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SURJECTIVITY OF PARTIAL DIFFERENTIAL OPERATORS IN CLASSES OF ULTRADIFFERENTIABLE FUNCTIONS OF ROUMIEU TYPE

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Dedicated to the memory of Professor Gottfried Köthe

INTRODUCTION

It is well known, that a partial differential operator $P(D)$ with constant coefficients is surjective in $C^\infty(\Omega)$ if and only if the open set Ω is P-convex (see e.g. [10], section 10.6), which means, that the supports of T and $P(-D)T$ have the same distance to the boundary of Ω for $T \in C^\infty(\Omega)'$. The proof of this result is easy, since $C^\infty(\Omega)$ is an F-space and the closed range theorem can be applied. The more difficult topological structure of the space $\mathscr{D}(\Omega)'$ of distributions does not allow the application of this theorem, nevertheless a similar characterization of the surjectivity of $P(D)$ in $\mathscr{D}(\Omega)'$ holds, involving also the singular supports of T and $P(-D)T$ for $T \in C^\infty(\Omega)'$ ([10], Corollary 10.7.10). In both cases, $P(D)$ is surjective, if Ω is convex.

The situation changes completely, if Gevrey classes $\Gamma^{\{s\}}(\Omega)$ of ultradifferentiable functions are considered. It was noticed by E. De Giorgi [9] and L. C. Piccini [15], that simple partial differential operators may not be surjective in the space of real analytic functions $\mathscr{A}(\mathbb{R}^N) = (\Gamma^{\{1\}}(\mathbb{R}^N))$, and L. Hörmander [11] then characterized the surjectivity of $P(D)$ in $\mathscr{A}(\mathbb{R}^N)$ by means of a Phragmen-Lindelöf principle valid on the characteristic variety of $P(-z)$. Sufficient conditions were proved by K. G. Andersson [1], L. Cattabriga [6] and L. Cattabriga and E. De Giorgi [8]. Nonquasianalytic Gevrey classes where considered by L. Cattabriga [7] and G. Zampieri [18]. Recently R. Braun, R. Meise and D. Vogt [4,5] used the results of D. Vogt [17] on the projective limit functor to characterize the surjectivity of $P(D)$ in nonquasianalytic classes $\mathscr{E}_{\{\omega\}}(\Omega)$ of ultradifferentiable functions of Roumieu type again by means of a certain Phragmen-Lindelöf principle. Their result implies the same characterization for surjective partial differential operators in H. Komatsu's ultradifferentiable functions $\mathscr{E}_{\{M_\alpha\}}(\Omega)$, if the conditions (1.2) (see section 1) and (M2) (of H. Komatsu [12]) are assumed.

One topological reason for the more difficult behaviour of partial differential operators in $\mathscr{E}_{\{\omega\}}(\Omega)$ and $\mathscr{E}_{\{M_\alpha\}}(\Omega)$ seems to be, that these spaces then are isomorphic to the $\mathbb{N}$-fold product of the dual of a power series space of *finite* type (see [3] and [14]), while the distributions are isomorphic to the $\mathbb{N}$-fold product of $(s)'$ ([16]), the space (s) of rapidly decreasing sequences being the model space of an *infinite* type power series space. Now it was observed in [14], that there is a change of type for the spaces $\mathscr{E}_{\{M_\alpha\}}(\Omega)$, in fact, for «large» ultradifferentiable classes defined by sequences like $M_a := e^{ca^b}$, $1 < b \leq 2$, $\mathscr{E}_{\{M_a\}}(\Omega)$ is isomorphic to the dual of a power series space of *infinite* type (see 1.3 for more examples). So the question arises, how the surjectivity of partial differential operators can be characterized for these classes of ultradifferentiable functions of Roumieu type and this problem will be solved in the present paper.

We generally assume, that $(M_a)_{a\geq 0}$ satisfies the conditions (M1), (M2') and (M3') of H. Komatsu [12] (see section 1). The main condition for the results of this paper now is the following:

$$\begin{aligned} &\forall k \geq 1\ \exists n \geq 1 : M(t) - km(t) \geq M(t/n) \\ &\qquad M(t/k) \geq M(t) - nm(t) \quad \text{for large } t \end{aligned} \tag{1.1}$$

(see section 1 for the notation and easy sufficient conditions). Assuming (1.1), it is shown, that the partial differential operators behave in $\mathscr{E}_{\{M_a\}}(\Omega)$ similarly as in $\mathscr{D}(\Omega)'$. In fact, we also consider the ultradistributions $\mathscr{D}_{(M_a)}(\Omega)'$ of Beurling type and we will prove, that the surjectivity of $P(D)$ in $\mathscr{E}_{\{M_a\}}(\Omega)$ is equivalent to the surjectivity of $P(D)$ in $\mathscr{D}_{(M_a)}(\Omega)'$ and it can be characterized by means of P-convexity and several (equivalent) P-convexity conditions for singular supports (see Corollary 3.4). It turns out, that the surjectivity of $P(D)$ is gouverned by singular support conditions involving $m(t)$ rather than $M(t)$, as could be guessed from the results of G. Björck [2] on the surjectivity of $P(D)$ in $\mathscr{D}_{(\omega)}(\Omega)'$. Also we get surjectivity in the spaces defined on convex sets. So the result is like in the classical case of distributions. This is most striking in the special case, where $M_a = e^{ca^2}$, $c > 0$, which is maximal with respect to condition (M2'). In fact our result implies, that in this case $P(D)$ is surjective in $\mathscr{E}_{\{M_a\}}(\Omega)$ (or in $\mathscr{D}_{(M_a)}(\Omega)'$), if and only if $P(D)$ is surjective in $\mathscr{D}(\Omega)'$.

1. PRELIMINARIES

The aim of this paper is to study the surjectivity of partial differential operators in spaces of ultradifferentiable functions and ultradistributions (defined by some sequence $(M_a)_{a\geq 0}$) in the sense of H. Komatsu [12]. The notation connected with these notions is that of [12], especially,

$$M(t) := \sup \left(ln\,|t|^{a} M_0/M_a \mid a \in \mathbb{N}_0 \right),\ t \in \mathbb{C},$$

is the function associated with $(M_a)_{a\geq 0}$,

$$m_a := M_a/M_{a-1} \quad \text{for } a \geq 1$$

$$m(t) := \max \{a \mid m_a \leq |t|\},\ t \in \mathbb{C}.$$

We will generally assume in this paper (without explicit reference), that $(M_a)_{a\geq 0}$ satisfies the conditions of logarithmic convexity (M1), stability under differential operators (M2') and non quasianalyticity (M3') of H. Komatsu, that is,

$$(M1) \qquad m_a \text{ is increasing}$$

$$(M2') \qquad m(t) \geq b\, ln t - 1/b \quad \text{for some } b > 0$$

$$(M3') \qquad (1/m_a)_{a\geq 1} \in l_1 .$$

The definition of ultradifferentiable functions and ultradistributions is now as follows ([12]):

