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LINEAR TRANSFORMATIONS OF TAUBERIAN TYPE IN NORMED SPACES

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Dedicated to the memory of Professor Gottfried M. Köthe

1. INTRODUCTION

Let $T : D(T) \subset X \to Y$ be a linear transformation where X and Y are normed spaces. We call T *Tauberian* if $(T'')^{-1}(Q\widehat{Y}) \subset \widetilde{D}(T)^{\wedge}$ where Q is the quotient map defined on Y'' with kernel $D(T')^{\perp}$. Bounded Tauberian operators in Banach spaces were studied by Kalton and Wilansky in [KW]. As Gonzalez and Onieva remark in [GO3], these operators appear in summability (see [GW]), factorization of operators [DFJP], [N], preservation of isomorphic properties of Banach spaces [N], the preservation of the closedness of images of closed sets [NR], the equivalence between the Radon-Nikodym property and the Krein-Milman property [S], and generalised Fredholm operators [T], [Y]. Classes of Tauberian operators related to a certain measure of weak compactness are investigated in [AT]. Other recent works are [AG] (which contains the solution of a problem raised in [KW]), [Gon1], [Gon2], [GO1], [GO2], [GO3], and [MP]. The present paper investigates unbounded Tauberian operators. This wider class is a natural object of study in any investigation concerning the second adjoint T'' of an unbounded operator, about which little seems to be known. Our main goal is Theorem 3.10 which implies as a corollary the following partial characterisation: *Let T' be continuous. Then T is Tauberian if and only if for each bounded subset B of $D(T)$, if TB is relatively $\sigma(Y, D(T'))$ compact (alternatively, relatively $D(T')$ -seminorm compact) then B is relatively $\sigma(\widetilde{D}(T), D(T)')$ compact.* This result contains the well known characterisation [KW; Theorem 3.2] for the classical case. Section 4 provides some examples and further properties of Tauberian operators; thus for example the usual closable ordinary differential operators defined between L_p spaces (see e.g. [Go1; Ch VI]) and their successive adjoints are all Tauberian (Corollaries 4.6 and 4.7). Section 5 looks at the continuous case.

2. PRELIMINARIES

The symbols $X, Y, Z, \ldots$ will denote normed spaces and the class of linear transformations $T : X \to Y$ will be denoted by $L(X,Y)$. We denote the domain, range and null space of T by $D(T), R(T)$ and $N(T)$ respectively. We call T *bounded* if T is continuous and $D(T) = X$. If X is a linear subspace of Y then J_X^Y denotes the operator in $L(X,Y)$ that is the natural injection of X into Y, and Q_X^Y denotes the operator in $L(Y, Y/X)$ that is the natural quotient map defined on Y with null space X. We denote the completion of X by $\widetilde{X}$, and the completion of $D(T)$ by $\widetilde{D}(T)$. We shall abbreviate $J_X^{\tilde{X}}$ to J_X and $Q_X^{\tilde{X}}$

to Q_X. The *adjoint* T' of T is the conjugate of $TJ^X_{D(T)}$ in the sense of [Gol; II.2.2]. Thus $T' \in L(Y', D(T)')$ and $T'' \in L(D(T)'', D(T')')$. The operator T is called an F_+*-operator* ([C1], [C2]) if there exists a finite codimensional subspace E of X for which the restriction $T|_E$ has a continuous inverse. If X and Y are complete and T is closed then $T \in F_+ \Leftrightarrow T \in \phi_+$. In general we have [C2] $T \in F_+ \Leftrightarrow T' \in \phi_- \Leftrightarrow T'' \in \phi_+$. The *graph* of T is the subspace of $X \times Y$ consisting of the subset $\{(x, Tx) : x \in D(T)\}$ and is denoted by $G(T)$. We shall write $||y||_{D(T')}$ for the seminorm $\sup\{|y'y| : y' \in D(T'), ||y'|| \leq 1\}$ $(y \in Y)$. We denote by B_X the unit ball of X $\{x \in X : ||x|| \leq 1\}$. Except where stated otherwise, Q will denote the quotient map defined on Y'' with null space $D(T')^{\perp}$. We shall freely identify $D(T')'$ with QY''.

