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ON PROJECTIONS ON SUBSPACES OF FINITE CODIMENSION

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Dedicated to the memory of Professor Gottfried Köthe

Let $(X, \|\ \|)$ be a Banach space. Let V be a subspace of codimension k. By $\lambda(V,X)$ we shall denote the infimum of the norms of linear continuous projections mapping X onto V:

$$(1) \qquad \lambda(V,X) = \inf\{\|P\| : P^2 = P, PX = V\}.$$

Let

$$(2) \qquad \overline{\lambda}_k(X) = \sup\{\lambda(V,X) : \text{codim } V = k\},$$

and

$$(3) \qquad \underline{\lambda}_k(X) = \inf\{\lambda(V,X) : \text{codim } V = k\}.$$

In [3], [4] it was shown that for spaces $L^p[0,1], 1 \leqq p \leqq +\infty$, we have

$$(4) \qquad \underline{\lambda}_1(L^p[0,1]) = \overline{\lambda}_1(L^p[0,1]) \leqq 2^{|\frac{2}{p}-1|}.$$

Thus a natural question arises about the extension of the equality (4) for $k > 1$. In the present note it will be shown that

$$(5) \qquad \underline{\lambda}_k(L^p[0,1] \leq \underline{\lambda}_1(L^p[0,1])$$

and

$$(6) \qquad \overline{\lambda}_k(L^p[0,1] > \overline{\lambda}_1(L^p[0,1])$$

for $k \geqq 2$ and p either close enough to 1 or sufficiently large.

Proposition 1. $\overline{\lambda}_k(L^p[0,1]) \leqq \underline{\lambda}_1(L^p[0,1]) (k = 2, 3, \ldots)$.

Proof. Let an integer $k > 1$ be fixed. Let V be a subspace consisting of those elements x that $\int_{(i-1)k}^{i/k} x(t)\,dt = 0 (i = 1, 2, \ldots, k)$, i.e.

$$(7) \qquad V = \left\{x \in L^p[0,1] : \int_{\frac{i-1}{k}}^{i/k} x(t)\,dt = 0, \qquad i = 1, 2, \ldots k\right\}.$$

Now we consider subspaces $X_i, i = 1, \dots, k$, consisting of those elements of $L^p[0,1]$ which have supports contained in the interval $\left[\frac{i-1}{k}, \frac{i}{k}\right]$, i.e.

$$X_i = \left\{ x \in L^p[0,1] : \sup x \subset \left[\frac{i-1}{k}, \frac{i}{k}\right] \right\}. \tag{8}$$

It is easy to check that $L^p[0,1]$ is a direct sum of $X_1, \dots, X_k$:

$$L^p[0,1] = X_1 + \dots + X_k \tag{9}$$

and for $x_i \in X_i$ $(i = 1, \dots, k)$ we have

$$\|x_1 + \dots + x_k\| = (\|x_1\|^p + \dots + \|x_k\|^p)^{1/p}. \tag{10}$$

Observe that $V \cap X_i$ is a subspace of codimension 1 in X_i. Thus for each $\varepsilon > 0$ there is a projection P_i mapping X_i onto $V \cap X_i$ with the norm $\|P_i\| \leq \underline{\lambda}_1(L^p[0,1]) + \varepsilon$.

Let

$$Px = P_1 x_1 + \dots + P_k x_k \text{ for } x = x_1 + \dots + x_k,\ x_i \in X_i.$$

Then P is a projection of X onto V and

$$\begin{aligned} \|P_x\| &= \left(\|P_1 x_1\|^p + \dots + \|P_k x_k\|^p\right)^{1/p} \leq \\ &\leq \left(\underline{\lambda}_1 \left(L^p[0,1] + \varepsilon\right)\right) \left(\|x_1\|^p + \dots + \|x_k\|^p\right)^{1/p} \leq \\ &\leq \left(\underline{\lambda}_1 \left(L^p[0,1] + \varepsilon\right) \|x\|\right). \end{aligned} \tag{11}$$

The arbitrary of ε implies that

$$\underline{\lambda}_k (L^p[0,1]) \leq \lambda (V, L^p[0,1]) \leq \underline{\lambda}_1 (L^p[0,1]). \tag{12}$$

□

Let

$$V = \left\{ x \in L^p[0,1] : \int_0^1 \sin 2\pi t x(t)\, dt = 0, \int_0^1 \cos 2\pi t x(t)\, dt = 0 \right\}. \tag{13}$$

Clearly, V is a subspace of $L^p[0,1]$, $1 < p < +\infty$ and codim $V = 2$.