Definition 1.1. *Let $\Omega \subset \mathbb{R}^N$ be open and $K \subset\subset \mathbb{R}^N$.*

a) ultradifferentiable functions of Beurling type

$$\mathscr{E}_{(M_a)}(\Omega) := \{f \in C^\infty(\Omega) \mid \forall K \subset\subset \Omega \forall C > 0 :$$
$$p_{C,K}(f) := \sup\left\{|D^\alpha f(x)| C^{|\alpha|}/M_{|\alpha|} \mid \alpha \in \mathbb{N}_0,\ x \in K\right\} < \infty\Big\}$$
$$\mathscr{D}_{(M_a)}(K) := \left\{f \in \mathscr{E}_{(M_a)}(\mathbb{R}^N) \mid \operatorname{supp} f \subset K\right\}$$
$$\mathscr{D}_{(M_a)}(\Omega) := \lim_{K\subset\subset\Omega} \operatorname{ind} \mathscr{D}_{(M_a)}(K)$$

b) ultradifferentiable functions of Roumieu type

$$\mathscr{E}_{\{M_a\}}(\Omega) := \left\{f \in C^\infty(\Omega) \mid \forall K \subset\subset \Omega \exists C < 1 : p_{C,K}(f) < \infty\right\}$$
$$\mathscr{D}_{\{M_a\}}(K) := \left\{f \in \mathscr{E}_{\{M_a\}}(\mathbb{R}^N) \mid \operatorname{supp} f \subset K\right\}$$
$$\mathscr{D}_{\{M_a\}}(\Omega) := \lim_{K\subset\subset\Omega} \operatorname{ind} \mathscr{D}_{\{M_a\}}(K).$$

We will also need ultradifferentiable functions defined by growth restrictions on the Fourier transformations (see [2] and [3]) using a positive weight function ω.

Definition 1.2. $\mathscr{D}_{(\omega)}(K) := \left\{f \in \mathscr{D}(K) \mid \forall n \geq 1 : \int |\hat{f}(x)| e^{n\omega(x)}\, dx < \infty\right\}$

$$\mathscr{E}_{(\omega)}(\Omega) := \left\{f \in C^\infty(\Omega) \mid \varphi f \in \mathscr{D}_{(\omega)}(K) \quad \text{for any } \varphi f \in \mathscr{D}_{(\omega)}(K)\right\}.$$

Any of the above spaces is endowed with its natural topology, duals are always considered with their strong topology. This is not a $(FS)-$ or (DFS)-topology for $\mathscr{D}_{(M_a)}(\Omega)'$ and $\mathscr{E}_{\{M_a\}}(\Omega)$. So the closed range theorem cannot be applied to show the surjectivity of partial differential operators in these spaces, making the problem considered here nontrivial.

The main condition on the sequence $(M_a)_{a\geq 0}$ is the following, which is related to the linear topological structure of the spaces $\mathscr{E}_{\{M_a\}}(\Omega)$ ([14]):

$$(1.1)\qquad \begin{aligned} &\forall k \geq 1\ \exists n \geq 1 : M(t) - km(t) \geq M(t/n) \\ &\qquad\qquad M(t/k) \geq M(t) - nm(t) \quad \text{for large } t. \end{aligned}$$

Indeed, this condition means, that $\mathscr{E}_{\{M_a\}}(\Omega)$ is isomorphic to the $\mathbf{N}$-fold product of the dual of a power series space of infinite type ([14], § 4) and this seems to be the basic reason for our results in section 3, which totally differ from the «classical» case of the Gevrėy-sequence $M_a := a^{as}$, $s \geq 1$ (see the introduction).

(1.1) implies that

$$(1.2)\qquad \exists C \geq 1 : m(2t) \leq Cm(t) \quad \text{for large } t$$

or equivalently:

$$(1.2')\qquad \exists C \in \mathbf{N} : 2m_a \leq m_{Ca} \quad \text{for large } a$$

(see [14], Proposition 4.1 and Theorem 4.4). Therefore we also get from (1.1) (by [14], Theorem 3.1)

$$(1.3)\qquad \begin{aligned} &\forall n \geq 1\ \exists k \geq 1 : M(nt) \leq M(t) + km(t) \\ &\qquad\qquad M(t) + nm(t) \leq M(kt) \quad \text{for large } t. \end{aligned}$$

The following condition ([14], (4.5)) is sufficient for (1.1) and can easily be verified in concrete cases:

$$(1.4)\qquad \exists C \geq 1\ \forall k \geq 1 : m(kt) \leq Cm(t) \quad \text{for large } t$$

or equivalently,

$$(1.4')\qquad \exists C \in \mathbf{N}\ \forall k \geq 1 : km_a \leq m_{Ca} \quad \text{for large } a.$$

(1.4') implies that

$$(1.5)\qquad a^s = O(m_a) \quad \text{for any } s,$$

that is,

$$M_a \geq (a!)^s \quad \text{for any } s \text{ and large } a.$$

So the classes $\mathscr{E}_{\{M_a\}}(\Omega)$ and $\mathscr{E}_{(M_a)}(\Omega)$ contain any of the Gevrey classes $\Gamma^{\{s\}}(\Omega)$, if (1.1) holds. The sequence $(M_a)_{a\geq 0}$ then is increasing rather fast and we are dealing with «large» classes of ultradifferentiable functions and «small» spaces of ultradistributions. This is made more precise by the following examples:

Let $M_a := e^{aG(\ln a)}$ with $G \in C^1(\mathbb{R}_+)$ and suppose that

$$(1.6) \qquad G(t) + G'(t) \leq Ce^t \quad \text{for some } C \quad \text{and large } t$$

$$(1.7) \qquad G' \quad \text{is increasing and unbounded}.$$

Then $(M_a)_{a\geq 0}$ satisfies (M1) and (1.4') (and therefore (M3') by (1.5)). (M2') follows from (1.6).

Examples 1.3. *a)* $M_a := e^{ca^b}$, $0 < c$, $1 < b \leq 2$.

b) $M_a := e^{ca(\ln a)^b}$, $0 < c$, $1 < b$.

c) $M_a := e^{ca(\ln a)(\ln a)^b}$, $0 < c$, $1 < b$.

With $b = 1$ (and $c > 1$) in Example 1.3b) or 1.3c) we get the Gevrey sequence, which certainly does not satisfy (1.1). So Example 1.3c) shows that sequences with (1.1) need not be very far away from the classical case.

The Fourier-Laplace transform of $T \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)'$ is defined by

$$\widehat{T}(z) := \mathscr{F}(T)(z) := \langle {}_xT, e^{-i\langle x,z\rangle}\rangle$$

with $\langle x, z\rangle := \sum x_i z_i$ for $x \in \mathbb{R}^N$ and $z \in \mathbb{C}^N$.

Let $K \subset \mathbb{R}^N$ be convex and compact and let

$$H_K(t) := \sup\{\langle x, t\rangle | x \in K\}$$

be its supporting function. Let

$$\mathscr{E}_{(M_a)}(K)' := \left\{T \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)' | \operatorname{supp} T \subset K\right\} \quad (\text{similarly for } \mathscr{E}_{\{M_a\}}(K)').$$

We then have the following Paley-Wiener theorems («P-W theorems» for short), if (1.1) holds.

