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RIESZ OPERATORS AND PERTURBATION IDEALS

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Abstract. *This paper is a general survey on Riesz operator theory on infinite dimensiona1 complex Banach spaces. We also outline a new approach to the study of the containment relationships which exists between some well-known perturbation ideals, starting from some recent characterizations, due to the author, of the class of all Riesz operators and of the ideal of all inessential operators.*

INTRODUCTION. In this paper we give, starting from some recent results of the author ([2], [3], [5]), a new presentation of *the theory* of *Riesz operators* on a Banach space and of *the perturbation ideals* associated with them. Riesz operators arise directly from one of the more classic sector of functional analysis: the spectral theory of compact operators. It is well-known that the spectrum of a linear compact operator defined on a Banach space, verifies a series of properties that, in literature, are known as «*Riesz-Schauder theory*» (see [26], Chap. VI). Some of these properties in the particular case of a compact integra1 operator lead to the so-called *Fredhohn alternative* for integra1 equations of second kind.

The class of Riesz operators $\mathscr{R}$ (E) on a Banach space *E* has been introduced in 1954 by Ruston ([48]) who defined such operators by considering as axioms some of the spectral properties of compact operators. Successively this class of operators has been studied and characterized by different authors (Heuser [25], Caradus [15], Dieudonné [19], West [61]). More recently such operators have been characterized geometrically by Smyth [52], by Murphy [35] and in terms of invariant closed subspaces by the author ([3], [5]).

Although we don't know if a Riesz operator admits proper closed invariant subspaces, these are an interesting tool for studying such a class of operators. In fact, as we shall see, the behaviour of a bounded operator on a infinite-dimensional closed invariant subspace together with a certain distribution of the spectrum, characterizes this type of operator. More precisely in [3] Riesz operators have been characterized «internally» as all bounded operators on a Banach space which do not admit on every closed invariant infinite-dimensiona1 subspace a continuous inverse and such that each spectral point different from zero is an isolated point. Similarly it is possible to give an «external» characterization in terms of quotient spaces and surjectivity ([5]). The study of the relationships which exist between the closed invariant subspaces and the Riesz operators will be the heart of §3 which also includes the decomposition property of a Riesz operator on a Hilbert space due to West ([60]). The class of all Riesz operators $\mathscr{R}$ *(E)* , whose general properties are given in §2, contains the class *S(E)* of all *strictly singular operators* introduced in [36] by Kato, in his treatment of the perturbation theory of semiFredholm operators, and the class C(E) of all *strictly cosingular* *operators* introduced in [42] by Pelczynski.

The reason of this inclusions, if one starts from the detinition of strictly singular operator and strictly cosingular operator, is not evident at first glance. Moreover the proof of them is really laborious (see [163, [44]). The «internal» and «external» characterizations make both inclusions more intuitive. In fact the proofs are reduced to the algebraic fact that S(E) and C(E) are both ideals of the Banach algebra $\mathscr{L}(E)$ of all bounded linear opcrators on *E* (§4). The Riesz operators, in the case dim *(E)* $= \infty$, do not form as do S(E) , C(E) and the set of all compact operators, an idea1 of $\mathscr{L}(E)$.

From many points of view the Φ -ideals, i.e. the two-sidcd ideals of operators contained in $\mathscr{R}(E)$, are more important of the same class $\mathscr{R}(E)$. For exarnple if we perturb a Fredholm operator *A* with an operator *K* belonging to a Φ -ideal, the index *(A + K)* does not change.

Probably from an abstract point of view, the most important Φ -ideal, in spite of its name, is *the* ideal *I(E)* of the *inessential operators* introduced in [38] by Kleinccke. It is meaningful the fact that this ideal is the starting-point of the extension of the Fredholm theory and of the Riesz theory in Banach algebras (see [9], or the recent papcr [8]). The idea1 of inessential operators $I(E)$ is maximal in $\mathscr{R}(E)$. We shall give some characterizations of I(E) which show a sort of symmetry of this ideal with respect to the defects ([5]).

The elaboration of such characterizations and of the relationships between the various Φ - ideals will form the centra1 part of $4. In the last paragraph we have included some interesting results of Weis and we have also included the measure of non strict singularity and the measure of non strict cosingularity introduced in [51] and [57] by Schechter and by Weis, respectively. Such measures imply the algebraic structure of S(E) and C(E) and shed light on the reason of some inclusions between those ideals and the perturbation ideals P+(E) , $P_-(E)$ defined later.

We conclude this paper by giving a general survey of the relationships between the various Φ -ideals in certain Banach spaces. Most of these relationships are still valid when we consider operators acting between different Banach spaces. For simplicity we have considered only endomorphisms.

We conclude this introduction by remarking that we have given only the proofs that we retained to be essential for avoiding a simple list of results. In particular some proofs given here seem to be new and are strictly connected with the characterization of I(E) given by the author.

§1. Throughout this paper we shall always suppose that *E* is an infinite dimensiona1 complex Banach space. By Z(E) we shall denote the Banach algebra of all bounded linear operators, by $\mathscr{F}(E)$ and X(E) we shall denote the ideal of all bounded *finite-rank operators* on *E* and *the* ideal of all *compact opemtors*, respectively.

We shall suppose that the reader possesses a good familiarity with the spectral theory of bounded operators and in particular with the so-called Riesz-Schauder theory for compact operators. Anyway a consistent part of the ideas involved here may be found in Heuser's book ([26]) or in the monography of Caradus-Pfaffenberger-Yood ([16]).

If $A \in \mathscr{L}(E)$, we shall denote by $\alpha(A)$ and $\beta(A)$ the *defects* of A, i.e. the dimension of the kemel ker (A) and the codimension of the image space $A(E)$, respectively.

An operator $A \in \mathscr{L}(E)$ is said to *be* a *Fredholm* operator if a(A) , $\beta(A)$ *are* both finite. The set $\Phi(E)$ of all Fredholm operators is a multiplicative semigroup of Z(E) . Moreover, denoting by $\pi : \mathscr{L}(E) \to \mathscr{L}(E) / \mathscr{K}(E)$ the canonical quotient map, we have

Theorem 1.1. *(Atkinson): $A \in \Phi(E)$ ifand only if $\pi(A)$ is invertible in $\mathscr{L}(E) / \mathscr{K}(E)$.*

Proof. See [26], Prop. 25.2.

The ideal X(E) in the previous theorem may be replaced, as we shall see later, by any Φ -ideal. The quotient algebra $\widehat{\mathscr{L}} = \mathscr{L}(E)/\mathscr{K}(E)$ provided with the norm $\|\hat{A}\| = \inf_{K \in \mathscr{K}(E)} \|A + K\|$ is a Banach algebra, known in the literature as the *Calkin algebra,* which has an important role in Fredholm theory.

In the following we shall use the following theorem due to Kato ([36]).

Theorem 1.2. *Let $A \in \mathscr{L}(E)$. If $\beta(A) < \infty$ then $A(E)$ is closed.*

The semigroup Q(E) is stable under certain perturbations; in fact denoting by ind(A) the index of $A(= \alpha(A) - p(A))$, *we* have

Theorem 1.3. *Let $A \in \Phi(E)$. Then*

(a) There exists an $\varepsilon = \varepsilon(A)$ such that $\|B\| < \varepsilon$ implies $A + B$ E-@(E) and $ind(A + B) = ind(A)$.

(b) For each $K \in \mathscr{K}(E)$ we have $A + K \in \Phi(E)$ and $ind(A + K) = ind(A)$.

Proof. See [26], §37.

Part (a) of the previous theorem is due to Dieudonné ([19]) whereas part (b) is due to Yood ([63]).

Let us now consider the so-called classes of *semiFredholm operators* $\Phi_+(E)$, $\Phi_-(E)$ defined as follows:

$$Q+(E) = \{A \in \mathscr{L}(E) : \alpha(A) < \infty, A(E) \text{ closed}\}$$

$$\Phi_-(E) = \{A \in \mathscr{L}(E) : \beta(A) < \infty\}.$$

The classes $\Phi_+(E)$, $\Phi_-(E)$ are multiplicative semigroups ([16], § 1.3) and clearly, Φ (E) = Q+(E) n $\Phi_-(E)$. It is possible to define an index on @+(E),@-(E) as follows. Let

$\infty - n = \infty$, $n - \infty = -\infty$ for each positive integer n; we define for each ***A*** $\in$ ***[P+(E)*** u $\Phi_-(E)$, $ind(A) = \alpha(A)$ ***-/3(A).***

For a proof of the following theorems we refer to the monographs [16], [24] or, for a proof which does not use the antipodal theorem of Borsuk, to the recent edition of Heuser book [27], Satz 82.4.

Theorem 1.4. *(Kato). If* ***A*** $\in$ ***<P+(E)*** $\cup\ \Phi_-(E)$ *, there* ***exists a*** $\varepsilon = \varepsilon(A) >$ ***0 such thatfor*** *each* $U \in \mathscr{L}(E)$ *which verifies* $\|U\| < \varepsilon$*, we have* $A + U \in \Phi_+(E) \cup \Phi_-(E)$ *,*

$$\alpha(A + U) \geq \alpha(A), \quad \beta(A + U) \geq \beta(A)$$

and

$$ind(A + U) \geq ind(A) .$$

Two important quantities, strictly connected to the defects $\alpha(A)$ and $\beta(A)$ of a linear operator A , are the lenghts of the chains

$$\ker(A) \text{ c } \ker(A^2) \text{ c } \ldots$$

$$A(E) \supset A^2(E) \supset \ldots$$

The *ascent* of ***A*** is the smallest positive integer p = ***p(A),*** whenever it exists, such that ker(***AP)*** = ker(A^{p+1}) . If such p does not exist we let p = + ∞. Analogously the *descent* of ***A*** is defined to be smallest integer g = ***g(A)*** , whenever it exists, such that $A^{q+1}(E) = A^q($ E) . If such g does not exist we let g = + ∞. It is possible to prove that if p and g are both finite then p = g ([26]).

