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HOMOGENEOUS PSEUDO-KÄHLERIAN MANIFOLDS: A HAMILTONIAN VIEWPOINT

A.T. HUCKLEBERRY

Dedicated to the memory of Professor Gottfried Köthe

Dorfmeister and Guan [DG] have recently observed that the classification of homogeneous pseudo-kähler manifolds and that of homogeneous Kähler manifolds is essentially the same. The goal of this note is to give a proof of their result by using elementary methods related to the *moment map.*

Throughout the present paper (M,ω) denotes a connected *symplectic manifold*, i.e. M is a connected differentiable manifold and ω is a non-degenerate 2-form on M with $d\omega = 0$. If M is in addition a complex manifold so that the associated almost complex structure J satisfies $\omega(JX,JY) = \omega(X,Y)$ for all vector fields X and Y, then (M,ω) is called a *pseudo-kählerian* manifold. Let $\mathrm{Aut}_{\mathcal{O}}(M)$ denote the group of biholomorphic transformations of M and

$$\mathrm{Aut}_{\mathcal{O}}(M,\omega) := \{g \in \mathrm{Aut}_{\mathcal{O}}(M) | g^*\omega = \omega\}.$$

Theorem (Dorfmeister and Guan). *Let M be a compact pseudo-kählerian manifold which is homogeneous under a Lie group $G \subset \mathrm{Aut}_{\mathcal{O}}(M,\omega)$. Then M is canonically biholomorphic to a product $Q \times T$ of a homogeneous rational manifold Q and a compact complex torus T.*

Remarks. (1) A *complex torus* T is by defintiion the quotient $\mathbb{C}^n/\Gamma$, where Γ is a discrete lattice of rank $2n$.

(2) A *homogeneous rational manifold* Q is a compact complex manifold which can be realized as an orbit of a linear group in some projective space. Equivalently, $Q = S/P$ where S is a complex semi-simple Lie group and P a parabolic subgroup, i.e. a subgroup of S which contains a maximal connected solvable subgroup (Borel group). Homogeneous rational manifolds are simply-connected and are therefore orbits of compact groups, $Q = K/L$. A quotient K/L carries a K-invariant complex structure which is projective algebraic if and only if $L = C(T)$, where $C(T)$ denotes the centralizer of a torus T in K. In this case a torus denotes a *connected closed abelian* subgroup of K. Such quotients are exactly the orbits appearing in the adjoint representation. Since K is a compact and $\mathfrak{k}$ carries an $\mathrm{Ad}(K)$-invariant metric, it follows that such orbits, i.e. orbits of the form $K/C(T)$, are exactly the orbits in the *coadjoint representation of* K.

(3) For semi-simple groups the above theorem is due to A. Borel ([B]). If M is assumed to be kählerian, i.e. the symmetric form $g(X,Y) :== \omega(X,JY)$ is definite, it has been proved by Y. Matsushima ([M]). In the kählerian case, the same result holds without assuming that

G preserves the symplectic structure. In other words, one only needs to know that M is $\mathrm{Aut}_{\mathcal{O}}(M)$ -homogeneous. This is a result of Borel-Remmert ([BR]).

1. EQUIVARIANT HOLOMORPHIC FIBRATIONS

If G is a *complex* Lie group and $H < G$ is a closed complex subgroup, then we have the *normalizer fibration* $G/H \to G/N$, where $N = N_G(H^0)$. The base is realizable as the $\mathrm{Ad}(G)$ -orbit of the subspace $\mathfrak{h}$ in the Graßmann manifold of subspaces of $\mathfrak{g}$ which have the same dimension as that of $\mathfrak{h}$. This method of coming from the abstract homogeneous manifold G/H to the concrete orbit G/N in a situation which can be treated by algebraic techniques is due to J. Tits ([T]).

If G is only a *real* Lie group of holomorphic transformations, then we define N to be the set of transformations g in the normalizer $N_G(H^0)$ which have the additional property that the induced right-action of g on G/H^0 is holomorphic. We refer to the fibration $G/H \to G/N$ as the $\mathfrak{g}$ *-anticanonical fibration* (1). This is a G-equivariant *holomorphic* fiber bundle and the base is likewise a G-orbit in a projective space.