The operator T is called *partially continuous* [CL1] if there exists a finite codimensional subspace E of X for which $T|_E$ is continuous.

Proposition 2.1. *If either (i) $D(T)$ is complete or (ii) T is partially continuous, then T' is continuous.*

Proof. (i) Let $D(T)$ be complete and let $y'_n \in D(T'), y'_n \to y' \in Y'$. Since $T'y'_n \in D(T)'$ and $\lim y'_n Tx = y'Tx$ for each x in the Banach space $D(T)$, it follows from the uniform boundedness principle that $y'T' \in D(T)'$, i.e. $y' \in D(T')$. Hence $D(T')$ is closed. Therefore T' is continuous [Gol; II.2.15].

(ii) See [CL1]. ■

3. TAUBERIAN OPERATORS

The two main results of this section are Theorems 3.7 and 3.10. The latter contains a characterisation of Tauberian operators with continuous adjoint.

Lemma 3.1. *$G(T')$ is a $\sigma(Y', Y) \times \sigma(D(T)', D(T))$ closed subset of $Y' \times D(T)'$. In particular, $G(T'')$ is $\sigma(D(T)'', D(T)') \times \sigma(D(T')', D(T'))$ closed.*

Proof. Let $(y'_\alpha, T'y'_\alpha) \to (y', x')$ (with the appropriate weak* topologies). Then for $x \in D(T), |y'Tx| = \lim |T'y'_\alpha x| = |x'x| \leq ||x'||\,||x||$, so $y'T$ is continuous on $D(T)$, i.e. $y' \in D(T')$. Weak* continuity now gives $T'y' = x'$. ■

Lemma 3.2. *[La]. The following statements are equivalent.*

(i) T' is continuous

(ii) $D(T')$ is $\sigma(Y', \tilde{Y})$ closed

(iii) $Q_E J_Y T$ is continuous, where $E = D(T')_{\perp \tilde{Y}}$.

Proof. Write $Q = Q_E, J = J_Y$. Since Q is bounded we clearly have $(QJT)' = (JT)'Q' = T'Q'$ where $D(Q') = [D(T')_{\perp \tilde{Y}}]^{\perp}$. By [Gol; II.2.8] QJT is continuous if and only if

$D(T'Q') = [D(T')_{\perp\tilde{Y}}]^{\perp}$ and the right hand side is the $\sigma(Y',\tilde{Y})$ closure of $D(T')$. Therefore (ii) $\iff$ (iii).

For the proof of the equivalence of (i) and (ii) the reader is referred to [La].

We shall include here an independent proof of the case when Y is separable because of its simplicity. Assume Y is separable. Then the $\sigma(Y',\tilde{Y})$ topology is metric. Let T' be continuous and let (y'_n) be a sequence in $D(T')$ such that $\lim \sigma(Y',\tilde{Y})\, y'_n = y'$. Then $T'y'_n$ is bounded and so by Alaoglu's theorem has a $\sigma(Y',\tilde{Y})$ convergent subsequence $T'y'_{n_k} \to x'$ say. By Lemma 3.1, $y' \in D(T')$. Thus (i) $\Rightarrow$ (ii). The converse is trivial by [Gol; loc. cit.].■

Lemma 3.3. *[La]. Let $S = Q_{D(T')_\perp}T$. Then*

(i) S is closable

(ii) $S'y' = T'y'$ for $y' \in D(T')$ and $D(S') = D(T')$

(iii) $S'' = T''$.