We recall that, if ***A*** = $I -$ ***K,*** $K \in$ ***X(E) ,*** then p(A) = ***g(A)*** $< \infty$.

Theorem 1.5. ***(Heuser*** *[26]). Let* ***A : E → E be linear, E a vector space. We*** *have*

(a) If $p(A)$ = ***g(A)*** < ∞ *then* $\alpha(A) = \beta(A)$ *.*

(b) If $\alpha($***A)*** $= \beta(A)$ ***and*** *one* ***of the chains has a finite*** *lenght then* ***p = g ,***

Now, let ***E*** be a Banach space, ***A*** $\in \mathscr{L}(E)$ and let us denote by ***a(A), p(A) the*** spectrum of ***A*** and the resolvent of ***A*** , respectively. It is well known from functional calculus that the «resolvent function» $\lambda \in$ ***p(A)*** $\mapsto R_\lambda = (\lambda I - A)^{-1}$ is holomorphic. A subset σ of ***a(A)*** is said to be a *spectral* set if σ is open and closed. Let Γ_σ be an integration path separating σ from the remaining part of the spectrum. To the set σ there is associated a *spectral* projection $P_\sigma \in \mathscr{L}(E)$, defined as follows

$$P_\sigma = \frac{1}{2\pi i} \int_{\Gamma_\sigma} (\lambda I - A)^{-1} d\lambda .$$

If σ isreduced toasinglepoint $\{\lambda_0\}$ we have [26], §50) that λ_0 isapoleof $R_\lambda = (\lambda I - A)^{-1}$ if and only if $\lambda_0 I -$ ***A*** possesses finite chains. In particular $\lambda_0 I -$ ***A*** is a Fredholm operator having finite chains if and only if the projection P_λ associated with λ_0 is a finite rank operator.

2. RIESZ OI'ERATORS

The class of all Riesz operators is defined by considering some spectral properties of compact operators. Let E bea complex Banach space.

Definition. ***A* ∈ *S'(E)* is said to be a Riesz** *operator* ***iffor*** *each* $\lambda \neq 0$

(a) ***XI-A*** $\in \Phi(E)$

(b) $\lambda I - A$ ***possesses finite chains.***

It is possible to simplify the definition given above. In fact, as we have seen , the quantities α, β, p, g of an operator are strictly correlated and it is possible to show that (a) implies (b) (see [26], $52). Then if we denote by $\mathscr{R}$ *(E)* the class of all Riesz operators we have:

(1) $$\mathscr{R}(E) = \{A \in \mathscr{L}(E) : \lambda I - A \in \Phi(E) \text{ for each } \lambda \neq 0\}.$$

It is possible to give a weaker characterization of $\mathscr{R}$ *(E)* . In fact (see [27], §105)

(2) $$\mathscr{R}(E) = \{A \in \mathscr{L}(E) : \lambda I - A \in \Phi_+(E) \text{ for each } \lambda \neq 0\}.$$

or

(3) $$\mathscr{R}(E) = \{A \in \mathscr{L}(E) : \lambda I - A \in \Phi_-(E) \text{ for each } \lambda \neq 0\}.$$

We recall that ***A* ∈ *S'(E)*** is said to be ***quasi-compact*** if there exists a positive integer n and an operator $K \in$ X(E) such that $||A^n - K|| < 1$. Because of the next theorem, Riesz operators are by some authors ([48]) called *asymptotically* ***quasi-compact.*** Let π denote the canonical quotient map $\mathscr{L}(E) \to \mathscr{L}(E) / \mathscr{K}(E)$ and let us denote by r(A) the spectral radius of A .

Theorem 2.1. ***(Ruston*** *[48]). Let* ***A*** $\in$ ***S(E) . A is a Riesz*** *operator* $\Leftrightarrow$ $\pi(A)$ ***is*** *quasi-***nilpotent in the Calkin algebra** $\mathscr{L}(E)/\mathscr{K}(E)$ ***. In other words if we let*** $\hat{A} = \pi(A)$***, then A*** $\in \mathscr{R}(A)$ *if and only if*

(4) $$r(\hat{A}) = \lim_{n,\infty} [\inf_{K \in \mathscr{K}(E)} ||A^n - K||]^{1/n} = 0.$$

Proof. By the Atkinson characterization $\lambda I - A \in \Phi(E)$ if and only if $\lambda I - A$ is invertible in $\mathscr{L}(E)/\mathscr{K}(E)$ **.**

Riesz operators have been introduced in 1954 by Ruston ([48]) who defined them in a different way from that given above, by considering some other properties of compact operators. Successively in [25] Heuser considered the class of all bounded operators on a Banach space which verify (a) and (b). This class of operators was called, by Heuser, *vollfinit*. Later, in 1965, *Caradus* ([15]) shows, by eliminating some redundances, that the operators introduce by Ruston and the *vollfinit* operators coincide.

Riesz operators present, as we shall see in the sequel, a spectral situation similar to that of compact operators. The next theorem gives some informations on the distribution of the spectral points of a Riesz operator and a characterization of such operators with the aid of spectral projections.

Theorem 2.2. *Let* $A \in \mathscr{L}(E)$. *We have*

(a) $A \in \mathscr{R}(E) \Leftrightarrow$ *each spectral point* $\lambda \neq 0$ *is isolated and the spectral projection* P_λ *associated with* λ *is a finite-rank operator.*

(b) The spectrum a(A) of a Riesz operator is finite or a sequence which converges to zero.

(c) If $\dim(E) = \infty, 0$ *belongs to* $\sigma(A)$.

Proof. See [26], $52.

From theorem 2.1 it follows immediately that quasi-nilpotent operators and, a fortiori, nilpotent operators are Riesz opcrators. Moreover also each compact operator on E belongs to $\mathscr{R}(E)$. It is natura1 to ask if $\mathscr{R}(E)$ is an ideal. This is true if dim(E) $< \infty$; in fact in such case we have $\mathscr{F}(E) = \mathscr{R}(E) = \mathscr{L}(E)$. If $\dim(E) = \infty, \mathscr{R}(E)$ is not an ideal, as the following examples show.

Let $E^2 = E \times E$, E any Banach space, and let us define S, T $\in \mathscr{L}(E^2)$ as follows

$$S(x_1, x_2) = (0, x_1) \text{ and } T(x_1, x_2) = (x_2, 0)$$

where $(x_1, x_2) \in E^2$.

We have $S^2 = T^2 = 0$ and hence S, T $\in \mathscr{R}(E)$. It is easy to verify that ST($x_1, x_2) =$ $(x_1, 0)$ and $TS(x_1, x_2) = (0, x_2)$, i.e. ST and TS arebothprojectionsof E^2 onto E.

Now, in any Banach space E, a projection P is a Riesz operator if P is a finite-dimensional operator (In fact in this case we have $\alpha(I - P) = \dim(P(E)) < \infty)$.

Since in the previous example we have assumed dim $(E) = \infty$, both products ST and TS do not belong to $\mathscr{R}(E)$. We observe also that $(S+T)^2 = I$ and hence $S+T \notin \mathscr{R}(E)$.

To prove that $\mathscr{R}$ (E) is not closed in general, there is an example, due to Kakutani (cf. [21], Ex. 3.15) of a sequence $\{A_k\}$ of Riesz opcrators, defined on the Hilbert space ℓ_2 , which converges to an opcrator $A \notin \mathscr{R}(\ell_2)$.

We also observe that, differently from compact operators, the range of a Riesz operator may be not separable.

Although $\mathscr{R}$ (E.) generally is not a closed ideal of $\mathscr{L}(E)$, the sum, the product and the limit of Riesz operators are still Riesz operators if we assume the «commutativity modulo X(E) ». Precisely, denoting by [A, ***BI the*** operator ***AB – BA, we*** have

Theorem 2.3. *Let E be a Banach space.*

(a) If $A,B \in \mathscr{R}(E)$ and $[A,B] \in \mathscr{K}(E)$ then $A+B \in \mathscr{R}(E)$.

(b) It $A \in \mathscr{R}(E), B \in \mathscr{L}(E)$ and $[A,B] \in \mathscr{K}(E)$ then $AB, BA \in \mathscr{R}(E)$.

(c) If ***{A,}*** *is* ***a sequence*** *of Riesz* ***operators*** *such that $\|A_n - A\| \to 0$,* ***$A \in$ S'(E) and*** *$[A_n, A] \in$* ***X(E) foreach n,*** *then $A \in \mathscr{R}(E)$.*

The following result of West ([61]) suggests, in a certain sense, the «size» of $\mathscr{R}$ (H) in $\mathscr{L}(H)$, where H is an Hilbert space. Denoting by $\mathscr{R}_0(H)$ the subalgebra of $\mathscr{L}(H)$ spanned by $\mathscr{R}$ ***(H) we*** have

Theorem 2.4. $\mathscr{R}_0(H) = \mathscr{L}(H)$.

Proof. See [61], theorem 6.2.