$T \in \mathscr{E}_{(M_a)}(K)'$ if and only if

$$(1.8) \quad \exists n \geq 1 \; \forall \varepsilon > 0 : |\widehat{T}(z)| \leq C_\varepsilon \exp\left(H_K(Im\ z) + \varepsilon|Im\ z| + M(z) + nm(z)\right).$$

Moreover, if $H(K, \varepsilon)$ denotes the space of entire functions bounded by the right hand side of (1.8) for some n, then

$$(1.9) \qquad \mathscr{F} \text{ is continuous from } \mathscr{E}_{(M_a)}(K)' \text{ into } H(K, \varepsilon)$$

$$(1.10) \qquad \mathscr{F}^{-1} \text{ is continuous from } H(B_{\varepsilon/2}, \varepsilon/2) \text{ into } \mathscr{E}_{(M_a)}(B_\varepsilon)',$$

where $B_\varepsilon := \{x \in \mathbb{R}^N \mid |x| \leq \varepsilon\}$ (to prove (1.10) use the closed graph theorem, (1.16) and (1.14)). Similarly we get from (1.1) (and [13]):

$T \in \mathscr{E}_{\{M_a\}}(K)'$ if and only if

$$(1.11) \quad \forall n \geq 1,\ \varepsilon > 0 : |\widehat{T}(z)| \leq C_{n,\varepsilon} \exp\left(H_K(Im\ z) + \varepsilon|Im\ z| + M(z) - nm(z)\right)$$

$\mathscr{E}_{(m)}(\Omega)'$ is defined in the following way:

$$(1.12) \qquad \mathscr{E}_{(m)}(\Omega)' := \left\{T \in \mathscr{E}_{(M_a)}(\Omega)' \mid \exists n \geq 1 : |\widehat{T}(z)| \leq ne^{n|Im\ z| + nm(z)}\right\}$$

So we take the Paley-Wiener theorems as a definition in this case. Of course this is the dual space of $\mathscr{E}_{(m)}(\Omega)$ under reasonable assumptions (e.g. as in [3] or [2]), but we will not need this in the present paper.

The Paley-Wiener theorems for ultradifferentiable functions are easier (see [12], Theorem 9.1): Let (1.1) be satisfied again.

$f \in \mathscr{D}_{\{M_a\}}(K)$ if and only if

$$(1.13) \qquad \exists n \geq 1 : |\widehat{f}(z)| \leq C_n e^{HK(Im\ z) - M(z) + nm(z)}$$

$f \in \mathscr{D}_{(M_a)}(K)$ if and only if

$$(1.14) \qquad \forall n \geq 1 : |\widehat{f}(z)| \leq C_n e^{HK(Im\ z) - M(z) - nm(z)}.$$

(1.15) Moreover, the topology of $\mathscr{D}_{\{M_a\}}(K)$ and $\mathscr{D}_{(M_a)}(K)$ may be defined by the L_1-norms or the sup-norms corresponding to the weights in (1.13) and (1.14), respectively.

For $T \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)'$ and $\varphi \in \mathscr{D}_{(M_a)}(\mathbb{R}^N)$ we have

$$(1.16) \qquad \langle T, \varphi \rangle = (2\pi)^{-N} \int \widehat{T}(x)\widehat{\varphi}(-x)\,\mathrm{d}x.$$

Indeed, both sides are continuous on $\mathscr{E}_{(M_a)}(\mathbb{R}^N)'_b$. For the right hand side this follows from (1.9), (1.14) and (M2'). This proves (1.16), since (1.16) holds for $T \in \mathscr{D}(\mathbb{R}^N)$, and $\mathscr{D}(\mathbb{R}^N)$ is dense in $\mathscr{E}_{(M_a)}(\mathbb{R}^N)'$ (use Theorem 6.10 in [12]).

2. CONVOLUTION AND SINGULAR SUPPORT

For $T \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)'$ and $H \in \mathscr{E}_{(m)}(\mathbb{R}^N)'$ we define convolution by

$$T * H := \mathscr{F}^{-1}(\mathscr{F}(T)\mathscr{F}(H)).$$

$T * H$ is defined by (1.8) and (1.12) and $T * H \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)'$. We also have

$$(2.1) \qquad \operatorname{supp}(T * \varphi) \subset \operatorname{supp} T + \operatorname{supp} \varphi$$

(use regularization and prove (2.1) for $H \in \mathscr{D}_{(M_a)}(\mathbb{R}^N)$ using a resolution of the identity and the P-W theorems). For $\varphi \in \mathscr{D}_{(M_a)}(\mathbb{R}^N)$ we get from (1.16):

$$(2.2) \qquad \langle T * H, \varphi \rangle = \langle \check{T} * H, \check{\varphi} \rangle = (2\pi)^{-N} \int \widehat{T}(x)\widehat{H}(x)\widehat{\varphi}(-x)\,\mathrm{d}x.$$

Definition 2.1. *Let F be one of the sheafs $\mathscr{D}'_{\{M_a\}}$, $\mathscr{E}_{(m)}$ or $\mathscr{E}_{(M_a)}$. For $T \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)'$ the F-singular support of T (denoted by $F - ss(T)$) is the complement of the largest open set Ω, such that $T|_\Omega \in F(\Omega)$.*

For a function L defined on $\mathbb{R}^N$ let

$$p_{\mathrm{d}}(L, \varphi) := \sup\left\{ |\widehat{\varphi}(x)| e^{L(x) + \mathrm{d}\, m(x)} \,|\, x \in \mathbb{R}^N \right\} \quad \text{for} \quad \varphi \in \mathscr{D}_{(M_a)}(\mathbb{R}^N).$$

For $T \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)'$ and $H \in \mathscr{E}_{(m)}(\mathbb{R}^N)'$ (and vice versa) let

$$B_T(H, \varphi) := \langle T * \check{H}, \varphi \rangle \quad \text{for} \quad \varphi \in \mathscr{D}_{(M_a)}(\mathbb{R}^N).$$

Let $B_\varepsilon(y) := \{x \in \mathbf{R}^N \mid |x-y| \le \varepsilon\}$ and $B_\varepsilon := B_\varepsilon(0)$. We consider the following estimate (with $\varepsilon > 0$, $x_0 \in \mathbf{R}^N$ and d fixed):

$$(2.3) \qquad \exists C(H) > 0\ \forall \varphi \in \mathscr{D}_{(M_a)}(B_\varepsilon) : |B_T(H,\varphi)| \le C(H) p_{\mathrm{d}}(L,\varphi).$$

Lemma 2.2. *Let (1.1) be satisfied.*

a) $x_0 \notin \mathscr{E}_{(m)} - ss(T)$ *for* $T \in \mathscr{E}_{(m)}(\mathbf{R}^N)'$, *if*

i) (2.3) holds with $L(x) = M(x)$ *for any* $H \in \mathscr{E}_{(M_a)}(B_\varepsilon(x_0))'$ *or if*

ii) (2.3) holds with $L(x) = -M(x)$ *for any* $H \in \mathscr{D}_{\{M_a\}}(B_\varepsilon(x_0))'$.

b) i) $x_0 \notin \mathscr{E}_{(M_a)} - ss(T)$ *for* $T \in \mathscr{D}_{\{M_a\}}(\mathbf{R}^N)$, *if (2.3) holds with* $L(x) = 0$ *for any* $H \in \mathscr{E}_{(M_a)}(B_\varepsilon(x_0))'$.

ii) $x_0 \notin \mathscr{D}'_{\{M_a\}} - ss(T)$ *for* $T \in \mathscr{E}_{(M_a)}(\mathbf{R}^N)'$, *if (2.3) holds with* $L(x) \doteq 0$ *for any* $H \in \mathscr{D}_{\{M_a\}}(B_\varepsilon(x_0))$.

