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AN ELEMENTARY PROOF FOR THE NON-PARALLELIZABILITY OF ORIENTED GRASSMANNIANS

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

Let $\tilde{G}_{n,k}$ denote the Grassmann manifold of oriented n-planes in $\mathbf{R}^{n+k}$. If $n = 1$ or $k = 1$ then $\tilde{G}_{n,k}$ is a sphere and the conditions for parallelizability are known by work of Kervaire and Milnor. Hence we assume that $n > 1$ and $k > 1$. Using Schubert calculus [2] I.D. Miatello and R.J. Miatello [4] proved that $\tilde{G}_{n,k}$ is stably parallelizable if and only if $(n, k) = (2,2)$ or $(n, k) = (3,3)$. Another proof was given by Sankaran and Zvengrowski [5] using real K-theory of complex projective spaces. The purpose of this note is to show that Pontrjagin classes lead to a very simple proof of the above result.

Over $\tilde{G}_{n,k}$ one has canonical vector bundles $\gamma_{n,k}$ and $\overline{\gamma}_{n,k}$ of dimension n and k respectively. Let $p_i = p_i(\gamma_{n,k}) \in H^{4i}(\tilde{G}_{n,k}) (1 \leqq i \leqq [n/2])$ be the Pontrjagin classes of $\gamma_{n,k}$ where $H^*(-)$ means cohomology with coeffizients in $\mathbb{Z}[1/2]$. If $\tau(\tilde{G}_{n,k})$ denotes the tangent bundle of $\tilde{G}_{n,k}$ we put $p_i(\tilde{G}_{n,k}) = p_i(\tau(\tilde{G}_{n,k}))$.

Theorem. *Let* $n > 1, k > 1$. *Then*
(a) $p_1(\tilde{G}_{n,k}) = (k - n)p_1$,
(b) $p_2(\tilde{G}_{4,4}) = 6p_1^2$.

This result implies the non-parallelizability:

Corollary [4]. *Let* $n > 1, k > 1$. $\tilde{G}_{n,k}$ *is stably parallelizable if and only if* $(n, k) = (2,2)$ *or* $(n, k) = (3,3)$.

Proof. It is well known that $\tilde{G}_{2,2}$ and $\tilde{G}_{3,3}$ are π-manifolds [4]. Furthermore $\tilde{G}_{3,3}$ is parallelizable. Now let $n \neq k$. Since $p_1 \in H^4(\tilde{G}_{n,k})$ has infinite order it follows from part (a) of the above Theorem that $p_1(\tilde{G}_{n,k}) \neq 0$. Finally we consider the case $n = k (k \geqq 4)$. Let $i : \tilde{G}_{4,4} \to \tilde{G}_{k,k}$ denote the canonical inclusion. Since $i^*\tau(\tilde{G}_{k,k})$ and $\tau(\tilde{G}_{4,4})$ are stably equivalent we obtain from part (b) of the Theorem $i^*p_2(\tilde{G}_{k,k}) = p_2(\tilde{G}_{4,4}) \neq 0$ and hence $p_2(\tilde{G}_{k,k}) \neq 0$. ∎

Another less elementary proof of the Corollary can be given using the roots of $SO(n)$ and Prop. 3.2 of [6].

2. PROOF OF THE THEOREM

Let F_s denote the flag manifold consisting of s-tuples $(V_1, \ldots, V_s)$ of mutually orthogonal 2-dimensional oriented subspaces of $\mathbb{R}^{2s}$. Over F_s are 2-dimensional oriented vector bundles $\xi_1, \ldots, \xi_s$ where the total space of ξ_i consists of pairs $((V_1, \ldots, V_s), x)$ with $x \in V_i$. Let $a_i \in H^2(F_s)(1 \leqq i \leqq s)$ denote the Euler class of ξ_i. Since $\xi_1 \oplus \ldots \oplus \xi_s$ is trivial we get $p_1(\xi_1 \oplus \ldots \oplus \xi_s) = 0$ and hence

$$\sum_{i=1}^{s} a_i^2 = 0. \tag{1}$$

There is a map $\pi : F_s \to \tilde{G}_{2t,2(s-t)}(1 \leqq t \leqq s-1)$ assigning to $(V_1, \ldots, V_s)$ the $2t$-dimensional oriented subspace $V_1 \oplus \ldots \oplus V_t$ of $\mathbb{R}^{2s}$. Using bundles of complex quadrics one sees easily that $\pi^* : H^*(\tilde{G}_{2t,2(s-t)}) \to H^*(F_s)$ is a monomorphism.