If E is a finite dimensional Banach space we have that $\mathscr{R}$ ***(E)*** $= \overline{\mathscr{R}(E)} = \mathscr{L}(E)$. Conversely if $\overline{\mathscr{R}\,(E)} = \mathscr{L}(E)$, the identity I is the uniform limit of a sequence of Riesz operators ***{A,}.*** Since $[A_n, I] = 0 \in$ X(E) we have, by the previous theorem, that $I \in \mathscr{R}$ ***(E)*** and so that also implies dim (E) $< \infty$. Hence $\mathscr{R}$ ***(E)*** is dense in $\mathscr{L}($ ***E)*** if andonlyif $\dim(E) < \infty$.

Let us now denote by $\mathrm{Hol}(\sigma(A))$ the class of all complex-valued functions which are holomorphic on an open set containing the spectrum $\sigma(A)$. The following result is due to West (see [61]).

Theorem 2.5. *If* ***$A \in \mathscr{R}$ (E) and*** *$f \in Hol($* ***a(A))*** *verifies $f($* ***0) = 0 then f(A)*** $\in$ $\mathscr{R}$ ***(E) . Conversely if*** *f* ***vanishes*** *only* ***in 0, then f(A)*** $\in \mathscr{R}$ ***(E) implies A*** $\in \mathscr{R}$ ***(E) .***

Let us consider a closed subalgebra ***B(E)*** of $\mathscr{L}(E)$ such that the identity $I \in$ ***B(E)*** and let $\mathscr{K}_B$ = X(E) $\cap$ ***B(E)*** . Moreover let us denote, if ***A*** $\in$ ***B(E)*** , by $\sigma_{B(E)}(A)$ the spectrum of ***A*** with respect to ***B(E)*** . It is well-known, from the theory of Banach algebras, that if a(A) is discrete then a(A) = $\sigma_{B(E)}($ *A)* . Let us denote by $\hat{A}_B$ the quotient image of ***A*** in ***B(*** $E)/\mathscr{K}_B$.

Theorem ***2.6. Let A*** $\in$ ***B(E) . Then A is a Riesz*** *operator* ***ifand*** *only* ***if*** $r(\hat{A}_B)$ ***= 0 .***

Proof. Let ***A*** $\in$ ***B(E)*** be a Riesz operator. Since ***u(A)*** **is** finite or denumerable, ***u(A)*** = $\sigma_{B(E)}($ ***A)*** . Let us denote by P_λ the spectral projection associated with $\lambda \neq 0$, $\lambda \in$ cr(A) **.**

P_λ is a finite rank operator and moreover it belongs to B(E) . Clearly, for each $\varepsilon > 0$, the set $\{\lambda \in$ a(A) $: |\lambda| > \varepsilon\}$ is finite, suppose $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

If we let $P = \sum_{k=l}^{n} P_\lambda$, we have $P \in \mathscr{K}_B$ and

$$\sigma(A - AP) = \sigma(A(I - P)) = \sigma(A) - \{\lambda_1, \ldots, \lambda_n\}$$

thus $r(A - AP) < \varepsilon$. Since $AP \in \mathscr{K}_B$ we also have

$$r(\hat{A}_B) \geq r(A \text{ -}AP) < \varepsilon$$

i.e. $r(\hat{A}_B) = 0$. Conversely each operator of B(E) invertible modulo K_B is necessarily invertible in $\mathscr{L}(E)/\mathscr{K}(E)$ and hence if $r(\hat{A}_B) = 0$, A is a Riesz operator.

Let us denote by $\ell_\infty(E)$ the vector space of all bounded sequences $\{x_n\}$ of elements of E. Clearly $\ell_\infty(E)$ is a normed space with respect to the norm $\|\{x_n\}\| = \sup \|x_n\|$. Let us denote by m(È) the subspace of $\ell_\infty(E)$ of all the sequences $\{x_n\}$ which verify the property that each subsequence $\{x_{n_k}\}$ of $\{x_n\}$ admits a subsequence which converges in E. $\ell_\infty(E)$ is a Banach space and m(E) is a closed subspace of $\ell_\infty(E)$. Moreover if $A \in \mathscr{L}(E)$ and $\{x_n\}$ c $\ell_\infty(E)$ (resp. $\{x_n\}$ c m(E)) then {As,} c $\ell_\infty(E)$ (resp. {As,} c m(E)). Let us denote by $\tilde{E}$ the quotient space $\ell_\infty(E)/m(E)$ and if $A \in \mathscr{L}(E)$ define $\tilde{A} \in \mathscr{L}(\tilde{E})$ by

$$\tilde{A}(\{x_n\} + m(E)) = \{As,\} + m(E).$$

Clearly $A \in$ X(E) if and only if $\tilde{A} = \tilde{0}$.

It is possible to represent the Calkin algebra $\widehat{\mathscr{L}} = \mathscr{L}(E)/\mathscr{K}(E)$ in $\mathscr{L}(\tilde{E})$ by means of the representation $\gamma : \widehat{\mathscr{L}} \to \mathscr{L}(\tilde{E})$ defined by $\gamma(\hat{A}) = \tilde{A}$. Such representation studied by Lebow and Schechter ([29]) is suggested in a natural way by the following result

Theorem 2.7. *Ler $A \in \mathscr{L}(E)$. We have*

(a) $\hat{A}$ is in vertible in $\widehat{\mathscr{L}}$ (i.e. $A \in \Phi(E)$) $\Leftrightarrow$ $\tilde{A}$ is invertible in $\mathscr{L}(\tilde{E})$.

(b) $r(\hat{A}) = 0$ (i.e. $A \in \mathscr{R}(E)$) $\Leftrightarrow$ $r(\tilde{A}) = 0$.

It is an open problem if the representation γ is irriducible or if $\gamma(\widehat{\mathscr{L}})$ is a closed subset of $\mathscr{L}(\tilde{E})$. Lebow and Schechter have shown that the last property is verified if the Banach space *E possesses the* so-called *Grothendieck approximation property* (see [29]). The proof of the last theorem is obtaincd by using a quotient technique due to Buoni, Harte, Wickstead (see [9], §3). Such a technique leads to a *geometrica1 characterization of Riesz operators* due to Murphy [35] and Smyth [52]. Before giving characterizations of this kind we recall that, if D is a bounded subset of E, the *measure of non compactness q(D)* is the infimum of $\varepsilon > 0$ such that D has a finite cover by open balls in E of radius ε.

Let us denote by U_E the closed unit ball of E. If $A \in \mathscr{L}(E)$, there exists a relationship between the quantities $\|\tilde{A}\|$ and $\mu(A(U_E))$. In fact the measure of non-compactness $\mu(A(U_E))$ is a seminorm on Z(E) which induces a norm in $\mathscr{L}(E)/\mathscr{K}(E)$. Moreover we have

$$\|\tilde{A}\| \geq 4\mu(A(U_E)) \geq 8\|\tilde{A}\|$$

(see [9], proof of Theorem 0.35, p. 12). This last inequality permits to characterize $\mathscr{R}(E)$ geometrically in the following way

Theorem 2.8. *Let $A \in \mathscr{L}(E)$. The following properties are equivalent*

(a) $A \in \mathscr{R}(E)$

(b) $\lim_{n,\infty} \mu(A^n(U_E))^{1/n} = 0$ *(Murphy [35])*

(c) For each $\varepsilon > 0$ there exists a positive integer n such that $A^n(U_E)$ admits a finite cover by open balls of radius ε^n. (Smyth [52]).

A Riesz operator may be characterized by means of the action on its commutant. We recall that the commutant of A is the subset of $\mathscr{L}(E)$ defined as follows

$$z(A) = \{B \in \mathscr{L}(E) : AB = BA\}.$$

It is easy to show that t(A) is a closed subalgebra of Z(E). Moreover if we consider the operator $M_A : z(A) \to z(A)$ defined by $M_A(B) = BA$, M_A is a bounded linear operator on the Banach space $z(A)$ and (see [9], §0) $\|A\| = \|M_A\|, \sigma(A) = \sigma(M_A)$.

Theorem2.9. *Let $A \in \mathscr{L}(E)$. Then $A \in \mathscr{R}(E) \Leftrightarrow M_A \in \mathscr{R}(z(A))$.*

It is possible to prove that if $A \in$ X(E), then M_A is compact on $z(A)$ ([9], Theorem 0.52). It is somewhat surprising the fact that the converse of such property generally is not true. In fact in [35] Murphy has found a counterexample of a non-compact operator A on ℓ_1 such that M_A is compact on $z(A)$.

In [55] Vala has studied the operator φ_A defined on $Z(E)$, A a fixed operator, by $\varphi_A(B) = ABA$ for each $B \in \mathscr{L}(E)$. Also the operator φ_A has the property to be a Riesz operator on Z(E) if and only if A is a Riesz operator on E. An equivalent property is valid for compact operators. Such properties suggest some natura1 extensions of the notion of compacmess and of «Riesz element» for elements of a Banach algebra.

Such notions, together with the theory of Fredholm elements of a Banach algebra, have been studied by different authors (Bonsall [IO], Alexander [6], Vala [55], Bames [9], Pearlman

[41], Smyth [53]). An excellent and unified account of the main results of this research ama is given by the monography [9].

We conclude this section by considering an important class of operators strictly connected with the class of Riesz operators.