Proof. a) I) We can assume without loss of generality, that $x_0 = 0$. Let

$$B_T : H \to B_T(H) := B_T(H,\cdot)$$

be the linear mapping from $\mathscr{E}_{(M_a)}(B_\varepsilon)'$ (resp. $\mathscr{D}_{\{M_a\}}(B_\varepsilon)$) into the dual of the normed space

$$E := \left(\mathscr{D}_{(M_a)}(B_\varepsilon), p_{\mathrm{d}}(L,\cdot)\right).$$

The graph of B_T is closed by (2.2). Since $\mathscr{E}_{(M_a)}(B_\varepsilon)'$ and $\mathscr{D}_{\{M_a\}}(B_\varepsilon)$ are (DFS)-spaces (by (M2')), B_T is continuous into E'. With

$$q_{-n}(\mp M, H) := \sup\left\{ |\widehat{H}(w)|\, e^{\mp M(w) - nm(w) - \varepsilon |Im\ w|/2} \mid w \in \mathbb{C}^N \right\}$$

we get the following estimates from (1.10) and (1.15):

(2.4)

$$\forall n \ge 1\ \exists C > 0\ \forall \varphi, H \in \mathscr{D}_{(M_a)}(B_{\varepsilon/2}) : |B_T(H,\varphi)| \le C q_{-n}(\mp M, H) p_{\mathrm{d}}(L,\varphi).$$

Here and in the following the upper sign corresponds to the cases a) i) and b) i).

II) Let $G_a := \prod_{j \le a} m_{[j/4]}$, where [ ] is the Gauß bracket. Then $(G_a)_{a \ge 0}$ satisfies (M3'). Using (M2') it is easy to see that

$$(2.5) \qquad G(t) \text{ is equivalent to } 4M(t).$$

Let $\varphi \in \mathscr{D}_{(G_a)}(B_{\varepsilon/2})$ be chosen such that $\varphi = 1$ near 0. For $z \in \mathbb{C}^N$ let

$$\varphi_z(x) := \varphi(x)e^{i\langle x,z\rangle}.$$

If $B_z(\xi) := [\varphi(T*\varphi_z)]\widehat{\ }(\xi) \in L_1(\mathbb{R}^N)$, then

$$(2\pi)^{-N}\int B_z(\xi)\mathrm{d}\xi = (\check{\varphi}T)\widehat{\ }(z)$$

by the Fourier inversion formula, since $T*\varphi_z \in \mathscr{D}(\mathbb{R}^N)$ for $T \in \mathscr{E}_{(M_a)}(\mathbb{R}^N)$ by (M2'). So we have to estimate the L_1-norm of $B_z(\xi)$ as a function of z to estimate $(\check{\varphi}T)\widehat{\ }(z)$.

III) Let $(V_a)_{a\geq 0}$ satisfy (M1) and let V be the associated function. Fix $j > 0$. (2.4) implies that

$$e^{V(z)+M(j\xi)}B_z(\xi) = \sup_{a,b}|(j\xi)^a\left[\varphi\left(T*z^b\varphi_z\right)\right]\widehat{\ }(\xi)|/\left(M_{|a|}V_{|b|}\right) \leq$$

$$\leq \sup_{c\leq a,l\leq b}\left|\left\langle\frac{T*D^{a-c}((D^l\varphi)_z)(2j)^{|a-c|}2^{|l|}}{M_{|a-c|}V_{|l|}},\frac{D^{b-l}((D^c\varphi)_{-\xi})(2j)^{|c|}2^{|b-l|}}{M_{|c|}V_{|b-l|}}\right\rangle\right| \leq$$

$$\leq C_1\sup_{a,b} q_{-n}\left(\mp M, D^b\left((D^a\varphi)_z\right)2^{|a|}(2j)^{|b|}/\left(V_{|a|}M_{|b|}\right)\right)\times$$

$$\times\sup_{a,b} p_{\mathrm{d}}\left(\pm M, D^a\left(\left(D^b\varphi\right)_{-\xi}(2j)^{|b|}2^{|a|}/\left(M_{|b|}V_{|a|}\right)\right)\right) =$$

$$= C_1\sup_{w\in\mathbb{C}^N}|\widehat{\varphi}(w-z)|\,e^{V(2(w-z))+M(2jw)\mp M(w)-nm(w)-\varepsilon|Im\ w|/2}\times$$

$$\times\sup_{x\in\mathbb{R}^N}|\widehat{\varphi}(x+\xi)|\,e^{M(2j(x+\xi))+V(2x)\pm L(x)+\mathrm{d}\,m(x)}.$$

a) i) Take $V = 0$ and $L = M$. (1.3) implies that for any j and n' there are n and j' such that

$$(2.6)\quad M(2jw) - M(w) - nm(w) \leq (j'-n)m(w) \leq -n'm(z) + (n-j')m(w-z)$$

$$M(x) + \mathrm{d}\,m(x) \leq M(\mathrm{d}''(x+\xi)) + M(\mathrm{d}''\xi).$$

The choice of φ now implies that for any j and n' (with fixed d")

$$|B_z(\xi)| \leq C_1 e^{M(\mathrm{d}''\xi)-M(j\xi)-n'm(z)}.$$

So $\check{\varphi}T \in \mathscr{D}_{(m)}(\mathbb{R}^N)$ by (M2') and II).

a) ii) Take $V_a := \prod_{j\leq a} m_{[j/2]}$ (i.e. V is equivalent to 2M) and $L = M$. (1.1) implies that for any j and n' there is n such that

$$(2.7)\quad M(2jw) + M(w) - nm(w) \leq 2M(w/(2n')) \leq 2M((w-z)/n') + 2M(z/n').$$

$$-M(x) + \mathrm{d}\,m(x) + V(2x) \leq -M(x) + \mathrm{d}\,m(x) + 2M(C_1 x) \leq$$
$$\leq M(\mathrm{d}'x) \leq M(\mathrm{d}''(x+\xi)) + M(\mathrm{d}''x).$$

By the choice of φ we get for any j and n' with fixed d"

$$|B_z(\xi)| \leq C_1 e^{-2M(z)+2M(z/n')+M(\mathrm{d}''\xi)-M(j\xi)}$$

This shows that $\check{\varphi}T \in \mathscr{D}_{(m)}(\mathbb{R}^N)$ as above.

b) Take $V = M$ and $L = 0$. By (1.3) we get

$$M(2x) + \mathrm{d}\,m(x) \leq M(\mathrm{d}''x)$$

(2.6) implies (in the case b) i)), that for any n' and j

$$|B_z(\xi)| \leq e^{-M(z)-n'm(z)+M(\mathrm{d}''\xi)-M(j\xi)}$$

(2.7) implies (in the case b) ii)) that for any n' and j

$$|B_z(\xi)| \leq e^{2M(z/n')-M(z)+M(\mathrm{d}''\xi)-M(j\xi)}$$

So b) follows in any case from the P-W theorems, (1.1) and (M2').

The main notion connected with the surjectivity of partial differential operators is now contained in the following definition, which generalizes the notion of P-convexity for singular supports ([10], Definition 10.7.1).