Proof. Write $Q = Q_{D(T')_\perp}$. We have $S' = T'Q' = T'J_{D(T')_\perp^\perp}$ and $D(S') = D(Q') \cap (Q')^{-1}(D(T')) = D(T')_\perp^\perp \cap D(T') = D(T')$, proving (ii). But $D(T')$ is a total subspace of $D(T')_\perp^\perp$. Consequently S is closable [Gol; II.2.11], proving (i). Finally, $S'' \in L(D(T)'', D(T')')$ and for $y' \in D(T')$ and $x'' \in D(S'')$ we have $x''S'y' = x''T'y'$ by (ii), whence $S'' = T''$. ■

The operator S of Lemma 3.3 corresponds to the «regular contraction» of T, and the subspace $D(T')_\perp$ to the «singularity» of T as defined by G. Köthe in [Ko].

Corollary 3.4. $T'' = (Q_{D(T')_{\perp\tilde{Y}}} J_Y T)''$

Proof. It is sufficient to observe that $(J_Y T)' = T'$. ■

Proposition 3.5. *Let (x_α) be a bounded net in $D(T)$ such that $\sigma(Y, D(T')) - \lim Tx_\alpha = O$. Then the set of $\sigma(D(T)'', D(T)')$ cluster points of $(\hat{x}_\alpha)$ is a nonempty subset of $N(T'')$.*

Proof. By Lemma 3.3 we may suppose that T is closable and thus that $\sigma(Y, D(T'))$ is Hausdorff [Gol; loc. cit.]. Since $\{\hat{x}_\alpha\}$ is a relatively $\sigma(D(T)'', D(T)')$ compact set, the net $(\hat{x}_\alpha)$ has a $\sigma(D(T)'', D(T)')$ convergent subnet, assumed to be itself, with limit $x'' \in D(T)''$ say. But $\sigma(D(T')', D(T')) - \lim T''\hat{x}_\alpha = Q(\sigma(Y, D(T')) - \lim Tx_\alpha)^\wedge = O$. Hence by Lemma 3.1, $x'' \in D(T'')$ and $T''x'' = O$, proving that $N(T'') \neq \emptyset$. The same argument applied to an arbitrary cluster point x'' shows that the set of such cluster points is contained in $N(T'')$. ■

Proposition 3.6. *Let E be a linear subspace of $\tilde{D}(T)$ containing $D(T)$. Then the following statements are equivalent:*

(i) $N(T'') \subset \hat{E}$

(ii) Every bounded net (x_α) *in* $D(T)$ *for which* $\sigma(Y, D(T')) - \lim Tx_\alpha = O$ *has a* $\sigma(E, D(T)')$ *convergent subnet.*

Proof. (i) $\Rightarrow$ (ii): Assume (i) and let (x_α) be a bounded net for which $\sigma(Y, D(T')) - \lim Tx_\alpha = O$. By proposition 3.5, $(\widehat{x}_\alpha)$ has a subnet which is $\sigma(D(T)'', D(T)')$ convergent to some point $x'' \in N(T'')$. Then $x'' = \widehat{x}$ where $x \in E$ and (ii) follows.

(ii) $\Rightarrow$ (i): Assume (ii) and let $T''x'' = O$. Choose a bounded net (x_α) in $D(T)$ such that $\sigma(D(T)'', D(T)') - \lim \widehat{x}_\alpha = x''$. By the weak* continuity of T'', we have $\sigma(D(T')', D(T')) - \lim T''\widehat{x}_\alpha = O$. Hence $(T''\widehat{x}_\alpha)y' \to 0 (y' \in D(T'))$, whence $\sigma(Y, D(T')) - \lim Tx_\alpha = O$. By hypothesis, (x_α) has a $\sigma(E, D(T'))$ convergent subnet, which we assume to be itself. Let $\sigma(E, D(T')) - \lim x_\alpha = x$ where $x \in E$. Then $\widehat{x} = \sigma(D(T)'', D(T)') - \lim \widehat{x}_\alpha = x'' \in \widehat{E}$. ∎

Theorem 3.7. *Let* E *be a linear subspace of* $\widetilde{D}(T)$ *containing* $D(T)$. *Consider the following two statements:*

(i) $N(T'') \subset \widehat{E}$

(ii) Every bounded sequence (x_n) *in* $D(T)$ *for which* $||Tx_n||_{D(T')} \to 0$ *has a subsequence weakly convergent to a point of* E.