2. SYMPLECTIC METHODS

Let (M,ω) be a symplectic manifold and G a Lie group of symplectic diffeomorphisms of M, i.e. a smooth action $G \times M \to M$ so that $g^*\omega = \omega$ for all $g \in G$. Let $\mathrm{Loc\,Ham}(M)$ be the set of smooth vector fields X on M such that $\mathcal{L}_X\omega = 0$. In other words, $X \in \mathrm{LocHam}(M)$ if and only if the local 1-parameter group G_t^X stabilizes ω, i.e. $(G_t^X)^*(\omega) = \omega$. In this situation we have the following diagram:

$$\begin{array}{ccccccc} 0 & \to & \mathbf{R} & \xrightarrow{i} & C^\infty(M) & \xrightarrow{\mathrm{sgrad}} & \mathrm{LocHam}(M) \\ & & & & & \lambda \nwarrow & \uparrow \alpha \\ & & & & & & \mathfrak{g} \end{array},$$

where i realizes the real numbers as constant functions, $\mathrm{sgrad}(f) = X_f$, where $i_{X_f}\omega = df$, and α is the natural Lie morphism arising from the G-action. The associated Lie algebra structure $\{,\}$ on $C^\infty(M)$ is defined by

$$\{f,g\} := \omega(\mathrm{sgrad}(f), \mathrm{sgrad}(g)).$$

It follows that $\mathrm{sgrad} : C^\infty(M) \to \mathrm{LocHam}(M)$ is a Lie morphism and we are confronted with a *lifting question.* Does there exist a Lie morphism $\lambda : \mathfrak{g} \to C^\infty(M)$ so that $\mathrm{sgrad} \circ \lambda = \alpha$.

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(1) Wedge products of $\mathfrak{g}$ -vector fields generate a G-stable subspace V of sections of the anti-canonical bundle. The $\mathfrak{g}$ -anticanonical map $M = G/H \to G/N$ is given by the associated map $M \to \mathbf{P}(V^*)$. (See [HO] for this and other details on the $\mathfrak{g}$ -anticanonical map).

If such a lifting exists, we refer to the G-action as a *Poisson action* (with respect to the lifting). In this case the G-equivariant dual map

$$\Phi : M \to \mathfrak{g}^*, \qquad \Phi(x)(\xi) = \lambda(\xi)(x),$$

is called the *moment map.* If every $\mathfrak{g}$ - field is the skew-gradient of some function, i.e. for every $\xi \in \mathfrak{g}$ the associated vector field ξ_M on M can be written $\xi_M = \mathrm{sgrad}(f_\xi)$, then the G-action is called a *hamiltonian* action.

The following is a list of elementary observations in the above setting (see [GS] for details): (1) If G is semi-simple, then there exists a unique lifting $\lambda : \mathfrak{g} \to C^\infty(M)$;

(2) The G'-action is hamiltonian, i.e. for every $\xi \in [\mathfrak{g}, \mathfrak{g}]$ there exists a function $f_\xi \in C^\infty(M)$ with $\mathrm{sgrad}(f_\xi) = \xi_M$;

(3) Suppose that $\xi \in \mathfrak{g}$ can be lifted. Then

$$\{x | df_\xi(x) = 0\} = \{x | \xi_M(x) = 0\}.$$

(4) If the G-action is Poisson with moment map $\Phi : M \to \mathfrak{g}^*$, then

$$\mathrm{Ker}(d\Phi_x) = \{v \in T_xM | \omega_x(v, w) = 0 \text{ for all } w \in T_xG(x)\} =: (T_xG(x))^<.$$

We refer to $(T_xG(x))^<$ as the *skew-orthogonal complement* to the tangent space of the G-orbit $G(x)$.

(5) If G is as in (4) and $G(x) = G/H$ is a *generic* orbit with moment fibering

$$\Phi|_{G(x)} : G/H \to G/J = G(\Phi(x)),$$

then $H^0 \trianglelefteq J^0$ and J^0/H^0 is abelian [2].

3. THE THEOREM OF DORFMEISTER AND GUAN

Let $M = G/H$ be as in the statement of the Theorem. Since the base $Q := G/N$ of the $\mathfrak{g}$-anticanonical fibration is a compact homogeneous rational manifold, $\pi_1(Q) = 1$ and a compact semi-simple group $K < G$ acts almost effectively and transitively on Q.

Lemma 1. *Every K-orbit in M is a section of the $\mathfrak{g}$-anticanonical fibration $M = G/H \to G/N = Q$.*

Proof. Let $Q \in Q$ and $L := \mathrm{Iso}_K\{q\}$. Since K is semi-simple, the K-action on M is Poisson. In particular, if $\xi \in \mathfrak{k}$ and T is the closure of the 1-parameter group $\exp \xi t$ in K, then [3]

$$\mathrm{Fix}_M(T) = \{x | \xi_M(x) = 0\} = \{x | df_\xi(x) = 0\} \neq \emptyset^{(3)}$$

[2] For complete details, see e.g. [HW].

[3] Since M is compact, the function f_ξ has critical points.

Now let T be a maximal torus in L. Recall that $\mathrm{Fix}_Q(T)$ is finite. Furthermore, since $N/H^0/H/H^0 = M/\Gamma$, where Γ is discrete, if $x \in \mathrm{Fix}_M(T)$ and F is the $\mathfrak{g}$-anticanonical fiber through x, then $F \subset \mathrm{Fix}_M(T)$. We choose $q \in Q$ to be the $\mathfrak{g}$-anticanonical image of such a point $x \in \mathrm{Fix}_M(T)$.

