^1(0,1,\mathbf{R}^n)$, we suppose that

$$(C2) \qquad \exists \overline{x} \in L_0^1(0,1,\mathbf{R}^n) : I_f(\overline{x}) \in \mathbf{R}$$

In [2], we proved the following existence theorems.

Theorem 1-[2]. *Let (M.B.G.C.) and (C1) be satisfied, then there is $x_1 \in L^1(0,1,\mathbf{R}^n)$ such that*

$$\lambda_1 = I_f(x_1) \leq I_f(x) \qquad \forall x \in L^1(0,1,\mathbf{R}^n).$$

Moreover

$$\lambda_1 = I_{f^{**}}(x_1) \leq I_{f^{**}}(x) \qquad \forall x \in L^1(0,1,\mathbf{R}^n).$$

Theorem 2-[2]. *Let (B.G.C.), (C.C**) and (C2) be satisfied then there is* $x_2 \in L_0^1(0,1,\mathbf{R}^n)$ *such that*

$$\lambda_2 = I_f(x_2) \leq I_f(x) \qquad \forall x \in L_0^1(0,1,\mathbf{R}^n).$$

Moreover there is $x_2' \in L_0^1(0,1,\mathbf{R}^n)$ *such that*

$$\lambda_2 = I_{f^{**}}(x_2') \leq I_{f^{**}}(x) \qquad \forall x \in L_0^1(0,1,\mathbf{R}^n).$$

In the sequel we constantly refer to the preceding notations and, in particular we reserve the names $x_1, x_2, x_2', \lambda_1, \lambda_2, \hat{x}, \overline{x}$ to be used in the sense specified in the previous statements.

THE RANGE OF I_f ON $L^1(0,1,\mathbf{R}^n)$

This section is devoted to describe the range of the integral functional I_f on the space $L^1(0, 1,\mathbf{R}^n)$.

We recall that, in [2], we proved the following

Theorem 3-[2]. *Let* f *satisfy (M.B.G.C.) and (C1) then*

$$R_1(I_{f^{**}}) \subset R_1(I_f) \subset [\lambda_1, +\infty).$$

A simple example shows that the assertions in theorem 3 cannot be precised if we only assume (M.B.G.C.) and (C1).

Indeed let us consider the following integrand

$$f : [0,1] \times \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$$

$$f(t,x) = \begin{cases} 1 - |x|, & -1 \leq x \leq 1 \\ +\infty, & \text{elsewhere} \end{cases}$$

we have

$$f^{**}(t,x) = \begin{cases} 0, & -1 \leq x \leq 1 \\ +\infty, & \text{elsewhere} \end{cases}$$

and $R_1(I_{f^{**}}) = \{0\}$.

On the other side if we choose $x(t) = 0$ we obtain

$$I_f(x) = 1.$$

Theorem 4. *Let* $\alpha, \beta \in R(I_f), \alpha \leq \beta$ *then* $[\alpha, \beta] \subset R(I_f)$. *A similar results holds for* $R(I_{f^{**}})$.

Proof. Let $x, y \in L^1(0,1,\mathbf{R}^n)$

$$\alpha = I_f(x), \qquad \beta = I_f(y).$$

and let us consider $\varphi : [0,1] \to \mathbf{R}$ defined as

$$\varphi(t) = \int_0^t f(s, y(s))\,\mathrm{d}s + \int_t^1 f(s, x(s))\,\mathrm{d}s.$$

φ is absolutely continuous and

$$\varphi(0) = \alpha, \qquad \varphi(1) = \beta.$$

So, $\forall \mu \in [\alpha, \beta]$ we can find $t_0 \in [0,1]$ such that $\varphi(t_0) = \mu$. Let us define

$$z(t) = \begin{cases} y(t), & t \in [0, t_0] \\ x(t), & t \in (t_0, 1] \end{cases}$$

then

$$I_f(z) = \varphi(t_0) = \mu, \qquad z \in L^1(0,1,\mathbf{R}^n)$$

and the theorem is proved. ∎

We prove now that, under suitable assumption $R_1(I_{f^{**}})$ is upper unbounded.

