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TENSORNORM TECHNIQUES FOR THE (DF)-SPACE PROBLEM

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *After the negative solution of Grothendieck's (DF)-space problem (is $E \otimes_\varepsilon F$ a (DF)-space if E and F are?) it is interesting to know which classes of (DF)-spaces admit a positive solution. In this paper it is shown how to apply some recently developed tensornorm techniques to this question.*

1. The reader is assumed to have a basic knowledge of the theory of tensornorms ([10], [4], [5]) and its extension to locally convex spaces ([11], [5]). Throughout this paper all tensornorms are assumed to be finitely generated, i.e., tensornorms in the sense of Grothendieck. The problem of $E \otimes_\varepsilon F$ being (DF) is somehow dual to the «problème des topologies» of Grothendieck: Is every $C \subset E\tilde{\otimes}_\pi F$ (where E, F are Fréchet spaces) contained in some $\overline{\Gamma(A \otimes B)}$ where $A \subset E$ and $B \subset F$ are bounded. Both problems were solved in the negative by Taskinen [15], [16]. More information about how these problems are related to each other, the reader finds in [1] and [2]. An alternative approach to the study of classes with a positive answer to the (DF)-space problem (within the setting of (DFT)-spaces) is given in [2].

2. If H is a finite dimensional Hilbert space and G a subspace of H (hence 1-complemented), then for every normed space E

$$E \otimes_\alpha G \overset{1}{\hookrightarrow} E \otimes_\alpha H \quad \text{and} \quad E \otimes_\alpha H \overset{1}{\twoheadrightarrow} E \otimes_\alpha H/G$$

is a metric injection or surjection, respectively. It follows exactly as in the proof for right-projective tensornorms ([5], section 20 and 35) that for all locally convex spaces E, F, G and all quotient mappings $Q : F \to G$ (i.e. surjective, continuous and open)

$$id_E \otimes Q : E \otimes_\alpha F \twoheadrightarrow E \otimes_\alpha G$$

is a quotient map – provided F (and hence G) is Hilbertizable.

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Proposition 1. *Let α be a tensornorm.*

(1) If E is a (DF)-space and $F = \operatorname{ind}_{n\to} H_n$ a separated inductive limit of pre-Hilbert spaces, then

$$E \otimes_\alpha F = \operatorname{ind}_{n\to} (E \otimes_\alpha H_n)$$

holds topologically.

(2) If E is a normed space and F a bornological, Hilbertizable (DF)-space, then $E\otimes_\alpha F$ is (DF).

(3) $E \otimes_\alpha F$ is a (DF)-space if E and F are bornological, Hilbertizable (DF)-spaces.

Proof. (1) follows from the fact that

$$E \otimes_\alpha (\bigoplus_n H_n) = \bigoplus_n (E \otimes_\alpha H_n)$$

(see [5], 35.7), that $\bigoplus_n H_n$ is Hilbertizable and the remark just made (see also [8]). (2) and (3) are a consequence of this, since a (DF)-space is Hilbertizable if and only if it has a basis (B_n) of bounded sets such that all $[\![B_n]\!]$ are pre-Hilbert spaces (see e.g. [13], 6.4.2). ∎

For $\alpha = \varepsilon$ a different proof of this result was first given by Hollstein [12].

3. For the non-Hilbertizable case recall first that a Banach space E is an $\mathscr{L}_\infty$-space (in the sense of Linderstrauss-Pelczsynski) if and only if $E \otimes_\varepsilon F$ is a (DF)-space whenever F is a (DF)-space (see [7] and also [5]). In particular (and this is the easy part of this result), $c_0 \otimes_\varepsilon F$ and $\ell_\infty \otimes_\varepsilon F$ are (DF)-spaces for each (DF)-space F. The proofs of the following results will essentially be based on three facts:

3.1. $\mathscr{L}_p$-local technique: Let E be a locally convex space, α and β tensornorms, and $1 \leq p \leq \infty$ such that

$$E \otimes_\alpha \ell_p = E \otimes_\beta \ell_p$$

holds topologically. Then

$$E \otimes_\alpha F = E \otimes_\beta F$$

holds topologically for each locally conves space $E \in \operatorname{space}(\mathscr{L}_p)$. (A locally convex E is said to be in $\operatorname{space}(\mathscr{L}_p)$ if for each $s \in cs(E)$ there is an $r \in cs(E)$ such that the canonical map $\widetilde{E}_r \to \widetilde{E}_s$ is p-factorable – i.e. its bidual factors through some $L_p(\mu)$). For a proof see [5], 35.5.

3.2. Traced tensornorms: Let G be a normed space, α and β tensornorms and $\delta := \alpha \otimes_G \beta$ the traced tensornorm of α and β along G. Then, for all locally convex spaces E and F, the tensor contraction

$$(E \otimes_\beta G') \otimes_\pi (G \otimes_\alpha F) \twoheadrightarrow E \otimes_\delta F$$

$$(x \otimes g') \otimes (g \otimes y) \rightsquigarrow \langle g', g \rangle x \otimes y$$

is a quotient map.

See [3] and [5], 35.3 for a proof. Recall that the definition of $\alpha \otimes_G \beta$ was that the tensor contraction is a metric surjection for all normed spaces E and F.