For $g \in L$ let $T^g := gTg^{-1}$, and let $g(t) \in L$ be a continuous curve with $g(0) = e$ and $g(1) = g$. Since $\mathrm{Fix}_M(T^{g(t)})$ varies continuously and q is an isolated point in $\mathrm{Fix}_Q(T^{g(t)})$ for all t, it follows that $F \subset \mathrm{Fix}_M(T^g)$. Since $L = \cup_{g\in L} T^g$, this implies that $F \subset \mathrm{Fix}_M(L)$. Thus for all $x \in F, \mathrm{Iso}_K\{x\} = \mathrm{Iso}_K\{q\} = L$, i.e. the K-orbits in M are sections of the $\mathfrak{g}$-anticanonical map. ∎

Lemma 2. *Assume that the isotropy group H is discrete, i.e. $M = G/\Gamma$. Then G is abelian and M is a torus.*

Proof. The G'-action is hamiltonian. Therefore, for every $\xi \in \mathfrak{g}'$

$$\{x|\xi_M(x) = 0\} = \{x|df_\xi(x) = 0\} \neq \emptyset.$$

Since the isotropy is discrete, it follows that G' acts trivially on M. ∎

Lemma 3. *The $\mathfrak{g}$-anticanonical map and the K-moment map are the same.*

Proof. Let $x_0 \in M$ and $m := \dim_{\mathbb{R}} K(x_0)$ for some $x_0 \in M$. Since all K-orbits in M are of this dimension, it follows that

$$\mathrm{rank}_x \Phi = m \qquad \text{for all } x \in M \qquad \text{(Property 4)}.$$

Futhermore, the fibration $\Phi : K(x) \to K(\Phi(x))$ is a torus bundle (Property 5). Since $L = \mathrm{Iso}_K\{x\}$ already contains a maximal torus (due to the fact that $K/L = Q$ is homogeneous rational), this map is finite. But the base is simply-connected. Hence $\Psi : K(x) \to K(\Psi(x))$ is injective for all $x \in M$. In other words, if F_Ψ is a Ψ-fiber thorugh a point $x_0 \in M$ with $\mathrm{Iso}_K\{x_0\} = L$, then $F_\Psi \cup K(x_0) = \{x_0\}$ and $F_\Psi = \{x \in M|\mathrm{Iso}_K\{x\} = L\}$. Since all K-orbits in M are sections of the $\mathfrak{g}$-anticanonical bundle, this is also the description of the $\mathfrak{g}$-anticanonical fiber F through x_0.

Corollary 1. *Let F be a fiber of the $\mathfrak{g}$-anticanonical fibration. Then $\omega|_F$ is non-degenerate.*

Proof. Let $x_0 \in F$. Since $F = F_\Psi$ (Lemma 3), it follows that $T_{x_0}F$ is the skew-orthogonal complement of $T_{x_0}K(x_0)$ in $T_{x_0}M$. Since ω is non-degenerate, it follows that $\omega|_{T_{x_0}F}$ is non-degenerate. ∎

Corollary 2. *The* $\mathfrak{g}$ *-anticanonical fiber is a torus.*

Proof. Since $\omega|_F$ is non-degenerate and $F = N/H^0/H/H^0 = M/\Gamma$, the desired result is that in Lemma 2. ∎

We conclude this note by giving a

Proof of the Theorem. Let S be the smallest complex Lie group in $\mathrm{Aut}_{\mathcal{O}}(M)$ which contains K. Then S is semi-simple and acts holomorphically and almost effectively on the base Q. Let $q \in Q$ and $P := \mathrm{Iso}_S\{q\}$. As before $L := \mathrm{Iso}_K\{q\}$. Now L acts trivially on the $\mathfrak{g}$-anticanonical fiber F over q. Let $L^{\mathbb{C}}$ be the smallest complex subgroup of S which contains L. It follows that $L^{\mathbb{C}}$ acts trivially on F. But $P = L^{\mathbb{C}} \cdot R_u(P)$, where $R_u(P)$ is the *unipotent radical* of P.

Let I be the ineffectivity of the P-action on F. Now 1) $I \trianglelefteq P$, 2) $I > L^{\mathbb{C}}$, and 3) $R_u(P)$ is a product of 1-dimensional subgroups which are normalized by a maximal torus of $L^{\mathbb{C}}$. Since the maximal torus action is non-trivial on each of these root groups, it follows that $I = P$. Thus the S-orbits in P are also sections. But they are *holomorphic* sections and consequently the $\mathfrak{g}$ -anticanonical bundle yields the product structure $M = F \times Q$. Since F is a torus (Corollary 2), we have the desired result. ∎

Remark. If we only assume that M is Kähler, then, by averaging ω over K, the above type of arguments give a proof of the Borel-Remmert Theorem (see [H]). ∎

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A.T. Huckleberry
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
D-4630 Bochum
Federal Republic of Ge