Proposition 2. *The linear operator*

$$(14)\qquad Px = x - 2\left(\int_0^1 \sin 2\pi\tau x(\tau)\,dt \cdot \sin 2\pi t + \int_0^1 \cos 2\pi\tau x(\tau)\,dt \cdot \cos 2\pi t\right)$$

mapping $L^p[0,1]$ *onto* V *is a projection with minimal norm.*

Proof. Let $T_s, 0 \leq s < 1$ be a family of isometries mapping $L^p[0,1]$ onto itself and defined as follows

$$(15)\qquad (T_s x)|_t = \begin{cases} x(t+s) & \text{if} \quad t+s \leq 1 \\ x(t+s-1) & \text{if} \quad t+s > 1. \end{cases}$$

Observe that the space V is invariant under T_s. Indeed,

$$(16)\qquad \int_0^1 \sin 2\pi t (T_s x)|_t dt = \int_0^1 \sin 2\pi (t-s) x(t)\,dt =$$
$$= \int_0^1 \sin 2\pi t \cos 2\pi s x(t)\,dt - \int_0^1 \cos 2\pi t \sin 2\pi s x(t)\,dt = 0$$

by the definition of V. In a similar way we can prove that

$$(16)\qquad \int_0^1 \cos 2\pi t (T_s x)|_t dt = 0.$$

By (15) and (16), $T_s V = V$ for $0 \leq s < 1$.

Let P_0 be an arbitrary linear projection onto V. Let

$$(17)\qquad P = \int_0^1 T_s P_0 T_s^{-1} ds.$$

The linear operator P has norm non greater than the norm of $||P_0||$. Indeed, T_s are isometries and $||T_s|| = 1 = ||T_s^{-1}||$. Therefore

$$(18)\qquad ||P|| \leq \int_0^1 ||T_s|| \cdot ||P_0|| \cdot ||T_s^{-1}||ds = ||P_0||.$$

Now we shall show that P is a projection on V. To begin with, we observe that linear combinations of functions $\{1, \sin 2 \cdot 2\pi t, \cos 2 \cdot 2\pi t, \ldots, \sin k \cdot 2\pi t, \cos k \cdot 2\pi t, \ldots\}$ are dense in the space V.

Since P_0 is a projection on V, we have

$$P_0 \sin k2\pi t = \sin k2\pi,\ tP_0 \cos k \cdot 2\pi t = \cos k \cdot 2\pi t \text{ for } k = 0, 1, 2, \ldots$$

and

$$\begin{aligned} P \sin k \cdot 2\pi t &= \int_0^1 T_s P_0 T_s^{-1} \sin k \cdot 2\pi t ds = \\ &= \int_0^1 T_s P_0 \sin k2\pi(t-2) ds = \\ &= \int_0^1 T_s P_0 [\sin k2\pi t \cdot \cos k2\pi s - \\ &- \cos k2\pi t \cdot \sin k \cdot 2\pi s]\, ds = \\ &= \int_0^1 T_s [\sin k2\pi t \cos k2\pi s - \cos k2\pi t \sin k2\pi s]\, ds = \\ &= \int_0^1 T_s \cdot T_s^{-1} \sin k2\pi ds = \sin k2\pi t. \end{aligned} \tag{19}$$

By a similar consideration $P \cos k \cdot 2\pi t = \cos k \cdot 2\pi t$.

Since linear combinations of functions $\{1, \sin 2 \cdot 2\pi t, \cos 2 \cdot 2\pi t, \ldots, \sin k \cdot 2\pi t, \cos k \cdot 2\pi t \ldots\}$ are dense in V, we obtain that

$$Px = x \text{ for } x \in V. \tag{20}$$

Let now calculate Px more precisely.