Lemma 5. *Let us suppose that (M.B.G.C.), (C.C**) and (C1) are satisfied; then* $\forall \alpha \in \mathbf{R}$ *there is* $\overline{y} \in L^1(0,1,\mathbf{R}^n)$ *such that*

$$\alpha \leq I_{f^{**}}(\overline{y}) \in \mathbf{R}.$$

Proof. Let $x_k \in \mathbf{R}^n, \|x_k\| \to +\infty$ and let us define

$$\vartheta_k(t) = f^{**}(t, x_k);$$

ϑ_k is a real valued, measurable function and we can define, $\forall n \in \mathbf{N}$

$$\vartheta_k^n(t) = \min\{\vartheta_k(t), n\}.$$

Since, by (M.B.G.C.), we have

$$f^{**}(t,v) \geq \langle p, v\rangle - f^*(t,p) \geq \langle p, v\rangle - \gamma(t)$$

$$\forall p \in \mathbf{R}^n \qquad \|p\| \leq r$$

and

$$f^{**}(t,v) \geq r\|v\| - \gamma(t), \qquad \forall v \in \mathbf{R}^n$$

we can assert that

$$\min\{r\|x_k\| - \gamma(t), n\} \leq \vartheta_k^n(t) \leq n$$

so, since the first and the last member is in $L^1(0,1,\mathbf{R})$, we can deduce that

$$\vartheta_k^n \in L^1(0,1,\mathbf{R}).$$

Let

$$E_n = \{t \in [0,1] : f^{**}(t,x_k) \leq n\} = \\ = \{t \in [0,1] : f^{**}(t,x_k) = \vartheta_k^n(t)\}$$

it results

$$\vartheta_k^n(t) = \begin{cases} f^{**}(t,x_k), & t \in E_n \\ n, & t \notin E_n \end{cases}$$

E_n is a sequence of measurable sets such that

$$E_{n+1} \supset E_n$$

and

$$\bigcup_{n \in N} E_n = [0,1].$$

So meas(E_n) is an increasing sequence and meas$(E_n) \to 1$.

Let us define

$$y_k^n(t) = \begin{cases} x_k, & t \in E_n \\ \hat{x}(t), & t \notin E_n \end{cases}$$

we have

$$\int_0^1 |f^{**}(t, y_k^n(t))| \mathrm{d}t = \\ = \int_{E_n} |f^{**}(t,x_k)| \mathrm{d}t + \int_{[0,1]\setminus E_n} |f^{**}(t,\hat{x}(t))| \mathrm{d}t$$

and

$$f^{**}(\cdot, y_k^n(\cdot)) \in L^1(0,1,\mathbf{R}^n).$$

Moreover

$$\int_0^1 \|y_k^n(t)\| \mathrm{d}t \geq \|x_k\| \mathrm{meas}(E_n)$$

and, by (3),

$$r\|y_k^n(t)\| - \gamma(t) \leq f^{**}(t, y_k^n(t)).$$

Now, when n is sufficiently large, we can assume that

$$\text{meas}(E_n) \geq \frac{1}{2}$$

so that

$$\begin{aligned} I_{f^{**}}(y_k^n) &= \int_0^1 f^{**}(t, y_k^n(t))\,dt \geq \\ &\geq r\int_0^1 \|y_k^n(t)\|dt - \int_0^1 \gamma(t)\,dt \geq \\ &\geq \frac{r}{2}\|x_k\| - \int_0^1 \gamma(t)\,dt \end{aligned}$$

and, when k is sufficiently large, we have

$$I_{f^{**}}(y_k^n) \geq \alpha.$$

■

The preceding result allows us to prove the following theorem

Theorem 6. *Let us suppose that (M.B.G.C.), (C.C.**) and (C1) hold; then*

$$R_1(I_f) = R_1(I_{f^{**}}) = [\lambda_1, +\infty).$$

Proof. By theorem 3 it is sufficient to prove that

$$R_1(I_{f^{**}}) = [\lambda_1, +\infty).$$

Let $\alpha \in \mathbf{R}, \alpha \geq \lambda_1$ then by lemma 5, there is $\overline{y} \in L^1(0,1,\mathbf{R}^n)$ such that