In general (i) $\Rightarrow$ *(ii). If* T' *is continuous then (ii)* $\Rightarrow$ *(i).*

Proof. (i) $\Rightarrow$ (ii): Assume (i) and let (x_n) be a bounded sequence in $D(T)$ such that $||Tx_n||_{D(T')} \to 0$. By proposition 3.6, (x_n) has a $\sigma(E, D(T)')$ cluster point $x \in E$. The same argument applied to arbitrary countable subsets of $\{x_n\}$ shows that $\{x_n\}$ is relatively $\sigma(E, D(T)')$ countably compact, hence relatively $\sigma(E, D(T)')$ sequentially compact, i.e. (x_n) has a $\sigma(E, D(T)')$ convergent subsequence.

Let T' be continuous. Assume (ii) and let $x'' \in N(T'')$ where $||x''|| = 1$. Choose a net (x_α) in $B_{D(T)}$ such that $\sigma(D(T)'', D(T)') - \lim \widehat{x}_\alpha = x''$. Then $\sigma(D(T')', D(T')) - \lim T''\widehat{x}_\alpha = O$. Let $C_\alpha = co\{x_\gamma : \gamma \geq \alpha\}$. Write $Q_1 = Q_{D(T')_{\perp \tilde{Y}}}$. We have $\lim y'Q_1 J_Y Tx_\alpha = \lim y'Tx_\alpha = \lim Q(Tx_\alpha)^\wedge y' = 0$ for every $y' \in D(T') = D(T')^{\perp}_{\perp \tilde{Y}} = (Q_1 \widetilde{Y})'$ (see Lemma 3.2). Since the $\sigma(Y, D(T'))$ and $D(T')$-seminorm closures of the convex set $Q_1 J_Y T C_\alpha$ coincides and contain O, there is a sequence (c^n_α) in C_α for which $||Q_1 J_Y Tc^n_\alpha||_{D(T')} = \sup\{|y'Q_1 J_Y Tc^n_\alpha| : y' \in B_{D(T')}\} = \sup\{|y'Tc^n_\alpha| : y' \in B_{D(T')}\} = ||Tc^n_\alpha||_{D(T')} \to 0$. By (ii) (c^n_α) has a subsequence, which we assume to be itself, which is weakly convergent to some point $c_\alpha \in E$. By Lemma 3.1, $\widehat{c}_\alpha \in D(T'')$ and $T''\widehat{c}_\alpha = O$. Now let (v_n) be an arbitrary sequence in the set $\{c_\alpha\}$. Since $v_n \in E \subset \widetilde{D}(T)$ there exists a sequence (u_n) in $D(T)$ such that $||u_n - v_n|| \leq \frac{1}{n}$. Then $||Q(Tu_n)^\wedge|| = ||T''(u_n - v_n)^\wedge|| \leq \frac{||T''||}{n} \to 0$ since T'' is bounded.

But $\|Q(Tu_n)^\wedge\| = \sup\{|(Tu_n)^\wedge y'| : y' \in B_{D(T')}\} = \|Tu_n\|_{D(T')}$. Hence by (ii), (u_n) has a subsequence (u_{n_k}) weakly convergent to $u \in E$ say. Now for $x' \in D(T)', |x'(u_n - v_n)| \leq |x'(u_n - u)| + |x'(v_n - u_n)| \leq |x'(u_n - u)| + \frac{\|x'\|}{n} \to 0$. Thus $\sigma(E, D(T)') - \lim v_n = u \in E$. This shows that $\{c_\alpha\}$ is relatively sequentially $\sigma(E, D(T)')$ compact and hence (by Eberlein's Theorem, see e.g. [F, Ch 3]) relatively $\sigma(E, D(T)')$ compact. Therefore (c_α) has a $\sigma(E, D(T)')$ convergent subnet, which we assume to be itself. Let $\sigma(E, D(T)') - \lim c_\alpha = c$. We claim that $\widehat{c} = x''$. Indeed if W is a closed convex $\sigma(D(T)'', D(T)')$ neighbourhood of x'' we can determine α_0 such that $\widehat{x}_\alpha \in W$ for $\alpha \geq \alpha_0$, and since $W \supset \overline{C}_\alpha^\wedge$ (norm closure) for $\alpha \geq \alpha_0$ we have $\widehat{c}_\alpha \in W$ for $\alpha \geq \alpha_0$. Consequently $x'' = \widehat{c}$. Since $c \in E$, (i) follows. ∎