Recall that P_0 can represented in the form (see, for example, [2])

$$P_0 x = x - \int_0^1 \sin 2\pi t x(\tau) d\tau \cdot x_s(t) - \int_0^1 \cos 2\pi t x(\tau) d\tau x_c(t), \tag{21}$$

where x_s, x_c are such that

$$\begin{aligned} \int_0^1 \sin 2\pi t x_s(t)\, dt = 1 = \int_0^1 \cos 2\pi t x_c(t)\, dt \\ \int_0^1 \sin 2\pi t x_c(t)\, dt = 0 = \int_0^1 \cos 2\pi t x_s(t)\, dt. \end{aligned} \tag{22}$$

Let P^1 be a projection operator defined as follows:

$$P^1 x = \int_0^1 \sin 2\pi\tau x(\tau)\, d\tau x_s(t) + \int_0^1 \cos 2\pi\tau x(\tau)\, d\tau x_c(t). \tag{23}$$

Let

$$\overline{P}x = \int_0^1 T_s P T_s^{-1} x ds. \tag{24}$$

We shall calculate $\overline{P}$:

$$\begin{aligned}
(25)\quad \overline{P}x &= \int_0^1 \left(\int_0^1 \sin 2\pi(\tau+s)x(\tau)d\tau\right)(T_s x_s)|_t ds+ \\
&+ \int_0^1 \left(\int_0^1 \cos 2\pi(\tau+s)x(\tau)d\tau\right)(T_s x_c)|_t ds = \\
&= \int_0^1 \left(\int_0^1 (\sin 2\pi\tau\cos 2\pi s + \cos 2\pi\tau \sin 2\pi s)x(\tau)d\tau\right)(T_s x_s)|_t ds = \\
&= \int_0^1 \left(\int_0^1 (\cos 2\pi\tau\cos 2\pi s - \sin 2\pi\tau \sin 2\pi s)x(\tau)d\tau\right)(T_s x_c)|_t ds = \\
&= \int_0^1 \left(\int_0^1 \sin 2\pi\tau x(\tau)d\tau\right)\cos 2\pi s(T_s x_s)|_t ds+ \\
&+ \int_0^1 \left(\int_0^1 \cos 2\pi\tau x(\tau)d\tau\right)\sin 2\pi s(T_s x_s)|_t ds+ \\
&+ \int_0^1 \left(\int_0^1 \cos 2\pi\tau x(\tau)d\tau\right)\cos 2\pi s(T_s x_c)|_t ds- \\
&- \int_0^1 \left(\int_0^1 \sin 2\pi\tau x(\tau)d\tau\right)\sin 2\pi s(T_s x_c)|_t ds = \\
&= \int_0^1 \sin 2\pi\tau x(\tau)d\tau \int_0^1 \cos 2\pi(u-t)x_s(u)du+ \\
&+ \int_0^1 \cos 2\pi\tau x(\tau)d\tau \cdot \int_0^1 \sin 2\pi(u-t)x_s(u)du+ \\
&+ \int_0^1 \cos 2\pi\tau x(\tau)d\tau \cdot \int_0^1 \cos 2\pi(u-t)x_c(u)du- \\
&- \int_0^1 \sin 2\pi\tau x(\tau)d\tau \cdot \int_0^1 \sin 2\pi(u-t)x_c(u)du = \\
&= \int_0^1 \sin 2\pi\tau x(\tau)d\tau \left(\int_0^1 \sin 2\pi u \sin 2\pi t x_s(u)du - \right. \\
&\left. - \int_0^1 \cos 2\pi t \cos 2\pi u x_s(u)du\right) +
\end{aligned}$$

$$+\int_0^1 \cos 2\pi\tau x(\tau)\,d\tau\left(\int_0^1 \sin 2\pi u \cos 2\pi t x_s(u)\,du-\right.$$

$$\left.-\int_0^1 \cos 2\pi u \sin 2\pi t x_s(u)\,du\right)+$$

$$+\int_0^1 \cos 2\pi\tau x(\tau)\,d\tau\left(\int_0^1 \cos 2\pi u \cos 2\pi t x_c(u)\,du+\right.$$

$$\left.+\int_0^1 \sin 2\pi u \sin 2\pi t x_c(u)\,du\right)-$$

$$-\int_0^1 \sin 2\pi\tau x(\tau)\,d\tau\left(\int_0^1 \sin 2\pi u \cos 2\pi t x_c(u)\,du-\right.$$

$$\left.-\int_0^1 \cos 2\pi u \sin 2\pi t x_c(u)\,du\right)=$$

$$=\left(\int_0^1 \sin 2\pi\tau x(\tau)\,d\tau\cdot\sin 2\pi t+\int_0^1 \cos 2\pi\tau x(\tau)\,d\tau \cos 2\pi t\right)\ .$$

Thus $\overline{P}$ is a projection on a subspace generated by $\{\sin 2\pi t, \cos 2\pi t\}$, and $P = I - \overline{P}$ is of form (14).