$$I_{f^{**}}(\overline{y}) \in \mathbf{R}, \qquad I_{f^{**}}(\overline{y}) \geq \alpha$$

and by theorem 4, we can assert that, since $I_{f^{**}}(x_1) = \lambda_1$, we have

$$\alpha \in R_1(I_{f^{**}})$$

and

$$R_1(I_{f^{**}}) \supset [\lambda_1, +\infty)$$

the opposite inclusion being obvious. ■

THE RANGE OF I_f ON $L_0^1(0,1,\mathbf{R}^n)$

This section is devoted to provide a description for $R_2(I_f)$. This problem is a bit more complicated than the preceding one becasue of the boundary condition $\int_0^1 x(t)\,\mathrm{d}t = 0$, which characterizes $L_0^1(0,1,\mathbf{R}^n)$ and make it a non decomposable space.

In [2] we proved a result which we recall here:

Theorem 7-[2]. *Let f satisfy (B.G.C.), (C.C.**) and (C1) then*

$$R_2(I_{f^{**}}) \subset R_2(I_f) \subset [\lambda_2, +\infty).$$

Under the same assumptions we are now able to prove that, in theorem 7, equality holds.

We begin with a lemma which is a sort of intermediate values theorem.

Lemma 8. *Let us assume that $\alpha, \beta \in R_2(I_f), \alpha \le \beta, \alpha = I_f(x_0), \beta = I_f(y_0)$, for some $x_0, y_0 \in L_0^1(0,1,\mathbf{R}^n)$ and let us suppose moreover that $x_0 - y_0$ is a function symmetric with respect to $t = \frac{1}{2}$, i.e. $(x_0 - y_0)(t) = -(x_0 - y_0)(1-t)$; then $[\alpha, \beta] \subset R_2(I_f)$.*

Proof. Let us define

$$\varphi(t) = \int_0^t f(s, y_0(s))\,\mathrm{d}s + \int_t^{1-t} f(s, x_0))\,\mathrm{d}s + \int_{1-t}^1 f(s, y_0))\,\mathrm{d}s;$$

then

$$\varphi(0) = \alpha, \qquad \varphi\left(\frac{1}{2}\right) = \beta$$

and, since φ is absolutely continuous, $\forall \mu \in [\alpha, \beta]$, we can find $t_0 \in \left[0, \frac{1}{2}\right]$ such that

$$\varphi(t_0) = \mu.$$

Let us define

$$z(t) = \begin{cases} y_0(t), & t \in [0, t_0] \cup [1 - t_0, 1] \\ x_0(t), & t \in (t_0, 1 - t_0). \end{cases}$$

We have

$$I_f(z) = \varphi(t_0) = \mu$$

and moreover

$$\int_0^1 z(t)\mathrm{d}t = \int_0^{t_0} y_0(t)\mathrm{d}t + \int_{t_0}^{1-t_0} x_0(t)\mathrm{d}t + \int_{1-t_0}^1 y_0(t)\mathrm{d}t =$$
$$= \int_0^1 y_0(t)\mathrm{d}t + \int_{t_0}^{1-t_0} (x_0(t) - y_0(t))\mathrm{d}t = 0$$

because $y \in L_0^1(0,1,\mathbf{R}^n)$ and by the assumption of symmetry of $x_0 - y_0$ with respect to $t = \frac{1}{2}$.

So $z \in L_0^1(0,1,\mathbf{R}^n)$ and $I_f(z) = \mu$ and the theorem is proven. ∎

Lemma 9. *Let us assume (M.B.G.C.), (C.C.**) and (C2); then* $\forall \alpha \in \mathbf{R}$ *there is* $\overline{y} \in L_0^1(0,1,\mathbf{R}^n)$ *such that*

$$\alpha \leq I_{f^{**}}(\overline{y}) \in \mathbf{R}$$

and $\overline{x} - \overline{y}$ *is a symmetric function with respect to* $t = \frac{1}{2}$. *We recall that* $\overline{x}$ *is the argument of minimum for* $I_{f^{**}}$.