3.3. If E and F are (DF)-spaces, $E \otimes_\pi F$ is a (DF)-space: This is well-known, see e.g. [14], p. 186.

4. Using the representation $\alpha/ = \varepsilon \otimes_{c_0} \alpha$ (see [5], 29.10) of the right-projective associate $\alpha/$ of the tensornorm α and looking at

$$(E \otimes_\alpha \ell_1) \otimes_\pi (c_0 \otimes_\varepsilon F) \twoheadrightarrow E \otimes_{\alpha/} F$$

immediately gives the

Remark. *If E is normed and F a (DF)-space, then $E \otimes_{\alpha/} F$ is a (DF)-space for all tensornorms α.*

This observation is the key to the following results. Since $E \otimes_\alpha \ell_1 = E \otimes_{\alpha/} \ell_1$ holds isometrically for all normed spaces E, the local technique from 3.1 gives that

$$E \otimes_\alpha F = E \otimes_{\alpha/} F$$

holds topologically for all locally convex spaces $F \in \text{space}(\mathscr{L}_1)$ and arbitrary E. The remark implies

Proposition 2. *If F is a (DF)-space in space$(\mathscr{L}_1)$, then*

$$E \otimes_\alpha F$$

is (DF) for each normed space E and each tensornorm α.

In particular, if $E, F \in$ space $(\mathscr{L}_1)$ are (DF)-spaces, then $E \otimes_\varepsilon C_2'$ and $C_2 \otimes_\alpha F$ are (DF)-spaces as well. The formula $\alpha = \alpha \otimes_{C_2} \varepsilon$ (see [5], 29.8)

$$(E \otimes_\varepsilon C_2') \otimes_\pi (C_2 \otimes_\alpha F) \twoheadrightarrow E \otimes_\alpha F$$

gives

Corollary 1. *If E and F are (DF)-spaces in space$(\mathscr{L}_1)$, then $E \otimes_\alpha F$ is a (DF)-space for all tensornorms α.*

The same reasoning and proposition 1 yields

Corollary 2. *If E is a (DF)-space in space$(\mathscr{L}_1)$ and F a bornological, Hilbertizable (DF)-space, then $E \otimes_\alpha F$ is (DF) for all tensornorms α.*

5. The proof of the next result is based on a well-known fact due to Levy and Kadec saying that for $1 \leq q \leq 2$ the sequence space ℓ_q is an isometric subspace of some $L_1(\mu_q)$ (see e.g. [5], section 24) or dually: ℓ_p for $2 \leq p \leq \infty$ is a quotient of some $L_\infty(\mu_p)$.

Proposition 3. *If F is a (DF)-space in space$(\mathscr{L}_\infty)$, then*

$$\ell_p \otimes_\varepsilon F$$

is a (DF)-space for $2 \leq p \leq \infty$.

Proof. Write ℓ_p as a quotient of some $L_\infty(\mu)$. By what has already been said $F \otimes_\varepsilon L_\infty(\mu)$ is a (DF)-space. Since $\ell_\infty \otimes_\varepsilon E = \ell_\infty \otimes_{\varepsilon/} E$ holds isometrically for each normed space E, local technique shows that

$$F \otimes_\varepsilon E = F \otimes_{\varepsilon/} E$$

holds topologically for each locally convex space E, and hence the conclusion follows from the fact that

$$F \otimes_\varepsilon L_\infty(\mu) = F \otimes_{\varepsilon/} L_\infty(\mu) \to F \otimes_{\varepsilon/} \ell_p = F \otimes_\varepsilon \ell_p$$

is a quotient mapping (see e.g. [5], 35.6). ∎

Using $\ell_2 \otimes_{w_2} \ell_\infty = \ell_2 \otimes_\varepsilon \ell_\infty$ and $w_2 = \varepsilon \otimes_{\ell_2} \varepsilon$ (see [5], 29.3) local technique and proposition 1 imply

Corollary. *Let E be a bornological, Hilbertizable (DF)-space and $F \in$ space$(\mathscr{L}_\infty)$ a (DF)-space, then $E \otimes_\varepsilon F$ is a (DF)-space.*

In [2] an example of an (LB)-space E_0 in space$(\mathscr{L}_\infty)$, and a reflexive Banach space F is given such that $E_0 \otimes_\varepsilon F$ is not (DF). The relation

$$(E_0 \otimes_\varepsilon \ell_1) \otimes_\pi (c_0 \otimes_\varepsilon F) \to E_0 \otimes_{\varepsilon/} F = E_0 \otimes_\varepsilon F$$

implies that $E_0 \otimes_\varepsilon \ell_1$ is not (DF). We do not know whether $E \otimes_\varepsilon F$ is (DF) if E and F are (DF)-spaces in space$(\mathscr{L}_\infty)$.

6. For bornological (DF)-spaces E and F the following table collects the main results:

$E\backslash^F$	1	2	∞
1	YES	YES	NO
2	YES	YES	YES
∞	NO	YES	?

where, for example, the YES in the first square means that $E \otimes_\varepsilon F$ is (DF) if E, F are in space($\mathscr{L}_1$) (Corollary 1).

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