Lemma 3.8. *Let $Q\widehat{y} \in T''B_{D(T)''}$. Then y belongs to the $\sigma(Y, D(T'))$ closure of $TB_{D(T)}$.*

Proof. By the weak* continuity of T'', we have $T''B_{D(T)''} = T''(\overline{B_{D(T)^\wedge}}^{w*}) \subset \overline{T''B_{D(T)^\wedge}}^{w*} = \overline{Q(TB_{D(T)})^\wedge}^{w*}$ where $w*$ signifies $\sigma(D(T')', D(T'))$. Now let $Q\widehat{y} \in T''B_{D(T)''}$. Then $Q\widehat{y} \in \overline{Q(TB_{D(T)})^\wedge}^{w*}$ and so there is a net (x_α) in $B_{D(T)}$ with $\sigma(D(T')', D(T')) - \lim Q(Tx_\alpha - y)^\wedge = O$. So for $y' \in D(T')$ we have $Q(Tx_\alpha - y)^\wedge y' = y'(Tx_\alpha - y) \to 0$. ∎

Lemma 3.9. *Let E be a linear subspace of $\widetilde{D}(T)$ containing $D(T)$. If $N(T'') \subset \widehat{E}$ and if $Q(\overline{TB_X}^\sigma)^\wedge \subset T''\widehat{E}$, where $\sigma = \sigma(Y, D(T'))$, then $(T'')^{-1}(Q\widehat{Y}) \subset \widehat{E}$.*

Proof. Assume the given condition holds and let $Q\widehat{y} = T''x''$. By Lemma 3.8, $y \in \overline{TB_X}^\sigma$, so $T''x'' \in T''\widehat{E}$. Thus $T''x'' = T\widehat{x}$ where $x \in E$, and then $x'' - \widehat{x} \in N(T'') \subset \widehat{E}$. Therefore $x'' \in \widehat{E}$. ∎

Theorem 3.10. *Let E be a linear subspace of $\widetilde{D}(T)$ containing $D(T)$. Consider the following statements:*

(i) $(T'')^{-1}(Q\widehat{Y}) \subset \widehat{E}$

(ii) For all bounded subsets B of $D(T)$, if TB is relatively $\sigma(Y, D(T'))$ compact then B is relatively $\sigma(E, D(T)')$ compact

(iii) For all bounded subsets B of $D(T)$, if TB is relatively $D(T')$-seminorm compact then B is relatively $\sigma(E, D(T)')$ compact.

Then (i) ⇒ (ii) ⇒ (iii). If T' is continuous then all three statements are equivalent.

Proof. (i) ⇒ (ii): Assume (i), let $B \subset D(T)$ be bounded and let TB be relatively $\sigma(Y, D(T'))$ compact. Let (x_α) be a net in B. Then $(\widehat{x}_\alpha)$ has a $\sigma(D(T)'', D(T)')$ convergent subnet, which we assume to be $(\widehat{x}_\alpha)$ itself. Write $x'' = \sigma(D(T)'', D(T)') - \lim \widehat{x}_\alpha$. The net (Tx_α) has a subnet (assumed to be itself) which is $\sigma(Y, D(T'))$ convergent to some point $y \in Y$. We have $(T''x_\alpha)y' \to y'y(y' \in D(T'))$. Hence by Lemma 3.1, $x'' \in D(T'')$ and $T''x'' = Q\widehat{y}$. Condition (i) now gives $x'' = \widehat{x}$ where $x \in E$ proving that (x_α) has a $\sigma(Y, D(T'))$ convergent subnet.