Definition. ***A bounded operator A on a Banach space E is said to be meromorphic if each spectral point $\lambda \neq 0$ is a pole of the resolvent $R_\lambda = (\lambda I - A)^{-1}$.***

Since a point λ is a pole of R_λ if and only if $\lambda I - A$ possessess finite chains, A is a meromorphic operator if and only if $p(\lambda I - A) = q(\lambda I - A) < \infty$ for each $\lambda \neq 0$. The spectrum of a meromorphic operator is finite ora sequence of spectral points which converge to 0. Moreover each $\lambda \neq 0$, $\lambda \in$ a(A) , is an eigenvalue of A, ([26]).

All Riesz operators are meromorphic. In a Hilbert space the spectral structure of a norma1 meromorphic operator presents some interesting properties. In fact (see [26], Prop. 70.2) each eigenvalue $\lambda \neq 0$ of A is a simple pole of R_λ and moreover

Theorem 2.10. ***Let $A \in \mathscr{L}(H)$ be norma1 and meromorhic, and let $A \neq 0$, $\{\lambda_1, \lambda_2, \ldots\}$ the set of all eigenvalues different from zero, ordered according to decreasing absolute values. If P_n is the spectral orthogonal projection of H onto $\ker(\lambda_n I - A)$, we have $A = \sum_{n=1}^{\infty} \lambda_n P_n$ (where the convergence is uniform).***

We observe that by the previous theorem each Riesz normal operator A is necessarily compact. In fact in such case we know that all P_n are finite rank operators.

3. RIESZ OPERATORS AND INVARIANT SUBSPACES

Also in this paragraph E shall denote a complex Banach space. In this section we shall study the relationships existing between Riesz operators and the closed invariant subspaces. The following theorem (due to West, [62]) is in a certain sense the starting point of the develop of the materia1 of this paragraph. Let M be a subspace of E. In the following we shall denote by $A|_M$, $A \in \mathscr{L}(E)$, the restriction of A on M.

Theorem 3.1. ***Let $A \in \mathscr{R}(E)$ and considera closed subspace M invariant under A . Then the restriction $A|_M$ of A on M is also a Riesz operator on M and $\sigma(A|_M) \subset \sigma(A)$.***

The last theorem leads to an important property of Riesz operators. In fact if $A \in \mathscr{R}(E)$ wehave,since $\lambda I - A \in \Phi(E)$ foreach $\lambda \neq 0$, $\lambda I' - A' \in \Phi(E')$ i.e. $A' \in \mathscr{R}(E')$. Conversely if $A' \in \mathscr{R}(E')$, A'' is a Riesz operator on E'' and since A is the restriction of A'' on the closed invariant subspace E of E'', so is also A , by the previous theorem.

Let us now consider the class

$\Omega_0(E) = \{A \in S(E)$: the restriction of A on each closed infinite dimensiona1 invariant subspace M for A, is not bijective) .

The class $\Omega_0(E)$ generally contains the class $\mathscr{R}(E)$. In fact let us suppose that $A \in \mathscr{R}$ (E) and $A \notin$ S&,(E) . Then there exists a closed infinite-dimensional invariant subspace M for A such that $A|_M$ admits inverse $(A|_M)^{-1}$. Since $A|_M \in \mathscr{R}(M)$ and the two operators $A|_M, (A|_M)^{-1}$ commute, the product $(A|_M) . (A|_M)^{-1} \in \mathscr{R}(M)$ and that impiies of course dim (M) $< \infty$, contradicting the assumption dim (M) = ∞.

Later we shall see that Ω_0 (E) and $\mathscr{R}(E)$ possesses the property of having the same, uniquely determined maximal ideal. For this reason one may suspect that $\mathscr{R}(E)$ and $\Omega_0(E)$ coincide.

The following example shows that generally we have $\Omega_0(E) \supsetneq \mathscr{R}$ (E) . Let G be a compact abelian group and denoting $G^\wedge$ the dual group of G. $G^\wedge$ is, as it is well-known, discrete. Let us denote by M(G) the Banach algebra of ali regular complex Borei measures on G with respect to the convolution and denoting L_1 (G) the usual L_1 -space with respect to the Haar measure of G. L,(G) may be embedded, via the Radon-Nikodym theorem, in M(G) as a closed ideal. If $\mu \in$ M(G) a *convolution operator* $T_\mu : L_1(G) \to L_1(G)$ is defined by $T_\mu f = \mu * f$ for each $f \in L_1$ (G) , where the symbol $(*)$ denotes the convolution on M(G).

The following results relate the spectral properties of the operator T_μ, with the measure-theoretical properties of $\mu \in$ M(G) .

1) T_μ is compact on L_1 (G) $\Leftrightarrow \mu \in L_1(G)$, i.e. μ is absolutely continuous with respect to the Haar measure on G (Akemann [7], Kitchen [37]).

2) If $\mu^k \in L_1(G)$ for some integer k then T_μ is a Riesz operator. Conversely if μ^k is singular with respect to the Haar measure for each k, then T_μ is not a Riesz operator (see proof of Theorem 2 of [4]).

Now, iet us consider a measure $\mu \in$ M(G) which verifies the following two properties

a) μ^k is singular with respect to he Haar measure, for each positive integer k.

b) $\lim_{n,\infty} \mu^\wedge(n) = 0$, where $\mu^\wedge$ denotes the Fourier-Stieltjes transform of μ defined by

$$\mu^\wedge(n) = \int_G e^{-inx} \, d\mu(x) \qquad \text{for each } n \in G^\wedge.$$

We observe that such a measure does exist (see [3]).

The property (a) implies of course that $T_\mu \notin \mathscr{R}(L_1(G))$ and (b) implies $T_\mu \in \Omega_0(L_1(G))$ (see [3]).

The following theorem characterizes $\mathscr{R}(E)$ in the class of all $\Omega_0(E)$ -operators (see [3], Theorem 1).

Theorem 3.2. *Let $A \in \mathscr{L}(E)$. Then $A \in \mathscr{R}(E)$ if and only if the following two conditions hold*

1) $A \in \Omega_0(E)$

2) $\sigma(A)$ is finite or a sequence which converges to zero.

It is possible to improve the last theorem. Let us consider the class

> $\Omega_1(E) = \{A \in S(E)$: *the* restriction $A|_M$ of A on any closed infinite-dimensional invariant subspace M for A does not admit a bounded inverse $(A|_M)^{-1} : A(M) \to M\}$.

Clearly $\Omega_1(E) \subset \Omega_0(E)$ and generally we have $\mathscr{R}(E) \subsetneq \Omega_1(E)$. In fact a well-known result of Read ([45]) proves that there exists a bounded operator A on ℓ_1 which does not admit a non trivial closed invariant subspace. Moreover $sp(A) = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$, therefore $A \notin \mathscr{R}(\ell_1)$ whereas we have trivially $A \in \Omega_1(\ell_1)$.

Theorem 3.3: *(Internal characterization of a Riesz operator). Let $A \in \mathscr{L}(E)$. Then $A \in \mathscr{R}(E)$ if and only if the following conditions hold:*

(a) $A \in \Omega_1(E)$.

(b) $\sigma(A)$ is finite or a sequence which converges to zero.

Proof. See Aiena [3].

Now, let us denote by M any closed invariant subspace for $A \in S(E)$ and let $[A]_M : E/M \to E/M$ denote the map defined, for each residual class $[x]_M$, by

$$[A]_M[x]_M = [Ax]_M, \text{ where } x \in [x]_M.$$

It is easy to see that, since M is invariant for A, the map $[A]_M$ is well-defined. Moreover if $\pi_M : E \to E/M$ denotes the quotient map, and N denotes a algebraic complement of $(\lambda I - A)(E)$, $\lambda \neq 0$, we have

$$\beta(\lambda[I]_M - [A]_M) \leq \dim(\pi_M(N)) \leq \dim(N) = \beta(\lambda I - A).$$

The last inequality, together with the characterization (3) of $\mathscr{R}(E)$, leads directly to the following theorem

Theorem 3.4. *If $A \in \mathscr{R}(E)$ and M is a closed invariant subspace for A, then $[A]_M$ is a Riesz operator on E/M.*

Let us consider the following class

$\Omega_2(E) = \{A \in \mathscr{L}(E) : [A]_M$ is not surjective for each closed infinite-codimensional invariant subspace M for $A\}$.

Since the two maps $\pi_M A : E \to E/M$ and $[A]_M$ have the same range we also have

$\Omega_2(E) = \{A \in \mathscr{L}(E) : \pi_M A$ is not surjective for each closed infinite-codimensional invariant subspace M for $A\}$.

The class R,(E) contains $\mathscr{R}(E)$ (see [5]). The same example of Read of an operator on ℓ_1, mentioned before, shows that generally $\Omega_2(E) \supsetneq \mathscr{R}(E)$. The classes C+(E) and Ω_2 (E) present some relationships analogues to those, found by Pelczynski ([42]), between the ideal of all strictly singular operatos and the ideal of all strictly cosingular operator (see §4). In fact we have

Theorem 3.5. *(Volkmann, Wacker [57]): Let E be any Banach space. Then*
(a) $A^* \in \Omega_2(E^*) \Rightarrow A \in \Omega_1(E)$.
(b) $A^* \in \Omega_1(E^*) \Rightarrow A \in \Omega_2(E)$.

The following characterization of the class $\mathscr{R}(E)$ among the $\Omega_2(E)$ operators is, in a certain sense, duel to that given in Theorem 3.3 (see Aiena [5]).