Definition 2.3. *Let $\Omega \subset\subset \mathbb{R}^N$ be open. Let E be contained in $\mathscr{E}_{(M_a)}(\Omega)'$ and let F be one of the sheafs $\mathscr{E}_{(M_a)}$, $\mathscr{D}_{\{M_a\}'}$ and $\mathscr{E}_{(m)}$.*

Ω is called P-convex for F-singular supports in E («P-convex for $(F,E) - ss$»), if for any $K \subset\subset \Omega$ there is $K' \subset\subset \Omega$ such that for any $T \in E$:

$$F - ss(P(-D)T) \subset K \Rightarrow F - ss(T) \subset K'.$$

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N$ be open and let $(M_a)_{a\geq 0}$ satisfy (1.1). If Ω is P-convex for $(\mathscr{E}_{(m)}, \mathscr{E}_{(m)'}) - ss$, then Ω is P-convex*

i) for $(\mathscr{E}_{(M_a)}, \mathscr{D}_{\{M_a\}}) - ss$ and

ii) for $(\mathscr{D}_{\{M_a\}}, \mathscr{E}_{(M_a)'}) - ss$.

Proof. Let $K \subset\subset \Omega$ and $K_\varepsilon = \{x \in \mathbf{R}^N | \, \mathrm{d}(x, K) \leq \varepsilon\}$. Let $T \in \mathscr{D}_{\{M_\alpha\}}(\Omega)$ (resp. $T \in \mathscr{E}_{(M_\alpha)}(\Omega)'$) and

$$\mathscr{E}_{(M_\alpha)} - ss(P(-D)T) \subset K \quad \left(\text{resp.}\, \mathscr{D}_{\{M_\alpha\}'} - ss(P(-D))T) \subset K\right)$$

with fixed $K \subset\subset \Omega$. If ε is so small that $K_{2\varepsilon} \subset\subset \Omega$ we get

$$T * H \in \mathscr{E}_{(m)}(\Omega)' \text{ and } P(-D)T * H | \llcorner K_{2\varepsilon} \subset \mathscr{E}_{(m)}(K_{2\varepsilon})$$

for any $H \in \mathscr{E}_{(M_\alpha)}(B_\varepsilon)'$ (resp. $H \in \mathscr{D}_{\{M_\alpha\}}(B_\varepsilon)$). By assumption this implies that

$$\mathscr{E}_{(m)} - ss(T * H) \subset (K_{2\varepsilon})' \subset\subset \Omega$$

for any $H \in \mathscr{E}_{(M_\alpha)}(B_\varepsilon)'$ (resp. $H \in \mathscr{D}_{\{M_\alpha\}}(B_\varepsilon)$). Let ε be so small that

$$\widetilde{K} := \left[(K_{2\varepsilon})'\right]_\varepsilon \subset\subset \Omega .$$

Let $x_0 \notin \widetilde{K}$. We can assume, that $x_0 = 0$. Then T satisfies the assumption of Lemma 2.1 b) for x_0, hence

$$\mathscr{E}_{(M_\alpha)} - ss(T) \subset \widetilde{K} \subset\subset \Omega \quad (\text{ resp.}\, \mathscr{D}'_{\{M_\alpha\}} - ss(T) \subset \widetilde{K} \subset\subset \Omega).$$

This proves the lemma.

3. SURJECTIVITY OF PARTIAL DIFFERENTIAL OPERATORS

In this section Ω always denotes an open set in $\mathbf{R}^N$ and $P(D)$ is a partial differential operator with constant coefficients.

Theorem 3.1. *Let $(M_\alpha)_{\alpha \geq 0}$ satisfy (1.1). Ω is P-convex for $(\mathscr{E}_{(m)}, \mathscr{E}'_{(m)}) - ss$ if $P(D)$ is surjective in*

i) $\mathscr{D}_{(M_\alpha)}(\Omega)'$ modulo $\mathscr{D}_{\{M_\alpha\}}(\Omega)'$

or in

ii) $\mathscr{E}_{\{M_\alpha\}}(\Omega)$ modulo $\mathscr{E}_{(M_\alpha)}(\Omega)$.

Proof. Let $G(\Omega) := \mathscr{D}_{(M_a)}(\Omega)'$ and $F(\Omega) := \mathscr{D}_{\{M_a\}}(\Omega)'$ in case i) (resp. $G(\Omega) := \mathscr{E}_{\{M_a\}}(\Omega)$ and $F(\Omega) := \mathscr{E}_{(M_a)}(\Omega)$ in case ii)). We generalize the proof of Theorem 10.7.6 in [10] (see also Theorem 3.4.11 in [2]). Let $G_0(\Omega)$ and $F_0(\Omega)$ be the sections in $G(\Omega)$ (and $F(\Omega)$, respectively) with compact support in Ω.

I) Choose an increasing sequence of sets $K_j \subset\subset \Omega$, which is cofinal for the compact sets in Ω. If Ω is not P-convex for $(\mathscr{E}_{(m)}, \mathscr{E}'_{(m)}) - ss$, there are $K \subset\subset \Omega$, $T_j \in \mathscr{E}_{(m)}(\Omega)'$, $x_j \in \Omega$ and $\varepsilon_j > 0$ such that we have (see the proof of Theorem 10.7.6 in [10]):

$$\mathscr{E}_{(m)} - ss(P(-D)T_j) \subset K \tag{3.1}$$

$$x_j \in \mathscr{E}_{(m)} - ss(T_j) \tag{3.2}$$

$$x_j \notin \left(K_j + B_{\varepsilon_j}\right) \cup \left(2B_{\varepsilon_k} + \text{ supp } T_k\right) \quad \text{for } j > k \tag{3.3}$$

$$\left(\text{supp } T_j + B_{\varepsilon_j}\right) \cup \left(K + 2B_{\varepsilon_1}\right) \subset\subset \Omega. \tag{3.4}$$

By the definition of $\mathscr{E}_{(m)}(\Omega)'$ there are $s_k > 0$ such that

$$\widetilde{p}_{-s_k}(0, T_k) := \int |\widehat{T}_k(x)| e^{-s_k m(x)} \mathrm{d}\, x < \infty. \tag{3.5}$$

Let $H_0 := 0$. If $H_{j-1} \in G_0(\mathbb{R}^N)$ is choosen, we choose α_{j-1} by the P-W theorems (1.8) and (1.13) such that

$$p_{-\alpha_{j-1}}\left(\mp M, H_{j-1}\right) := \sup_{x \in \mathbb{R}^N} |\widehat{H}_{j-1}(x)| e^{\mp M(x) - \alpha_{j-1} m(x)} < \infty. \tag{3.6}$$

Here and in the following the upper sign corresponds to case i). We can assume that

$$\alpha_j \text{ is increasing and unbounded}.$$

Since $x_j \in \mathscr{E}_{(m)} - ss(T_j)$, we can apply Lemma 2.2 a) and obtain $H_j \in G_0(B_{\varepsilon_j}(x_j))$ such that

$$\forall C_1 > 0 \; \exists \varphi \in \mathscr{D}_{(M_a)}(B_{\varepsilon_j}) : \; |\langle T_j * \check{H}_j, \varphi \rangle| \geq C_1 p_{\beta_j}(\pm M, \varphi), \tag{3.7}$$

where $\beta_j := C(s_j + \alpha_{j-1})$ with the constant C from (1.2).