(ii) $\Rightarrow$ (iii): This implication follows trivially on comparing topologies.

Now let T' be continuous and assume (iii). Write $D = D(T')_{\perp \widehat{Y}}, Q_1 = Q_D, J = J_Y$ and $S = Q_1 JT$. Then S is continous by Lemmas 3.2 and 3.3, and $S'' = (JT)'' = T''$ by Lemma 3.3. Let $Q_1 Jy \in \overline{SB_X}$ (where $y \in Y$) and choose a sequence (x_n) in $B_{D(T)}$ with $Sx_n \to Q_1 Jy$. Then $\|Tx_n - y\|_{D(T')} = \sup\{|y'(JTx_n - Jy + D)| : y' \in D(T')\} = \|Sx_n - Q_1 Jy\| \to 0$. Hence $\{Tx_n\}$ is relatively $D(T')$-seminorm compact. By (iii) $\{x_n\}$ is relatively $\sigma(E, D(T)')$ compact and hence relatively $\sigma(E, D(T)')$ sequentially compact. Hence there exists $x \in B_E$ and a subsequence (x_{n_k}) which is $\sigma(E, D(T)')$ convergent to x. Then $S''\widehat{x} = Q(Q_1 Jy)^\wedge$. This shows that $Q(\overline{SB}_X^\wedge) \subset S''(\widehat{E})$. Lemma 3.2 and 3.3 show that $\sigma(Q_1 JY, D(T'))$ is the weak topology of $Q_1 JY$ (the range space of S). Since the norm and weak closures of SB_X coincide, we have $Q(\overline{SB}_X^\sigma)^\wedge \subset S''(\widehat{E})$ (where σ denote $\sigma(Q_1 JY, D(T'))$. The above sequential argument also shows that condition (ii) of Theorem 3.7 is satisfied for the operator S, whence $N(S'') \subset \widehat{E}$. Hence by Lemma 3.9 and (ii) of Lemma 3.3, $(T'')^{-1}(Q\widehat{Y}) \subset \widehat{E}$. ∎

Corollary 3.11. *Let T be Tauberian and let B be a bounded subset of $D(T)$ for which TB is relatively weakly compact. Then B is relatively $\sigma(\widetilde{X}, X')$ compact.*

4. EXAMPLES AND FURTHER PROPERTIES OF TAUBERIAN OPERATORS

As an immediate consequence of Theorem 3.7 we have:

Proposition 4.1. *If T is Tauberian then $\widetilde{N}(T)$ is reflexive.*

Proposition 4.2. *Let $\gamma(T) > 0$. Then $\widetilde{N}(T)$ is reflexive if and only if $N(T'') = \widetilde{N}(T)^\wedge$.*

Proof. Since $\gamma(T) > 0$ we have (see e.g. [CL2]):
$N(T'') = R(T')^{\perp D(T)''} = (N(T)^{\perp D(T')})^{\perp D(T)''} = N(T)''$. Now $\widetilde{N}(T)$ is reflexive if and only if $\widetilde{N}(T)^\wedge = N(T)'' = N(T'')$. ∎

It is well known that bounded ϕ_+-operators in Banach spaces are Tauberian. The connection between F_+-operators and Tauberian operators in the general sense will now be investigated.

We prove the generalization of [KW; Theorem 4.2] for F_+-operators:

Theorem 4.3. *Let T be Tauberian. The following are equivalent*

(i) $T \in F_+$

(ii) $T|_R \in F_+$ for all subspaces R of $D(T)$ with reflexive completion.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. To prove (ii) $\Rightarrow$ (i), assume $T \notin F_+$. By [Gol; III 1.9] there exists an infinite dimensional subspace W of $D(T)$ for which T/W is precompact. Hence $J_Y T B_W$ is a relatively compact subset of $\widetilde{Y}$. By Theorem 3.10, B_W is relatively $\sigma(\widetilde{D}(T), D(T)')$ compact, hence relatively $\sigma(\widetilde{W}, W')$ compact. Thus $\widetilde{W}$ is reflexive and hence by (ii) $T|_W \in F_+$. But then $T|_W$ cannot be precompact since W is infinite dimensional. Therefore (ii) $\Rightarrow$ (i). ∎

The normed space X will be called *very irreflexive* (VIR) if X contains no infinite dimensional subspace with reflexive completion.