Theorem 3.6. ***(external characrerization). Let $A \in \mathscr{L}(E)$. Then $A \in \mathscr{R}(E)$ ifand only if the following two conditions are verified***
(a) $A \in \Omega_2(E)$
(b) a(A) is finite ora sequence which converges to zero.

It is well-known that if E is a Banach space having dimension ≥ 2 and $A \in X(E)$, then there exists a closed subspace of E invariant under A . This result, known in literature as the Theorem of Aronszajn and Smith, is also an immediate consequence of the famous result of Lomonosov ([31]) which establishes the existence of a proper hyperinvariant subspace (i.e. invariant for each operator which commutes with A) for any compact operator. If A is a non quasi nilpotent Riesz operator, each spectral point $\lambda \neq 0$ is an eigenvalue and hence there exists a $x \neq 0$ such that Tx = Xx. Then it follows, if dim($E) \geq 2$, that the unidimensional subspace M spanned by x is a proper closed invariant subspace for A . Unfortunately if A is a quasi-nilpotent Riesz operator, we don't know if a closed proper subspace invariant for A does exist. Now, let $K \in$ X(E) and Q a quasi-nilpotent operator.

If $\pi : \mathscr{L}(E) \to \mathscr{L}(E)/\mathscr{K}(E)$ denotes the canonical quotient map we have

$$\sigma(\pi(K + Q)) = \sigma(\pi(Q)) \subset \sigma(Q) = \{0\}$$

i.e. K + Q is a Riesz operator. In [61] West has shown that such decomposition property characterizes all Riesz operators on a .complex Hilbert space. In fact we have

Theorem 3.7. ***Let H be a complex Hilbert space. Then $A \in \mathscr{R}(H)$ if and only if there exists a $K \in X(H)$ and a quasi-nilpotent operator Q such that $A = K + Q$.***

It is an open problem whether the decomposition of the last theorem is always possible for a Riesz operator defined on a Banach space. The method used by West for decomposing a Riesz operator on a Hilbert space is analogous to the process of super-diagonalizing a matrix and then splitting it into the sum of diagonal and nilpotent matrices. In fact a Riesz operator on a Hilbert space is super-diagonalizable along the direct sum of the ranges of the spectral projections associated with the non zero spectral points and the remainder of such a process is the quasi-nilpotent operator Q . Such method presents some analogies witht the super-diagonalization process of a compact operator due to Ringrose ([46]), which depends essentially on the above mentioned theorem of Aronszajn and Smith. We also have observed that an equivalent result is not known for Riesz operators and without itone cannot expect a complete theory of super-diagonalization for such a class of operatos. Recently Davison and Herrero ([IS]) have shown that a West-type decomposition is valid for Riesz operators acting on a large class of Banach spaces including c_0 and ℓ^p, $1 \leq p < \infty$.

Definition. ***A Banach space E will be said to have a finite dimensiona1 p-block decomposition ($1 \leq p \leq \infty$) provided there are finite dimensiona1 subspaces E_n such that***

1) For each $x \in E$, there is a unique sequence $\{x_n\} \subset E_n$ such that $x = \sum_{n=1}^{\infty} x_n$ (norm con vergence)

2) $$\|x\| = \left(\sum_{n=1}^{\infty}\|x_n\|^p\right)^{1/p} \quad (1 \leq p < \infty),\ \text{or}\ \|x\| = \sup_n \|x_n\| \ (p = \infty).$$

Examples of such spaces are the spaces c_0 and ℓ^p($1 \leq p < \infty$), taking $E_n = \mathbb{C}$ forall n= $1, 2, \ldots$. We have ([18])

Theorem 3.8. ***Let E be a Banach space having the finite-dimensional p-block decomposition property. Then every Riesz operator A decomposes as a sum $K + Q$, where $K \in X(E)$ and Q is quasi-nilpotent.***

We conclude this paragraph by giving other two characterizations of Riesz operators due to Dieudonné ([19]) and West ([62]), respectively. The first one involves invariant subspaces (see also [21], Theor. 3.20).

Theorem 3.9. ***Let E be a Banach space. Then $E \in \mathscr{R}(E)$ if and only if the following condition is verified:***

For each $\lambda \in a(A)$, $\lambda \neq 0$ there exist two closed invariant subspaces M_λ, N_λ such that $E = M_\lambda \oplus N_\lambda$, $\dim(N_\lambda) < \infty$, the restrictions $(\lambda I - A)|N_\lambda$, $(\lambda I - A)|M_\lambda$ are nilpotent and bijective, respectively.

The second characterization is obtained decomposing for each $\lambda \in p(A)$ the resolvent $R_\lambda = (\lambda I - A)^{-1}$.

Theorem 3.10. *Let $A \in S'(E)$. A is a Riesz operator ifand only iffor each $\lambda \in p(A)$, $\lambda \neq 0$, we have $(\lambda I - 1)^{-1} = C(X) + B(\lambda^{-1})$ where C(X) is compact, $B(z)$ is an entire function and $C(\lambda)B(\lambda^{-1}) = B(\lambda^{-1})C(\lambda)$.*

4. Φ -IDEALS

We have observed that if dim $(E) = \infty$ then the class of all Riesz operators is not an idea1 of $\mathscr{L}(E)$. The class $\mathscr{R}$ *(E)* certainly contains the ideal (P(E) of all finite-rank operators and the closed ideal X(E) of all compact operators. If H is a separable Hilbert space a well-known result of Calkin ([14]) establishes that X(H) is the unique closed two-sided idea1 of $\mathscr{L}(H)$. This result has been, successively, extended to the spaces $\ell^p(1 \leq p < \infty)$ and c_0 by Gohberg-Markus-Feld'man ([23]). In a Banach space E the ideal structure of $\mathscr{L}(E)$ may be very complicated and a complete classification of all the ideals of $\mathscr{L}$(E) can be impossible (see [15], §5.3). In general $\mathscr{R}$ *(E)* contains severa1 closed ideals. A classica1 example of an ideal contained in $\mathscr{R}$ (E), which arise from applied work with operators, is the ideal W of the weakly compact operators on $E = L_1[0, 1]$. In fact if $A \in W$, we have $A^2 \in \mathscr{K}(L_1[0, 1])$ ([65, p. 322,323) and the Ruston characterization of Riesz operators implies that $W \subset \mathscr{R}$ *(L, [0, 1])*. In [23] there is given, in the cases $E = L_p[0, 1]$ $1 \leq p < 2$, $E = L_\infty[0, 1]$, $E = C[0, 1]$, an other example of ideal of $\mathscr{L}(E)$ contained in $\mathscr{R}$ *(E)* and containing X(E), the so-called idea1 of all strictly singular operators (or Kato operators).

Definition. *A two-sided idea1 J* ***of%(E)*** *, E a Banach space, is said to be a Φ – idea1 if*

(a) $J \supset \mathscr{F}(E)$

(b) $I - A \in \Phi(E)$ *for each* $A \in J$.

Clearly if A belongs to a Φ -ideal J, $I - \lambda A \in$ <D(E) for each λ, hence A is a Riesz operator. It is possible to show that in the Atkinson characterization of Fredholm operators and in the Ruston characterization of $\mathscr{R}$ *(E)*, $\mathscr{K}$*(E)* may be replaced by each closed Φ - idea1 (see [26], §51, §52). Moreover it is possible to prove that the closure of a Φ -ideal remains a Φ -ideal of $\mathscr{L}$*(E)* ([26], §51).

Next, we want to introduce the *ideal* I(E) *of inessential operators* on *E,* introduced in [38] by Kleinecke, which plays for many aspects a fundamental role in the theory of Riesz operators. To define I(E), first let us denote by $\mathscr{A}$ any Banach algebra with identity $e \neq 0$. The (Jacobson) *radical* of $\mathscr{A}$ is the intersection of all primitive ideals of $\mathscr{A}$ (or, also, the intersection of all the left (or right) maximal ideals of $\mathscr{A}$ and, since $\mathscr{A}$ possesses an identity, it is possible to prove that ([9])

(1) Rad ($\mathscr{A}$) = $\{x \in \mathscr{A}$: e $-$ ax(or e $-$ sa) is invertible for each $a \in \mathscr{A}\}$

Denoting by $\pi : \mathscr{L}(E) \to$ %'(E)/..%(E) = $\mathscr{L}$ the canonica1 quotient homomorphism, finally we define

(2)
$$I(E) = \pi^{-1}(\text{Rad}(\widehat{\mathscr{L}})).$$

It is easy to verify that I(E) is the uniquely determined ideal of Riesz operators. Each Φ -ideal is contained in I(E) . Moreover by Atkinson chamcterization of Φ (E) and (1) it follows that

(3)
$$I(E) = \{A \in \mathscr{L}(E) : I - UA \in \Phi(E), \quad \forall U \in \mathscr{L}(E)\}$$

and simmetrically

(4)
$$I(E) = \{a \in \mathscr{L}(E) : I - AU \in \Phi(E), \quad \forall U \in \mathscr{L}(E)\}.$$

Now, let us retum to the abstract situation. Let $\mathscr{S}$ be a subset of the Banach algebra $\mathscr{A}$. We define ***the perturbation class associated with*** $\mathscr{S}$ as the set

$$P(\mathscr{S}) = \{A \in \mathscr{A} : a + s \in \mathscr{S}, \text{ foreach } s \in \mathscr{S}\}.$$

It is easy to show that if $\alpha\mathscr{S} \subset \mathscr{S}, \alpha \neq 0$, then $P(\mathscr{S})$ is a subspace of $\mathscr{A}$. Moreover if $\mathscr{S}$ is open in $\mathscr{A}$ then $P(\mathscr{S})$ is closed ([29], Lemma 2.1). Let us denote by $\mathscr{G}$ the group of all invertible elements of $\mathscr{A}$ and let us suppose that $\mathscr{S}$ verifies the condition $\mathscr{G}\mathscr{S} \subset \mathscr{S}$. If we take $a \in \mathscr{G}$, $b \in P(\mathscr{S})$, $s \in \mathscr{S}$, we have

$$ab + s = a(b + a^{-1}s) \in \mathscr{S}$$

hence $P(\mathscr{S})$ is a left ideal of $\mathscr{A}$. Analogously the condition $\mathscr{S}\mathscr{G} \subset \mathscr{S}$ implies that P(Y) is a right ideal.