Let $H := \sum H_j$. Then $H \in G(\Omega)$, since the sum defining H is locally finite by (3.3).

II) Since $P(D)$ is surjective in $G(\Omega)$ modulo $F(\Omega)$, there are $W \in G(\Omega)$ and $h \in F(\Omega)$ such that $P(D)W = H + h$, that is,

$$\langle W, P(-D)\Psi \rangle = \langle H, \Psi \rangle + \langle h, \Psi \rangle \text{ for any } \Psi \in \mathscr{D}_{(M_o)}(\Omega).$$

If $\varphi \in \mathscr{D}_{(M_o)}(B_{\varepsilon_k})$, then $T_k * \varphi \in \mathscr{D}_{(M_o)}(\Omega)$ by (3.4), (2.1) and the P-W theorem (1.14), and we obtain from (2.1) and (3.3):

$$\langle W, (P(-D)T_k) * \varphi \rangle = \sum_{j \le k} \langle H_j, T_k * \varphi \rangle + \langle h, T_k * \varphi \rangle$$

(2.2) implies:

$$(*) \qquad \langle T_k * \check{H}_k, \check{\varphi} \rangle = -\langle h, T_k * \varphi \rangle - \sum_{j \le k} \langle H_j, T_k * \varphi \rangle + \langle W, (P(-D)T_k) * \varphi \rangle.$$

III) By (3.4) we can choose $\Phi \in \mathscr{D}_{(M_o)}(\Omega)$ such that $\Phi = 1$ near supp $(T_k * \varphi)$. Then $\Phi h \in F_0(\Omega)$ and (2.2) and the P-W theorems (1.11) and (1.14) imply

$$|\langle h, T_k * \varphi \rangle| = |\langle \Phi h, T_k * \varphi \rangle| \le C_k p_{\beta_k}(\pm M, \varphi).$$

By (3.5), (3.6) and (2.2) we get

$$\sum_{j \le k-1} |\langle H_j, T_k * \varphi \rangle| \le C'_k p_{\beta_k}(\pm M, \varphi).$$

IV) Choose $\chi \in \mathscr{D}_{(M_o)}(K + B_{\varepsilon_1})$ such that $\chi = 1$ near K. Then

$$T_{k,1} := (1 - \chi) P(-D) T_k \in \mathscr{D}_{(m)}(\Omega) \text{ and } p_{\mathrm{d}}(0, T_{k,1}) < \infty \text{ for any } \mathrm{d}$$

by (3.1). By (3.4) choose $\Phi \in \mathscr{D}_{(M_o)}(\Omega)$ such that $\Phi = 1$ near supp $(T_{k,1} * \varphi)$. Since $\Phi W \in G_0(\Omega)$ we get

$$p_{\mathrm{d}_o}(\mp M, W\Phi) < \infty \text{ for some } \mathrm{d}_0.$$

This implies that

$$|\langle W, T_{k,1} * \varphi\rangle| = |\langle W\Phi, T_{k,1} * \varphi\rangle| \leq C"_k p_{\beta_k}(\pm M, \varphi).$$

Choose $\widetilde{\Phi} \in \mathscr{D}_{(M_a)}(\Omega)$ by (3.4) such that $\widetilde{\Phi} = 1$ near $K + B_{\varepsilon_1}$. Then there are $d_i > 0$ (independent of k!) such that

$$p_{-d_1}(\mp M, \widetilde{\Phi} W) < \infty \quad \text{and} \quad \widetilde{p}_{-Cs_k} - d_2(0, \chi P(-D)T_k) < \infty$$

by (M2'), (1.2) and (3.5), since by (2.2)

$$\mathscr{F}(\chi P(-D)T_k)|_{\mathbb{R}^N} = \widehat{\chi} * P(-x)\widehat{T}_k$$

C is the constant from (1.2). This implies that

$$|\langle W, \left(P(-D)T_k - T_{k,1}\right) * \varphi\rangle| = |\langle \widetilde{\Phi} W, \chi P(-D)T_k * \varphi\rangle| \leq$$

$$\leq C_k p_{Cs_k + d_1 + d_2}(\pm M, \varphi) \leq C_k p_{C(s_k + \alpha_{k-1})}(\pm M, \varphi) =$$

$$= p_{\beta_k}(\pm M, \varphi) \quad \text{for large } k,$$

since α_k increases to ∞.

V) Finally, (*), III) and IV) imply that for large k there is C_k such that

$$|\langle T_k * \check{H}_k, \check{\varphi}\rangle| \leq C_k p_{\beta_k}(\pm M, \varphi) \quad \text{for any } \varphi \in \mathscr{D}_{(M_a)}(B_\varepsilon),$$

contradicting the choice of H_k in (3.7).

In analogy to the Theorems 10.7.8 and 10.7.6 in [10] we can now characterize the relative surjectivity for $P(D)$. Surprisingly, the relative surjectivity of $P(D)$ in $\mathscr{D}_{(M_a)}(\Omega)'$ modulo $\mathscr{D}_{\{M_a\}}(\Omega)'$ and in $\mathscr{E}_{\{M_a\}}(\Omega)$ modulo $\mathscr{E}_{(M_a)}(\Omega)$ are equivalent, if (1.1) holds.

Theorem 3.2. *Let $(M_a)_{a\geq 0}$ satisfy (1.1). The following are equivalent:*

i) $P(D)$ is surjective in $\mathscr{D}_{(M_a)}(\Omega)'$ modulo $\mathscr{D}_{\{M_a\}}(\Omega)'$

ii) $P(D)$ is surjective in $\mathscr{E}_{\{M_a\}}(\Omega)$ modulo $\mathscr{E}_{(M_a)}(\Omega)$

iii) Ω is P-convex for $(\mathscr{E}_{(m)}, \mathscr{E}'_{(m)}) - ss$

iv) Ω is P-convex for $(\mathscr{E}_{(M_a)}, \mathscr{D}_{\{M_a\}}) - ss$

v) Ω is P-convex for $(\mathscr{D}_{\{M_a\}'}, \mathscr{E}'_{(M_a)} - ss$

Proof. «i) $\Rightarrow$ iii)» and «ii) $\Rightarrow$ iii)» by Theorem 3.1

«iii) $\Rightarrow$ iv)» and «iii) $\Rightarrow$ v)» by Lemma 2.4

«iv) $\Rightarrow$ i)» and «v) $\Rightarrow$ ii)» is proved similarly as Theorem 10.7.8 of [10]. We only give some details concerning «v) $\Rightarrow$ ii)». Since $\mathscr{E}_{\{M_a\}}(\Omega)$ is reflexive (e.g. by the results of [14]), we only have to show (in analogy to (10.7.16)' in [10]), that for any $f \in \mathscr{E}_{\{M_a\}}(\Omega)$ there are $C > 0$, a continuous seminorm q on $\mathscr{E}_{\{M_a\}}(\Omega)'$ and a sequence $\Phi_r \in \mathscr{E}_{(M_a)}(\Omega)$ with locally finite supports such that

$$(3.8) \qquad |f(v)| + p(v) \leq C(q(P(-D)v) + \sum |\Phi_r(v)|) \quad \text{for any} \quad v \in \mathscr{E}_{\{M_a\}}(\Omega)'$$

where (e.g.)