Corollary 4.4. *Let $D(T)$ be VIR. Then*

$$T \ Tauberian \ \Rightarrow T \in F_+ .$$

Recall [Kat] the definition $\gamma(T) = \sup\{\gamma : \|Tx\| \geq \gamma d(x, N(T)) \text{ for } x \in D(T)\}$.

Theorem 4.5. *Let $T \in F_+$ and $\gamma(T) > 0$. Then T is Tauberian.*

Proof. Denoting $\sigma(D(T')', DT'))$ by σ we shall verify that

$$R(T'') \subset \overline{Q(R(T)^\wedge)}^\sigma \tag{1}$$

Let $x'' \in D(T'')$ and let (x_α) be a net in $D(T)$ such that $\sigma(D(T)'', D(T)') - \lim \widehat{x}_\alpha = x''$. Then $\sigma - \lim T''\widehat{x} = T''x''$ by weak* continuity and (1) follows. We next verify that

$$\overline{Q(R(T)^\wedge)}^\sigma = Q(R(T)^{\perp\perp}). \tag{2}$$

Let $z' \in \overline{Q(R(T)^\wedge)}^\sigma (= Q(R(T)^\wedge)_\perp^\perp)$. Choose $y'' \in Y''$ such that $z' = Qy''$. For $y' \in R(T)^\perp = N(T') \subset D(T')$ we have $y''y' = z'y' = 0$ since $y' \in (Q(R(T)^\wedge)_\perp$. Thus $y'' \in R(T)^{\perp\perp}$ whence $z' \in QR(T)^{\perp\perp}$. Now suppose that $y'' \in R(T)^{\perp\perp}$ and let $y' \in (Q(R(T)^\wedge))_\perp$. Then $y \in R(T) \Rightarrow y'y = \widehat{y}y' = (Q\widehat{y})y' = 0$. Thus $y' \in R(T)^\perp$. Then $(Qy'')y' = y''y' = 0$ (since $y' \in D(T')$). Therefore $Qy'' \in (Q(R(T)^\wedge))_\perp^\perp = \overline{Q(R(T)^\wedge)}^\sigma$ and (2) is established. Next we show that

$$Q\widehat{y} \in R(T'') \Rightarrow y \in \overline{R(T)}. \tag{3}$$

Assume $Q\widehat{y} \in R(T'')$. By (1) and (2) there exists $y'' \in R(T)^{\perp\perp}$ for which $\widehat{y} - y'' \in D(T')^\perp$. Then $y' \in R(T)^\perp = N(T') \Rightarrow 0 = \widehat{y}y' - y''y' = y'y$. Thus $y \in R(T)_\perp^\perp = \overline{R(T)}$ proving (3).