Furthermore

$$\pi^*(\gamma_{2t,2(s-t)}) = \bigoplus_{i=1}^{t} \xi_i, \qquad \pi^*(\overline{\gamma}_{2t,2(s-t)}) = \bigoplus_{j=t+1}^{s} \xi_j \tag{2}$$

and hence

$$\pi^* p_1(\gamma_{2t,2(s-t)}) = \sum_{i=1}^{t} a_i^2. \tag{3}$$

From [1], §10 we know that

$$p(\xi_i \otimes \xi_j) = 1 + 2(a_i^2 + a_j^2) + (a_i^2 - a_j^2)^2. \tag{4}$$

Recall that $\tau(\tilde{G}_{n,k}) = \gamma_{n,k} \otimes \overline{\gamma}_{n,k}$ [3], thus we obtain from (2)

$$\begin{aligned} \pi^*(\tau(\tilde{G}_{2t,2(s-t)})) &= \pi^*(\gamma_{2t,2(s-t)}) \otimes \pi^*(\overline{\gamma}_{2t,2(s-t)}) = \\ &= \bigoplus_{i=1}^{t} \bigoplus_{j=t+1}^{s} (\xi_i \otimes \xi_j). \end{aligned} \tag{5}$$

Now we are ready to prove part (a) of the Theorem. Let us begin with the case $n = 2t$, $k =$

$2(s-t)$. If $\tau = \tau(\tilde{G}_{2t,2(s-t)})$ we obtain from (1), (3), (4) and (5)

$$\begin{aligned}\pi^* p_1(\tau) &= p_1(\pi^*(\tau)) = \\ &= \sum_{i=1}^{t} \sum_{j=t+1}^{s} p_1(\xi_i \otimes \xi_j) = \\ &= 2 \sum_{i=1}^{t} \sum_{j=t+1}^{s} (a_i^2 + a_j^2) = \\ & 2(s-t) \sum_{i=1}^{t} a_i^2 + 2t \sum_{j=t+1}^{s} a_j^2 = \\ &= 2(s-2t) \sum_{i=1}^{t} a_i^2 + 2t \sum_{\ell=1}^{s} a_\ell^2 = \\ &= 2(s-2t)\pi^* p_1 = (k-n)\pi^* p_1 .\end{aligned}$$

Since π^* is a monomorphism we get assertion (a).

If $n+k \equiv 1 \mod 2$ we obtain assertion (a) by a similar calculation using the flag manifold of s-tuples of mutually orthogonal 2-dimensional oriented subspaces of $\mathbb{R}^{2s+1}$.

Finally we consider the case $n, k \equiv 1 \mod 2$. If $i : \tilde{G}_{n,k-1} \subset \tilde{G}_{n,k}$ denotes the inclusion then

$$i^*(\tau(\tilde{G}_{n,k})) = \tau(\tilde{G}_{n,k-1}) \oplus \gamma_{n,k-1} .$$

Hence we get

$$\begin{aligned} i^* p_1(\tilde{G}_{n,k}) &= p_1(\tilde{G}_{n,k-1}) + p_1(\gamma_{n,k-1}) = \\ &= (k-n-1) p_1(\gamma_{n,k-1}) + p_1(\gamma_{n,k-1}) = \\ &= (k-n) i^* p_1(\gamma_{n,k}) .\end{aligned}$$

This finishes the proof of assertion (a) because $i^* : H^4(\tilde{G}_{n,k}) \to H^4(\tilde{G}_{n,k-1})$ is a monomorphism.

In order to prove assertion (b) we use the flag manifold F_4. Recall [3] that

$$\tau(F_4) = \bigoplus_{1 \leqq i < j \leqq 4} (\xi_i \otimes \xi_j) .$$

Hence we get from (5)

$$\tau(F_4) = \pi^*(\tau(\tilde{G}_{4,4})) \oplus (\xi_1 \otimes \xi_2) \oplus (\xi_3 \otimes \xi_4) .$$

Since F_4 is stably parallelizable we obtain

$$\pi^* p(\tilde{G}_{4,4}) p(\xi_1 \otimes \xi_2) p(\xi_3 \otimes \xi_4) = 1. \tag{6}$$

It follows from (2) that for $1 \leqq i \leqq 2$

$$\pi^* p_i = \sigma_i(a_1^2, a_2^2), \pi^* \overline{p}_i = \sigma_i(a_3^2, a_4^2) \tag{7}$$

where σ_i denotes the i-th elementary symmetric function. Using (4), (6), (7) we get

$$\begin{aligned}
\pi^* \overline{p}_2(\tilde{G}_{4,4}) &= p_2(\xi_1 \otimes \xi_2) + p_1(\xi_1 \otimes \xi_2) p_1(\xi_3 \otimes \xi_4) + p_2(\xi_3 \otimes \xi_4) = \\
&= (a_1^2 - a_2^2)^2 + 4 \sum_{i=1}^{2} \sum_{j=3}^{4} a_i^2 a_j^2 + (a_3^2 - a_4^2)^2 = \\
&= \sum_{i=1}^{4} a_i^4 + 4 \sum_{1 \leqq i<j \leqq 4} a_i^2 a_j^2 - 6(a_1^2 a_2^2 + a_3^2 a_4^2) = \\
&= p_1^2 \left(\bigoplus_{i=1}^{4} \xi_i \right) + 4 p_2 \left(\bigoplus_{i=1}^{4} \xi_i \right) - 6 \pi^*(p_2 + \overline{p}_2) = \\
&= -6 \pi^* p_1^2
\end{aligned}$$

and hence $\overline{p}_2(\tilde{G}_{4,4}) = -6 p_1^2$. Assertion (b) follows from

$$p_2(\tilde{G}_{4,4}) = p_1^2(\tilde{G}_{4,4}) - \overline{p}_2(\tilde{G}_{4,4}) = 6 p_1^2.$$

This completes the proof of the Theorem.

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