$$p(v) := \sup \left\{ |\langle f, v\rangle| \mid f \in \mathscr{E}_{(M_a)}(\Omega),\ \forall K \subset\subset \Omega : p_{1,K}(f) \leq 1 \right\}$$

(see Definition 1.1). Notice that for $K = \emptyset$ we may take $K' = \emptyset$ in 3.2v) by the Malgrange-Ehrenpreis lemma. The space V_j (see Lemma 10.7.9 in [10]) is now defined by

$$V_j := \left\{ v \in \mathscr{E}_{(M_a)}(\Omega)' \mid \operatorname{supp} v \subset K'_{j+1}, P(-D)v \in \mathscr{D}_{\{M_a\}}\left(K_{j-1}\right)', p(v) < \infty \right\}$$

with the seminorms

$$\left\{ p(v), \widetilde{p}(P(-D)v) \mid \widetilde{p} \text{ a continuous seminorm on } \mathscr{D}_{\{M_a\}}\left(K_{j-1}\right)' \right\}.$$

V_j is an (F)-space and $\mathscr{D}_{\{M_a\}}(K_{j-1})'$ is an (FS)-space. Moreover, $\mathscr{E}_{(M_a)}(\Omega \setminus K'_{j-1})$ contains a sequence, which is dense in $\mathscr{E}_{\{M_a\}}(\Omega \setminus K'_{j-1})$ (by the sequence space representation proved in [14], which is based on Fourier series). For this part of the proof of «iv) $\Rightarrow$ i)» we need, that $\mathscr{E}_{\{M_a\}}(\Omega \setminus K'_{j-1})'$ contains a sequence, which is dense in $\mathscr{D}_{(M_a)}(\Omega \setminus K'_{j-1})'$. This also follows from [14]. This is enough to complete the proof of (3.8) as in Theorem 10.7.8 in [10].

The sequence $M_a = e^{ca^2}$ is maximal with respect to our general assumption (M2'). It was first noticed by D. Vogt [16] (see also [14]), that we have the following linear topological isomorphisms in this case:

$$\mathscr{E}_{\{M_a\}}(\Omega) \simeq \mathscr{D}_{(M_a)}(\Omega)' \simeq D(\Omega)' \simeq \left((s)'\right)^{\mathbb{N}}$$

where (s) is the space of all rapidly decreasing sequences. As a corollary of Theorem 3.2 we obtain the unexpected result, that the relative surjectivity of partial differential operators also coincides for these three sheaves:

Corollary 3.3. *Let $M_a := e^{ca^2}$, $c > 0$. The following are equivalent:*

i) $P(D)$ *is surjective in* $\mathscr{D}_{(M_a)}(\Omega)'$ *modulo* $\mathscr{D}_{\{M_a\}}(\Omega)'$

ii) $P(D)$ *is surjective in* $\mathscr{E}_{\{M_a\}}(\Omega)$ *modulo* $\mathscr{E}_{(M_a)}(\Omega)$

iii) $P(D)$ *is surjective in* $D(\Omega)'$ *modulo* $C^\infty(\Omega)$

iv) Ω *is* P*-convex for singular supports.*

Proof. P-convexity for singular supports (see Definition 10.7.1 in [10]) is just $P-$ convexity for $(C^\infty, C'^\infty) - ss$ in our notation, which is the same as P-convexity for $(\mathscr{E}_{(m)}, \mathscr{E}'_{(m)}) - ss$, since $m(t)$ is equivalent to $ln\, t$ for $M_a = e^{ca^2}$. $(M_a)_{a\geq 0}$ satisfies (1.1) by Example 1.3. So the equivalence of i), ii) and iv) is contained in Theorem 3.2, and the equivalence of iii) and iv) is the classical case considered in [10].

The main result of this paper now is the following corollary of Theorem 3.2, which characterizes the surjectivity of $P(D)$ in $\mathscr{E}_{\{M_a\}}(\Omega)$ and $\mathscr{D}_{(M_a)}(\Omega)'$.

Corollary 3.4. *Let* $(M_a)_{a\geq 0}$ *satisfy (1.1). The following are equivalent:*

i) $P(D)$ *is surjective in* $\mathscr{D}_{(M_a)}(\Omega)'$

ii) $P(D)$ *is surjective in* $\mathscr{E}_{\{M_a\}}(\Omega)$

iii) Ω *is* P*-convex and* P*-convex for* $(\mathscr{E}_{(m)}, \mathscr{E}'_{(m)}) - ss$, *for* $(\mathscr{E}_{(M_a)}, \mathscr{D}_{\{M_a\}}) - ss$ *or for* $(\mathscr{D}'_{\{M_a\}}, \mathscr{E}'_{(M_a)}) - ss$.

Proof. «ii) $\Rightarrow$ iii)» The mapping

$$L : \mathscr{E}_{(M_a)}(\Omega) \times C^\infty(\mathbf{R}^N) \to C^\infty(\Omega), (f, g) \to f + g|_\Omega, \tag{3.9}$$

is surjective by the closed range theorem. Since $\mathbf{R}^N$ is P-convex, (3.9) and ii) imply, that $P(D)$ is surjective in $C^\infty(\Omega)$, hence Ω is P-convex. The remaining statement follows from Theorem 3.2.

«iii) $\Rightarrow$ ii)» P-convexity implies by regularization:

$$\forall K \subset\subset \Omega\ \exists K' \subset\subset \Omega\ \forall T \in \mathscr{E}_{(M_a)}(\Omega)' : \text{ supp } P(-D)T \subset K \Rightarrow \text{ supp } T \subset K'.$$

So $P(D)$ is surjective in $\mathscr{E}_{(M_a)}(\Omega)$ by the closed range theorem. ii) now follows from Theorem 3.2.

«i) $\Rightarrow$ iii)» For every $f \in \mathscr{D}_{\{M_a\}}(\Omega)'$ there is $T \in \mathscr{D}_{(M_a)}(\Omega)'$ such that $P(D)T = f$. As in Theorem 10.6.6 of [10] one proves that this implies that

$$\forall K \subset\subset \Omega\ \exists K' \subset\subset \Omega . \forall \varphi \in \mathscr{D}_{(M_a)}(\Omega) : \text{ supp } P(-D)\varphi \subset K \Rightarrow \text{ supp } \varphi \subset K'.$$

This implies P-convexity by means of regularization.

«iii) $\Rightarrow$ i)» Since the mapping

$$L : \mathscr{D}_{\{M_a\}}(\mathbb{R}^N)' \times C^\infty(\Omega) \to \mathscr{D}_{\{M_a\}}(\Omega)', (f,g) \to f|_\Omega + g,$$

is surjective and $P(D)$ is surjective in $\mathscr{D}_{\{M_a\}}(\mathbb{R}^N)'$ by the usual (F)-space arguments, P-convexity of Ω implies the surjectivity of $P(D)$ in $\mathscr{D}_{\{M_a\}}(\Omega)'$. Now i) follows from Theorem 3.2.