As remarked in Section 2, $T \in F_+ \iff T'' \in \phi_+$. In particular $N(T'')$ is finite dimensional. Let P be a bounded projection defined on $D(T'')$ with range $N(T'')$ and let $S = I - P$ where I is the identity on $D(T'')$. Write $W = (T''|_{R(S)})^{-1}$. By the Closed Graph Theorem, W is continuous. Suppose that $T''x'' = Q\widehat{y}$, where $y \in Y$. We have $T''Sx'' = T''(x'' - Px'') = Q\widehat{y}$ and $Sx'' = WT''Sx''$. Hence $Sx'' \in W(Q\overline{\widehat{Y}}) \subset WQ(\overline{R(T)})^\wedge$ (by (3)) $\subset \overline{WQR(T)^\wedge}$ (by the continuity of Q and W) $= \overline{WT''D(T)^\wedge} = \overline{WT''SD(T)^\wedge + WT''PD(T)^\wedge} = \overline{WT''SD(T)^\wedge} = \overline{SD(T)^\wedge} \subset S\widetilde{D}(T)^\wedge$ (since S is idempotent and open). Since $\gamma(T) > 0$ we have $N(T'') = R(T')^{\perp D(T)''} = (N(T)^{\perp D(T)'})^{\perp D(T)''} = N(T)'' = N(T)^\wedge$ (see e.g. [CL2]). Consequently $x'' \in N(T)^\wedge + S\widetilde{D}(T)^\wedge$. Thus $x'' = S\widehat{x} + \widehat{n}$ where $x \in \widetilde{D}(T)$ and $n \in N(T)$. But $P\widehat{x} \in N(T'') = N(T)^\wedge$. So $x'' = \widehat{x} + \widehat{n} - P\widehat{x} \in \widetilde{D}(T)^\wedge + N(T)^\wedge \subset \widetilde{D}(T)^\wedge$ as required. ∎

The proof of statements (1) - (3) above are due to A.I. Gouveia.

Corollary 4.6. *Let X, Y be Banach spaces and T a closed operator. Then $T \in \phi_+ \Rightarrow T$ is Tauberian.*

Corollary 4.7. *Let $T \in F_+$ with codim $\overline{R(T)} < \infty$. Then T' and T'' are Tauberian.*

Proof. Indeed T' (and hence T'') is Fredholm by [C3; Proposition 3.3]. Hence the result by Corollary 4.6. ∎

It is well known [KW] that in the classical case an operator with closed range is Tauberian if and only if its null space is reflexive. For the general case we have:

Proposition 4.8. *Let $R(T)$ be $\sigma(Y, D(T'))$ closed. The following are equivalent:*

(i) T is Tauberian

(ii) $N(T'') \subset \widetilde{D}(T)^\wedge$.

Proof. We have (i) $\Rightarrow$ (ii) trivially.

Next assume (ii). Let $T''x'' = Q\widehat{y}$. Write $\sigma = \sigma(Y, D(T'))$. Then $y \in \overline{TB_X}^{\sigma} \subset R(T)$ by Lemma 3.8. So $y = Tx (\exists x \in D(T))$. Then $T''(x'' - \widehat{x}) = O$ whence $x'' - \widehat{x} \in N(T'') \subset \widetilde{D}(T)$. Hence $x'' \in \widetilde{D}(T)^\wedge$. Thereforfe (ii) $\Rightarrow$ (i). ∎

Corollary 4.9. *Let $\gamma(T) > 0$ and let $R(T)$ be $\sigma(Y, D(T'))$ closed. then T is Tauberian if and only if $\widetilde{N}(T)$ is reflexive.*

Proof. Combine Proposition 4.1, 4.2 and 4.8. ∎

5. SPECIAL CASE: CONTINUOUS OPERATORS IN NORMED SPACES

Let T be continuous and let $\tilde{T}$ denote the closure of $J_{\tilde{Y}}^{-1} T J_X$, i.e. the continuous extension to $\tilde{D}(T)$ of T regarded as an element of $L(\tilde{X}, \tilde{Y})$. With the natural identification of the isometric spaces X' and $\tilde{X}'$, we have $\tilde{T}' = T'$.

Proposition 5.1. *Let T be continuous. Then T is Tauberian if and only if $\tilde{T}$ is Tauberian.*

Proof. Immediate from $\tilde{D}(\tilde{T}) = D(\tilde{T}) = \tilde{D}(T)$ and $T'' = \tilde{T}''$. ∎

Proposition 5.2. *If T is continuous then $T \in F_+ \Rightarrow T$ is Tauberian.*

Proof. Immediate from Proposition 5.1 upon observing that $T \in F_+ \Rightarrow \tilde{T} \in \phi_+$. ∎

Corollary 5.3. *Let T be continuous and let $D(T)$ be VIR. Then T is Tauberian if and only if $T \in F_+$.*

Proof. Combine Proposition 5.2 with Corollary 4.4. ∎

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