Since Q,(E) verifies, whenever $\mathscr{A} = \mathscr{L}(E)$, the conditions considered above on $\mathscr{S}$, the perturbation class $P(\Phi$ (E)) is a closed idea1 of Z(E) , which we shall call ***Riesz ideal.***

Now let us denote by $\overline{\beta}(A)$ the closed defect of A , i.e. the codimension of $\overline{A(E)}$ ***. The*** following results are due to Schechter ([50]) (part (a)), and Lebow and Schechter ([29]) (part (b)), respectively. The proofs may also be found in [163, p. 70-7 1.

Theorem 4.1. *(a)* $A \in \Phi_+(E) \Leftrightarrow \alpha(A - K) < \infty$ *for each* $K \in X(E)$

(b) $A \in \Phi_-(E) \Leftrightarrow \overline{\beta}(A - K) < \infty$ *for each* $K \in X(E)$.

In the following theorems we reassume some characterizations of $P(\Phi(E))$ the proof of which depends on the last theorem.

Theorem 4.2. *Let* $A \in \mathscr{L}(E)$. *The following properties are equivalent*

(a) $A \in P(\Phi(E))$

(b) $\alpha(A - S) < \infty$ *for each* $S \in \Phi(E)$

(c) $\overline{\beta}(A - S) < \infty$ *foreach* $S \in$ Q>(E)

(d) $\beta(A - S) < \infty$ *for each* $S \in \Phi(E)$.

Proof. (a) $\Leftrightarrow$ (b) is due to Schechter ([50]). For the equivalence (a) $\Leftrightarrow$ (c) see Aiena [5]. That (a) $\Rightarrow$ (d) follows directly from the inclusion $P(\Phi(E))$ c $\{A \in \mathscr{L}(E) : \beta(A-S) < \infty\}$. That (d) $\Rightarrow$ (c) from the inequality $\overline{\beta}(A - S) \leq \beta(A - S)$.

Theorem 43. *Let* $A \in \mathscr{L}(E)$. *The following properties are equivalent*

(a) $A \in P(\Phi(E))$

(b) $\alpha(I - UA) < \infty$ *for each* $U \in \mathscr{L}(E)$

(c) $\overline{\beta}(I - AU) < \infty$ *for each* $U \in \mathscr{L}(E)$

(d) $\beta(I - AU) < \infty$ *for each* $U \in \mathscr{L}(E)$.

Proof. For the equivalence (a) $\Leftrightarrow$ (b) (due to Pietsch, [43]) see [2]. The equivalence (a) $\Leftrightarrow$ (c) is given in Aiena [5]. Now let $A \in P(\Phi(E))$ and $U \in \mathscr{L}(E)$. Since $P(\Phi(E))$ is an ideal, $AU \in P(\Phi(E))$ and since $I \in$ Q,(E) we have $I - AU \in \Phi(E)$ and thus $\beta(I - AU) < \infty$. Then we have (a) $\Rightarrow$ (d). The implication (d) $\Rightarrow$ (c) follows directly from the inequality $\overline{\beta}(I - AU) < \beta(I - AU)$.

Theorem ***4.4.*** $P(\Phi(E)) = I(E)$ ***(Schechter*** *[50])*.

Proof. First we observe that $P(\Phi(E))$ is a Φ -ideal. In fact S(E) c $P(\Phi(E))$ by Theorem 4.2, since $I \in \Phi(E)$, we have $I\text{-}A \in$ <D(E) foreach $A \in P(\Phi(E))$. Hence $P(\Phi(E)) \subset$ I(E) . Conversely if $A \in$ I(E) , we have (by characterization (4) of I(E)) $I - AU \in \Phi(E)$ for each $U \in \mathscr{L}(E)$ and hence $\beta(I - AU) < \infty$ for each $U \in \mathscr{L}(E)$. By characterization (d) of Theorem 4.3. we have then I(E) c $P(\Phi(E))$.

Let us now consider the following two classes of *Atkinson operators*

$$\Phi_\alpha = \{A \in \mathscr{L}(E) : \alpha(A) < \text{CO}, A(E)\ \textit{complemented}\}$$

$$\Phi_\beta = \{A \in \mathscr{L}(E) : P(A) < \infty, \ker(A)\ \textit{complemented}\}$$

Denoting by $\Phi_{\alpha\beta}$ the class

$$\Phi_{\alpha\beta} = \{A \in \mathscr{L}(E) : A \in \Phi_\alpha \cup \Phi_\beta; ind(A) \in \mathbb{Z}\}$$

where $\mathbb{Z}$ is the set of integers, we have ([SO]).

Theorem 4.5. $I(E) = P(\Phi_\alpha) = P(\Phi_\beta) = P(\Phi_{\alpha\beta})$.

We have seen in the previous paragraph how it is possible to characterize the class of Riesz operators among the $\Omega_1(E)$ -operators. It is natural to ask what relationships exist between I(E) and the classes Ω_1 (E) , Ω_2 (E) . The next theorem shows such relationships, in other words we have an intemal and an extemal characterization of I(E) .

Theorem 4.6. *(Aiena [2], [5]): **Let E be a complex Banach space. Then***

(a) I(E) is the uniquely determined maximal ideal of the class Ω_1 (E) , Each ideal of Ω_1 (E) -operators is a Φ -ideal.

(b) I(E) is the uniquely detennined maximal ideal of the class Ω_2 (E) . Each ideal of Ω_2 (E) -operators is a Φ -ideal.

We have already remarked that in a separable Hilbert space H , $\mathscr{L}(H)$ admits only a closed ideal which necessarily coincides with $\mathscr{K}(H)$. Generally that is not true in a not separable Hilbert space. In each case we are interested in the simpler problem of determining the Φ -ideals. We shall see now that in a Hilbert space there exists only a closed Φ -ideal.

First we reassume some definitions and some familiar results valid on Hilbert spaces (see the monograph [47]). An operator $U \in \mathscr{L}(H)$ is said to be a *partial isometry* if there exists a closed subspace M of H such that

$$\|Ux\| = \|x\| \quad \text{for each } x \in M \text{ and } Ux = 0 \text{ for each } x \in M^\perp$$

where $M^\perp$ is the orthogonal in the Hilbert space sense.

If A^* is the Hilbert adjoint of $A \in$ S(E) , the operator AA^* is selfadjoint and positive, therefore it has sense to define the operator $|A| = (AA^*)^{1/2}$. We recall that, by the *polar **decomposition** theorem*, there exists, for each $A \in \mathscr{L}(H)$, a partial isometry U such that $A = U|A|$ and $|A| = U^*A$.

Theorem 4.7. ***An Hilbert space H admits only one closed Φ -ideal. This idea1 coincides necessarily with X(H) .***

Proof. We need only to prove that X(H) $\supset$ *I(H)* . Let $A \in I(H)$. By the polar-decomposition theorem there exists a partial isometry U such that $|A| = U^*A$, *thus, I(H)* being an ideal, we have $|A| \in I(H)$. The operator $|A|$ is self-adjoint and hence $|A| \in$ X(H) (as alredy has been remarked, each self-adjoint Riesz operator is a compact operator). Then it follows that also $A = U \,.\, |A| \in X(H)$.

As we have seen, the situation of Φ -ideals in a Banach spaces E generally may be rather complicated. Let now consider the following two perturbation classes $P_+(E) = P(\Phi_+(E))$ and $P_-(E) = P(\Phi_-(E))$.

These perturbation classes have been introduced in [23] by Gohberg, Markus and Feld'man. Since Q?+(E) and $\Phi_-(E)$ are open subsets of $\mathscr{L}(E)$ and since for each invertible operator $A \in \mathscr{L}(E)$ we have $A \cdot \Phi_+(E) \subset \Phi_+(E)$ and $A \cdot \Phi_-(E) \subset \Phi_-(E)$, we have that the two perturbation classes $P_+(E)$ and $P_-(E)$ are closed ideals of $\mathscr{L}(E)$. Moreover, since $\Phi_+(E) \supset \Phi(E)$ and $\Phi_-(E) \supset \Phi(E)$ we have that $P(\Phi) = I(E)$ contains $P_+(E)$ and $P_-(E)$.

Theorem **4.8.** *$P_+(E)$ and $P_-(E)$ are Φ -ideals. Moreover we have*

$$\mathscr{K}(E) \subset P_+(E) \subset I(E)$$

$$\mathscr{K}(E) \subset P_-(E) \subset I(E)$$

Proof. For the inclusions X(E) $\subset P_+(E)$ and X(E) $\subset P_-(E)$ see [16], §4.4. .