Proof. Let $x_k \in \mathbf{R}^n, ||x_k|| \to +\infty$ and let us define

$$\widetilde{x}_k(t) = \begin{cases} \overline{x}(t) + x_k, & t \in \left[0, \frac{1}{2}\right] \\ \overline{x}(t) - x_k, & t \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Let us moreover set

$$\vartheta_k(t) = f^{**}(t, x_k(t)),$$

ϑ_k is a measurable function and we define

$$\vartheta_k^n(t) = \min\{\vartheta_k(t), n\}.$$

Exactly as in lemma 5 we have $\vartheta_k^n \in L^1(0,1,\mathbf{R}^n)$.

Let us define

$$E_n = \{t \in [0,1] : f^{**}(t, \widetilde{x}_k(t)) \leq n\} =$$
$$= \{t \in [0,1] : f^{**}(t, \widetilde{x}_k(t)) = \vartheta_k^n(t)\}$$

and

$$F_n = E_n \cup \{t \in [0,1] : 1 - t \in E_n\} =$$
$$= E_n \cup (\{1\} - E_n).$$

F_n is a measurable set and, since, as in lemma 5, $\text{meas}(E_n) \to 1$, we have

$$1 \geq \text{meas}(F_n) \geq \text{meas}(E_n) \to 1$$

and

$$\text{meas}(F_n) \to 1$$

too.

Moreover F_n is a set symmetric with respect to $\frac{1}{2}$.

Let us define

$$y_k^n(t) = \begin{cases} \widetilde{x}_k(t), & t \in F_n \\ \overline{x}(t), & t \notin F_n. \end{cases}$$

We have

$$f^{**}(\cdot, y_k^n(\cdot)) \in L^1(0,1,\mathbf{R}^n)$$

and moreover

$$\int_0^1 \|y_k^n(t)\| \mathrm{d}t \geq \int_{F_n} \|\widetilde{x}_k(t)\| \mathrm{d}t = \|x_k\| \text{meas}(F_n) - \|\overline{x}\|_{L^1(0,1,\mathbf{R}^n)}.$$

So, as in Lemma 5, we can find, for n and k sufficiently large, y_k^n such that

$$\alpha \leq I_{f^{**}}(y_k^n) \in \mathbf{R}.$$

It only remain to prove that

$$\int_0^1 y_k^n(t) \mathrm{d}t = 0.$$

Indeed

$$\int_0^1 y_k^n(t) \mathrm{d}t = \int_0^1 \overline{x}(t) \mathrm{d}t + \int_{F_n} (\overline{x}(t) - \widetilde{x}_k(t)) \mathrm{d}t = 0$$

since $\overline{x} - \widetilde{x}_k$ is a symmetric function and F_n is a symmetric set with respect to $\frac{1}{2}$. ∎

Theorem 10. *Let us assume that (B.G.C.), (C.C.**) and (C2) hold; then*

$$R_2(I_{f^{**}}) = R_2(I_f) = [\lambda_2, +\infty).$$

Proof. Let $\alpha \geq \lambda_2$; by lemma 9, we can find $\overline{y}$ such that

$$\overline{y} \in L_0^1(0,1,\mathbf{R}^n), \qquad I_{f^{**}}(\overline{y}) \in \mathbf{R}$$

and $\overline{x} - \overline{y}$ is a symmetric function with respect to $\frac{1}{2}$.

So, by lemma 8, we can conclude that

$$\alpha \in R_2(I_{f^{**}}) \quad \text{and} \quad R_2(I_{f^{**}}) \supset [\lambda_2, +\infty).$$

The opposite inclusion being obvious we can conclude by means of theorem 7. ∎

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Received March 23, 1988.
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