The surjectivity of $P(D)$ is gouverned by $m(t)$ and not by $M(t)$. If $(M_a)_{a\geq 0}$ and $(\widetilde{M}_a)_{a\geq 0}$ have equivalent functions $m(t)$ and $\widetilde{m}(t)$, the surjectivity of $P(D)$ in $\mathscr{E}_{\{M_a\}}(\Omega)$ and in $\mathscr{E}_{\{\widetilde{M}_a\}}(\Omega)$ is equivalent, though the classes $\mathscr{E}_{\{M_a\}}(\Omega)$ and $\mathscr{E}_{\{\widetilde{M}_a\}}(\Omega)$ may be the distinct (see the remark after Corollary 3.5).

The equivalence of i) and iii) (except for the statement on $(\mathscr{D}_{\{M_a\}'}, \mathscr{E}_{(M_a)'}) - ss$) holds in general, if only (1.2) is assumed. Only $\mathscr{D}_{\{M_a\}}(\mathbb{R}^N)$ has to be substituted by the space $\widetilde{\mathscr{D}}$ of all ultradistributions φ such that

$$|\widehat{\varphi}(z)| \leq Ce^{-M(z)+nm(z)+n|Im\ z|} \quad \text{for some } n.$$

This is not the space $\mathscr{D}_{\{M_a\}}(\mathbb{R}^N)$ of H. Komatsu in general. In fact, if $(M_a)_{a\geq 0}$ satisfies (M2), then $\widetilde{\mathscr{D}}$ coincides with $\mathscr{E}_{(M_a)}(\mathbb{R}^N)'$ and $\mathscr{E}_{(m)}(\mathbb{R}^N)$ is just $\mathscr{E}_{(M_a)}(\mathbb{R}^N)$. So the notions of P-convexity for $(\mathscr{E}_{(m)}, \mathscr{E}'_{(m)}) - ss$ and $(\mathscr{E}_{(M_a)}, \widetilde{\mathscr{D}}) - ss$ both coincide with P-convexity for $(\mathscr{E}_{(M_a)}, \mathscr{E}'_{(M_a)}) - ss$ and the equivalence of i) and iii) then is just the result, which also can be obtained from the paper of G. Björck [2]. One also sees, that the relative growth of $M(t)$ and $m(t)$ is important, when changing from condition (1.1) to condition (M2). Of course, ii) is not equivalent to i) and iii) in general by the results of R. Braun, R. Meise and D. Vogt [4, 5].

Corollary 3.5. *Let* $M_a = e^{ca^2}$, $c > 0$. *The following are equivalent:*

i) $P(D)$ *is surjective in* $\mathscr{D}_{(M_a)}(\Omega)'$

ii) $P(D)$ *is surjective in* $\mathscr{E}_{\{M_a\}}(\Omega)$

iii) $P(D)$ *is surjective in* $\mathscr{D}(\Omega)'$

iv) Ω *is* P*-convex and* P*-convex for singular supports.*

Proof. This follows from Corollary 3.4 in the same way, as Corollary 3.3 follows from Theorem 3.2.

Notice that the ultradifferentiable classes for $M_a = e^{ca^2}$ are distinct for different c, while the surjectivity of $P(D)$ coincides, since $m(t)$ is equivalent for any c in this case. The same remark applies to general $(M_a)_{a\geq 0}$ satisfying (1.1) and the sequences $N_a := \prod_{j\leq a} m_{[j/c]}$, $c > 0$, since N is equivalent to cM in this case.

Proposition 3.6. *Let $(M_a)_{a\geq 0}$ satisfy (1.1).*

Let $K \subset\subset \mathbb{R}^N$ be convex and $\varphi \in \mathscr{D}_{\{M_a\}}(\mathbb{R}^N)$. Then $\mathscr{E}_{(M_a)} - ss(\varphi) \subset K$ if and only if there is $\mathrm{d} > 0$ such that for any $n > 0$ there is $C_n > 0$ such that

$$|Im\ z| \leq n(m(z)+1) \Rightarrow |\widehat{\varphi}(z)| \leq C_n e^{H_K(Im\ z) - M(z) + \mathrm{d}\, m(z)}. \tag{3.10}$$

Proof. The necessity of (3.10) follows similar as in Theorem 7.3.8 in [10]. The proof of sufficiency of (3.10) is even simpler than in [10], since we can use the Fourier inversion formula and Cauchy's integral formula directly: Let $x_0 \notin K$ and $A > 0$. We have by (1.3)

$$M(Az) \leq M(z) + C_1 m(z) \text{ for some } C_1.$$

With C from (1.2) choose $\eta \in \mathbb{R}^N$ such that

$$H_K(\eta) - \langle \eta, x_0\rangle + C(C_1 + \mathrm{d}) < -B,$$

where B is chosen by (M2') such that

$$e^{-Bm(t)} \in L_1(\mathbb{R}^N).$$

We may substitute $m(t)$ by an equivalent C^1-function also denoted by $m(t)$. Let $\Gamma := \{x + i\eta m(x) \mid x \in \mathbb{R}^N\}$.

$$\varphi^{(a)}(x_0) = (2\pi)^{-N} \int_\Gamma z^a \varphi(z) e^{i\langle z, x_0\rangle} \mathrm{d}\, z.$$

Using (3.10) we get

$$\sup\left\{|\varphi^{(a)}(x_0)|A^{|a|}/M_{|a|} \mid a \in \mathbb{N}_0\right\} \leq$$

$$\leq C_2 \int e^{H_K(\eta m(x)) + (C_1 + \mathrm{d}) m(x + i\eta m(x)) - \langle \eta, x_0\rangle m(x)} \mathrm{d}\, x \leq$$

$$\leq C_3 \int e^{(H_K(\eta) + C(C_1+\mathrm{d}) - \langle \eta, x_0\rangle) m(x)} \mathrm{d}\, x \leq C_4 \int e^{-Bm(x)} \mathrm{d}\, x < \infty,$$

since (1.2) and (M3') imply by Lemma 4.1 in [12], that

$$m(x+i\eta m(x)) \leq C_5 + m(2x) \leq C_6 + Cm(x) \text{ for any } x.$$

This shows that $\varphi \in \mathscr{E}_{(M_a)}(K)$, since the choice of η is locally uniform with respect to x_0.

A standard application of Proposition 3.6 now shows, that $P(D)$ is surjective in $\mathscr{E}_{\{M_a\}}(\Omega)$, if Ω is convex and if $(M_a)_{a\geq 0}$ satisfies (1.1). Of course, this is in contrast to the case, where $(M_a)_{a\geq 0}$ satisfies (M2) ([4, 5]).

Corollary 3.7. *Let $(M_a)_{a\geq 0}$ satisfy (1.1) and let Ω be convex. Then $P(D)$ is surjective in $\mathscr{D}_{(M_a)}(\Omega)'$ and in $\mathscr{E}_{\{M_a\}}(\Omega)$.*

Proof. Ω is P-convex. Let $f \in \mathscr{D}_{\{M_a\}}(\Omega)$ and let $K \subset\subset \Omega$ be convex. Let $\mathscr{E}_{(M_a)} - ss(P(-D)f)$ be contained in K. Then $P(-z)\widehat{f}$ satisfies (3.10) by Proposition 3.6 and the Malgrange-Ehrenpreis lemma shows, that f also satisfies (3.10), so $\mathscr{E}_{(M_a)} - ss(f)$ is contained in K by 3.6 again, and Ω is P-convex for $(\mathscr{E}_{(M_a)}, \mathscr{D}_{\{M_a\}}) - ss$. The conclusion now follows from Corollary 3.6.

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