In [23] it is shown that if $E = \ell_q \oplus L_p[-1,1]$, where $1 < p < q < 2$, then the inclusion $P_-(E) \subset$ I(E) and X(E) $\subset P_-(E)$ are proper. Moreover if E' is the dual of E we have $P_+(E') \neq P_-(E')$ and $\mathscr{K}(E') \neq P_+(E')$.

The following theorem provides two characterizations of the ideals $P_+(E)$ and $P_-(E)$, respectively. A proof of part (a) may be found in [50]. The characterization of $P_-(E)$ seems to be new.

Theorem **4.9.** *Let $A \in \mathscr{L}(E)$. Then*

(a) $A \in P_+(E) \Leftrightarrow \alpha(A - S) < \infty$ for each $S \in \Phi_+(E)$

(b) $A \in P_-(E) \Leftrightarrow \overline{\beta}(A - S) < \infty$ for each $S \in \Phi_-(E)$

(c) $A \in P_-(E) \Leftrightarrow \beta(A - S) < \infty$ for each $S \in \Phi_-(E)$.

Proof. (b) If $A \in P_-(E)$, by the definition of $P_-(E)$ we have $\beta(A - S) < \infty$ for each $S \in \Phi_-$, hence $\overline{\beta}(A - \mathrm{S}) < \infty$. Conversely let $A \notin P_-(E)$. Then there exists $S \in \Phi_-(E)$ such that $A - S \notin \Phi_-(E)$. By theorem 4.1 there exists a $K \in$ X(E) such that $\overline{\beta}(A - S - \mathrm{K}) = \infty$. Taking $\mathrm{C} = S - K$ we have $\overline{\beta}(A - C) = \infty$ and since X(E) $\subset P_-(E)$ we also have $\mathrm{C} \in \Phi_-(E)$. Therefore also $\Leftarrow$ has been proved.

The equivalence (c) it follows from the inequality $\overline{\beta}(A - S) \leq \beta(A - S)$. ∎

Now we want to introduce other two important Φ -ideals: the ideal of all *strictly singular operators,* introduced in [36] by Kato in his perturbation theory for closed, densily defined, semiFredholm operators on a Banach space, and the idea1 of all *strictly cosingular operators* introduced in [42] by Pelczynski.

Definition. *Let E and F be Banach spaces.* $A \in \mathscr{L}(E, F)$ *is said to be strictly singular if for each closed injìnite-dimensiona1 subspace M of E the restriction* $A|_M$ *does not admit a bounded inverse.*

The following deep result due to Kato (see [24], §III.2) shows the relationship which exists between compactness and strictly singularity.

Theorem 4.10. *Let* $A \in Z(E)$. *Then the following properties are equivalent:*

(a) A is strictly singular

(b) For each closedinfinite-dimensiona1 subspace M of E there exists an infinite-dimensional closed subspace $N \subset M$ *such that* $A|_N$ *is compact.*

(c) For each $\varepsilon > 0$ *and each infinite-dimensiona1 subspace M of E there exists a closed subspace* $N \subset M$, $\dim(N) = \infty$, *such that the restriction of A on N has norm* $\leq \varepsilon$.

(d) For each closed subspace M , $\dim(M) = \infty$, *the restriction* $A|_M \notin \Phi_+(M, E)$.

Let us denote by S(E) the set of all strictly singular operators. It is useful, in strict analogy with the measure of non-compactness, to introduce for each $A \in \mathscr{L}(E)$ a number which quantifies the property «$A \notin S(E)$ ». Such a number has been introduced in [51] by Schechter as follows.

Definition. *Let* $A \in \mathscr{L}(E)$. *We define measure of non-strictly singularity the number*

$$\Delta(A) = \sup_{M} \inf_{N \subset M} \|A|_N\|$$

where the sup is taken over all infinite-dimensiona1 closed subspaces N, M **of** *E. If we let* $\Gamma_+(A) = \inf_M \|A|_M\|$ *we have, directly from the dejinitions,*

$$\Gamma_+(A) \leq \Delta(A) \leq \|A\|. \tag{5}$$

Moreover (see [50]) we have

Theorem 4.11. *Let* $A, B \in \mathscr{L}(E)$. *Then*

(a) $A \in \Phi_+(E) \Leftrightarrow \Gamma_+(A) > 0$

(b) $A \in S(E) \Leftrightarrow \Delta(A) = 0$

(c) $\Gamma_+(A + B) \leq \Gamma_+(A) + \Gamma_+(B)$

(d) $\Delta(A+B) \leq \Delta(A) + \Delta(B)$.

In particular (5), together the inequalities (c) and (d), implies that Γ_+ and A are two continuous seminorms on $\mathscr{L}(E)$. Moreover (b) and (d) imply that S(E) is an ideal of $\mathscr{L}(E)$.

We also observe that S(E) **is** closed. In fact if $\{A_n\}$ is a sequence of strictly singular operators which converges to $A \in \mathscr{L}(E)$, then A $(A,) \to$ A $(A) = 0$, i.e. $A \in S(E)$.

Theorem 4.12. ***S(E)* is a Φ *-ideal.***

Proof. S(E) contains X(E) and it is contained in Ω_1 (E) . Moreover S(E) is an ideal hence, by Theorem 4.6, S(E) $\subset$ I(E) .

The following example shows that generally X(E) $\not\supseteq S$(E) .

Let $K : L_1[0,1] \to L_1[0,1]$ beanintegraloperatordefinedasfollows

$$(Kf)(s) = \int_0^1 k(s,t)f(t)\mathrm{d}t \tag{6}$$

where the kemel $k(s,t)$ is bounded and measurable in [0, 1] x [0,1] . Such operator is weakly compact (see Dunford-Schwarz [22], IV.8.11). In [42] Pelczynski has shown that the ideal J of all weakly compact operators on E = L_1 [0, 1] coincides with S(E) . To obtain an example of a non compact operator $A \in$ S(E) it suffices to prove that there exists a kemel k(s, t) such that the integral operator (6) is not compact. An example of kemel which satisfies this property may be found in [24] (Example III.3.10).

In [32] Markus and Russu have shown that for each non reflexive Banach space having a symmetric base (except ℓ_1 and c_0) we have X(E) $\neq$ S(E) . The same happens for any Lorentz-sequence space except ℓ_p. In [33] Milman and in [60] Weis have shown that for $E = L_p(\Omega,\mu),(\Omega,\mu)$ σ-finite, and for $E = C[0,1]$ we have $S(E) = P(\Phi(E))$.

Definition. ***A Banach space E is said to be subprojective iffor each closed subspace M of E*** , dim ***(M)*** $= \infty$, ***there exists a closed infinite dimensional subspace N*** $\subset$ ***M and a continuous projection of E onto N .***

Subprojective spaces were investigated by Whitley ([63]) and examples of subprojective spaces are ℓ_p , $1 \leq p < \infty$, c_0 , L_p[0, 1] where $2 \leq$ p $< \infty$ and each Hilbert spaces.

The following result is due to Pfaffenberger ([40]). The easy proof given here is obtained by using Theorem 4.6.

Theorem 4.13. ***Let E be subprojective. Then S(E) = I(E) .***

Proof. We have only to prove the inclusion S(E) $\supset$ ***I(E)*.**

Let us suppose that there exists $A \in$ I(E) such that $A \notin$ S(E) . Then there exists a closed infinite-dimensional subspace M such that the restriction $A|_M : M \to A(M)$ is a homeomorphism. Since E is subprojective there exists a complemented subspace V , dim $(V) = \infty$, such that $V \subset A(M)$. Let us denote by U a topological complement of V and define $T \in \mathscr{L}(E)$ as follows

$$T = \begin{cases} (A|_M)^{-1} & \text{on } V \\ 0 & \text{on } U. \end{cases}$$

If we let $S = AT$, we have $S \in I(E), S|_V = (AT)|_V = I|_V$ contradicting Theorem 4.6. ∎

The following result shows that generally I(E) $\neq$ S(E) . (Clearly that also happens for the examples of Banach spaces considered after Theorem 4.9).

Theorem *4.14. (Weis [60]). Suppose that E has two subspaces M, N such that*

(a) both are isomorphic to c_0 or the same ℓ_p for some $1 \leq p < \infty$

(b) M is complementedin E

(c) N contains no infinite-dimensional subspace complemented in E . Then I(E) $\neq$ $\neq S(E)$.

In the following we list some examples of Banach spaces which verify the conditions (a), (b), (c) of the last theorem.

(a) Let E = C(K) where K is the disjoint union of the one-point compactification $\widetilde{\mathbf{N}}$ of N and the Stone-Cech-compactification $\beta\mathbf{N}$ of N , (see [60], $4).

(b) Let $E = L_p(0,\infty) + L_p(0,\infty)$ where $1 < p < q < 2$ and

$$||f|| = \inf\{||h||_p + ||g||_q : f = h + g\}$$

([60], §4).

(c) Let $E = U_{c,d}$ be an Orlicz sequence space defined as in [30], Theorem 4.b.12, where $1/d + 1/c = 1$.

Other examples of Banach spaces E which verify S(E) $\neq$ *I(E) are* given in [32] by Markus and Russu.

In [57] the following quantitative version of Theorem 1.3 has been proved.

Theorem 4.15. *Let A,B $\in \mathscr{L}(E)$. Then, if $\Delta(A) <$* **T+(B), we** *have $A + B \in \Phi_+(E)$ and ind(A + B) = ind(A) .*

Later we will give an analogous of the last theorem for the set $\Phi_-(E)$. We remark that othér quantitative versions of Theorem 1.3 may be obtained by using some quantities strictly connected with the measure of non-compactness (see [17]).