Recall that, by (18), we have

$$||P|| \leq ||P_0||$$

for an arbitrary projection P_0. Therefore P is a projection with minimal norm. □

Proposition 3. *In the space* $L^1[0,1]$ *the operator* P *given by formula (14) has the norm non less than* $1+\frac{4}{\pi}$ *:*

$$||P|| \geq 1+\frac{4}{\pi}. \tag{26}$$

Proof. First we shall show that the norm of the projection

$$\overline{P}x = 2\left(\int_0^1 \sin 2\pi\tau x(\tau)\,d\tau \sin 2\pi t+\int_0^1 \cos 2\pi\tau x(\tau)\,d\tau \cos 2\pi t\right)$$

is greater than 1. Let ε be an arbitrary positive number. Let $\delta > 0$ be chosen in such a way that

$$\begin{aligned}\left|\sin 2\pi t - \sin\frac{\pi}{4}\right| < \varepsilon\\ \left|\cos 2\pi t - \cos\frac{\pi}{4}\right| < \varepsilon\end{aligned}\quad \text{for } \left|t-\frac{1}{4}\right| < \delta.$$

Let

$$x_\varepsilon(t) = \begin{cases} \dfrac{1}{2\delta} & \text{if} \quad \left|t - \dfrac{1}{4}\right| \le \delta \\ 0 & \text{if} \quad \left|t - \dfrac{1}{4}\right| > \delta \end{cases} \tag{27}$$

It is easy to check that $||x_\varepsilon|| = 1$. Observe that

$$\int_0^1 \sin 2\pi\tau x_\varepsilon(\tau)\,d\tau = \int_0^1 \cos 2\pi\tau x_\varepsilon(\tau)\,d\tau > \frac{1}{\sqrt{2}} - \varepsilon \tag{28}$$

Hence

$$\begin{aligned}
||\overline{P}x_\varepsilon|| &= 2a\int_0^1 |\sin 2\pi t + \cos 2\pi t|\,dt = \\
&= 2a\cdot\left[\int_0^{3/8}(\sin 2\pi t + \cos 2\pi t)\,dt + \int_{7/8}^{0}(\sin 2\pi t + \cos 2\pi t)\,dt - \right.\\
&\left. - \int_{3/4}^{7/8}(\sin 2\pi t + \cos 2\pi t)\,dt\right] = 4a\int_{-1/8}^{3/8}(\sin 2\pi t + \cos 2\pi t)\,dt = \\
&= 4a\frac{1}{2\pi}\left[-\cos 2\pi t|_{-1/8}^{3/8} + \sin 2\pi t|_{1/8}^{3/8}\right] = 4a\frac{1}{2\pi}\cdot 2\sqrt{2} = \\
&= \frac{4\sqrt{2}\cdot a}{\pi} > \frac{4\sqrt{2}}{\pi}\cdot\left(\frac{1}{\sqrt{2}} - \varepsilon\right) = \frac{4}{\pi} - \frac{4\sqrt{2}}{\pi}\varepsilon.
\end{aligned}$$

Thus, the arbitrariness of ε implies that

$$||\overline{P}|| \ge \frac{4}{\pi}.$$

By Babenko-Pričugov theorem [1], we find

$$||P|| \ge 1 + \frac{4}{\pi}.$$

Theorem 1. *The following inequality holds for* p *sufficiently close to 1:*

$$||P||_{L^p[0,1]} > 2$$

where P is defined by (14).

Proof. Consider $||x_\varepsilon||_{L^p[0,1]}$ and $||Px_\varepsilon||_{L^p[0,1]}$, where x_ε given by (27) are continuous functions of p. Since

$$\frac{||Px_\varepsilon||_{L^1[0,1]}}{||x_\varepsilon||_{L^1[0,1]}} \geq 1 + \frac{4}{\pi} > 2,$$

we get the theorem, □

Theorem 2. *For q sufficiently large,*

$$||P||_{L^q[0,1]} > 2.$$

Proof. By the form of P, (see (14)). Clearly, the operator P^* conjugate to P is of the same form and

$$||P|| = ||P^*||.$$ □

Finally, we obtain

Theorem 3. *The following inequality holds for p either sufficiently close to 1 or sufficiently large:*

$$\overline{\lambda}_2(L^p[0,1]) > \overline{\lambda}_1(L^p[0,1]).$$

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