An immediate consequence of the previous Theorem and Theorem 4.11 is the following result

Corollary ([23]). *For each Banach space E, S(E) $\subset$ P+(E) .*

It is a long standing open question whether S(E) = *P+(E)* . This question, as the otber relative to the class *P_(E)* and the ideal of strictly cosingular operators, is of interest because a positive answer would be a topological characterization of the class *P+ (E)* . Of course, whenever S(E) = I(E) , the ideal *P+(E)* coincides with S(E) (For example, as we have seen, for $E = \ell_p, E = L_p[0,1], E = C[0,1]$). In [60] Weis has proved that $P_+(E)$ $= S(E)$ fora large class of Banach spaces, including most classical Banach spaces and he reduced the general question to some other long unsolved problems in Banach space theory. To expose the result of Weis we give first the following

Definition. ***A Banach space E is said to be weakly compactly generated (w.c.g.) ifthe linear span of some weakly compact subset of E is dense in E.***

All reflexive and all separable Banach space are w.c.g. Also $L_1(\Omega, \mu)$, if (Ω, μ) is a-finite, verifies this property (see [20], Chap. 5).

Theorem 4.16. ***(Weis [60]). Let E be w.c.g. Then S(E) = P+(E) .***

Now we want introduce another class of operators, defined in [42] by Pelczynski, by dualizing the concept of injectivity which appears in the definition of strict singularity, with the concept of surjectivity.

Definition. ***$A \in \mathscr{L}(E, F)$, E, F Banach spaces, is said to be strictly cosingular iffor each injìnite-codimensional closed subspace M of E the operator $\pi_M A : E \to F/M$ is not surjective.***

An important example of strictly cosingular operator is the injection $J : c_0 \to \ell_\infty$ (see [42]). Let us denote by C(E) the set of all strictly cosingular operators on *E.* In [42] it has been shown that between S(E) and C(E) there are the same relationships already observed for the classes $\Omega_1(E)$, $\Omega_2(E)$. More precisely if $A^* \in C(E)$ (respectively $A^* \in S(E)$) then $A \in S(E)$ (respectively $A \in C(E)$).

The following theorem is, in a certain sense, dual to Theorem 4.9.

Theorem ***4.17. (Vladimirski, see [44]). Let $A \in \mathscr{L}(E)$. Then the following properties are equivalent:***

(i) $A \in C(E)$

(ii) ***For each infinite-codimensional closed subspace M, there exists a closed infinite-codimensional subspace $N \supset M$ such that $\pi_N A : E \to F/N$ is a compact map.***

(iii) ***For each $\varepsilon > 0$ and each closed infinite-codimensional subspace M there exists a closed subspace $N \supset M$, codim (N) = ∞, such that $\|\pi_N A\| \le \varepsilon$.***

(iv) For each closed infinite-codimensional subspace $M \subset E$, $\pi_M A \notin \Phi_-(E, E/M)$.

The set C(E) contains the ideal X(E) (In fact if $A \in X(E)$, for each closed subspace M, codim$(M) = \infty$, we have $\pi_M A \in \mathscr{K}(E, E_M)$).

In analogy with the strictly singular operators it is possible, for any $A \in \mathscr{L}(E)$, *to* define a number which quantifies the property «$A \notin C(E)$ ».

Definition. ***Let $A \in \mathscr{L}(E)$. We define measure of not strictly cosingularity the number***

$$\Delta'(A) = \sup_M \inf_{N \supset M} \|\pi_N A\|$$

where M, N are closed subspaces having infinite codimension.

If we take $\Gamma_-(A) = \inf_M \|\pi_M A\|$, M a closed space having infinite-codimension, we have from the definition

$$\Gamma_-(A) \leq \Delta'(A) \leq \|A\|.$$

The following theorem is dual to Theorem 4.11.

Theorem 4.18. *([58]). Let* ***A,*** $B \in \mathscr{L}(E)$. ***Then we have***

(a) $A \in \Phi_-(E) \Leftrightarrow \Gamma_-(A) > 0$

(b) $A \in C(E) \Leftrightarrow \Delta'(A) = 0$

(c) $\Gamma_-(A + B) \leq \Gamma_-(A) + \Gamma_-(B)$

(d) A’(A + B) $\leq$ A’(A) + A’(B).

By the previous theorem and directly from the definitions it follows that the quantities Γ_- and A’ are both seminorms on $\mathscr{L}(E)$. Moreover C(E) is a closed ideal. It is of interest to observe that the Fredholm radius $r_\Phi(A)$ of ***A***

r_Φ ***(A)*** = the spectral radius of the image of ***A*** in $\mathscr{L}(E)/\mathscr{K}(E)$,

verifies the following properties ([58])

$$r,(A) = \lim_{n,\infty} \Delta(A^n)^{1/n} = \lim_{n,\infty} \Delta'(A^n)^{1/n}.$$

Theorem 4.19. C(E) ***is a*** Φ *-ideal.*

Proof. We have X(E) c C(E) c $\Omega_2(E)$. Moreover ***C(E)*** is an ideal, hence $C(E)$ C I(E) (Theorem 4.6).

In [33] Milman has shown that for ***E*** = $L_1[0, 1]$ we have C(E) **= *S(E)*** = W, where W is the ideal of all weakly compact operators on ***E.*** Therefore C(E) $\neq$ ***X(E)*.** If ***E*** = $L_p[0, 1]$, where $p = 1$ or $2 \leq p < \infty$, we have C(E) = S(E) = ***I(E)*** ([33]) and the same equalities are true whenever $1 < p < 2$ (Weis [59], or ***E*** = $C[0, 1]$ ([33]).

Definition. ***A Banach space E is*** *said* ***to be superprojective iffor*** *each* ***closed*** *subspace* ***M of E,*** *codim* ***(M)*** = ∞, *there exists* ***a closed complemented*** *subspace* ***N*** $\supset$ ***M having*** *infinite* ***codimension.***

Examplex of superprojective spaces are Hilbert spaces, all spaces ℓ_p, $1 < p < \infty$, and the spaces $L_p[0, 1]$, $1 < p < 2$ (see [63]). The spaces $C[0, 1]$ and B(S) = space of all bounded functions on a infinite set S, are neither subprojective nor superprojective.

To prove the next theorem we shall use Theorem 4.6.

Theorem 4.20. *If E is superprojective then $C(E) = I(E)$.*

Proof. We have only to show that $I(E) \subset C(E)$. Suppose that there exists $A \in I(E)$ such that $A \notin C(E)$. Then there exists a infinite-codimensional closed subspace M such that $\pi_M A : E \to E/M$ is surjective. It is easy to verify that $A^{-1}(M)$ is an infinite-codimensional subspace of E. Since E is superprojective there exists an infinite-codimensional subspace $V \supset A^{-1}(M)$ such that $E = V \oplus U$, V complemented. We have codim(V) = dim $(U) = \infty$, moreover since $\pi_M A$ is surjective, for any $\hat{y} \in E/M$ there exists an $x \in E$ such that $\hat{y} = (\pi_M A)(x)$. If $x = v + u, v \in V, u \in U$, let define $S : E/M \to E$ by $S(\hat{y}) = u$. Since $V \supset A^{-1}(M) = \ker(\pi_M A)$, S is well defined and taking $T = S\pi_M$, we have $(AT)|_U = I|_U$. Since $B = AT \in I(E)$ this contradicts Theorem 4.6. .

The following examples show that generally $C(E) \neq I(E)$.

Examples. (a) Let $F = \ell_q \oplus L_p$, $1 < p < q < 2$. F is superprojective, hence by the previous theorem $C(F) = I(F)$. In [23] it has been proved that there exists an operator $A \in I(F)$ such that $A \notin S(F)$. The dual A' of A is not strictly cosingular (see [42], Prop. 3). Since F is reflexive, taking $E = F'$, it follows that $A' \in S(E)$ (see [42], Proposition 3.b). Moreover since E is subprojective we have $S(E) = I(E)$ and therefore $A' \in I(E)$.

(b) Let $E = L_p(0,\infty) \cap L_q(0,\infty)$ provided with the norm

$$||f|| = \max(||f||_p, ||f||_q).$$

If we take p, q such that $2 < q < p < \infty$, we have $C(E) \neq I(E)$.

This result follows by duality from the example (b) which follows Theorem 4.14 (see [60]).

(c) Let $E = U_{c,d}$ defined as in (c) after Theorem 4.14. The relationship $C(E) \neq I(E)$ follows by duality from Theorem 4.13 and from $U'_{c\,d} \approx U_{c,d}$ (see the remark following 3.b.12) in [30]).

Theorem 4.21. *Let $A, B \in \mathscr{L}(E)$. If $\Delta'(A) < \Gamma_-(B)$ then $A + B \in \Phi_-(E)$ and moreover $ind(A + B) = ind(A)$.*

Proof. See [58].

If $A \in C(E)$ and $B \in \Phi_-(E)$ we have $\Delta'(A) = 0$ and $\Gamma'(B) > 0$, hence, by the previous theorem, we obtain the following result of Vladimirski ([56]).

Corollary. *If E is a Banach space, $C(E) \subset P_-(E)$.*

Also for the Φ-ideals $C(E)$ and $P_-(E)$ it is an open question if they coincide. As for S(E) Weis has proved in [60] the following result.

Theorem 4.22. *If E is w.c.g. then $C(E) = P_-(E)$.*

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