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ON DIFFERENT TYPES OF NON-DISTINGUISHED FRÉCHET SPACES

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Dedicated to the memory of Professor Gottfried Köthe

A Fréchet space F is called distinguished if its strong dual F_b' is barrelled or, equivalently, bornological. This means that the canonical representation of F as the (reduced) projective limit $\text{proj}_{\leftarrow n} F_n$ of Banach spaces F_n leads to the representation $\text{ind}_{n\rightarrow} F_n'$ of F_b' as the inductive limit of the dual spectrum $(F_n')_n$. Thus, in some sense, distinguishedness of a Fréchet space prevents some «pathology»; and it may be for that reason that Dieudonné, Schwartz and Grothendieck introduced this notion. On the other hand, recent examples of Taskinen [11] and Bonet-Taskinen [7] show that some natural Fréchet spaces of analysis like $C^\ell(\Omega) \cap L^1(\Omega)$ are not distinguished, and hence it appears natural to study the class of the non-distinguished Fréchet spaces in more detail.

The first classical examples of non-distinguished Fréchet spaces are due to Grothendieck-Köthe and Komura, respectively. The Köthe-Grothendieck example was an echelon space $\lambda_1(A)$ on $\mathbb{N} \times \mathbb{N}$. The distinguishedness of $\lambda_1(A)$ in terms of the Köthe matrix A was finally characterized by Bierstedt and Bonet [2] based on previous work by Bierstedt and Meise [3]; see also Vogt [13] for a short proof. This characterization allowed Bastin and Bonet [1] to show that all non-distinguished echelon spaces $\lambda_1(A)$ share the bad behaviour of the Köthe-Grothendieck example, that there exists a locally bounded linear form on $\lambda_1(A)_b'$ which is not continuous.

Komura's example of a non-distinguished Fréchet space differs from the previous one: He exhibited a Fréchet space F for which each locally bounded linear form on F_b' is continuous, but which is nonetheless not distinguished [9]. Not too many examples of this type were known. (However, see [12] for a related example.)

In this paper, we investigate new examples of non-distinguished Fréchet spaces and determine their type of non-distinguishedness in the above sense. We first concentrate on the so-called spaces of «Moscatelli-type». In a second part, we consider sequence spaces and function spaces and, among other things, classify the examples previously given by Bonet and Taskinen [7]. Although we mainly concentrate here in the presentation and analysis of examples, we also expect that our study provides a better understanding of non-distinguished Fréchet spaces.

NOTATIONS AND TERMINOLOGY

Given a dual pair (E, F), we denote by $\sigma(E,F)$, $\mu(E,F)$ and $\beta(E,F)$ the weak, the Mackey and the strong topology, respectively. For a locally convex space E, let E^* denote its algebraic dual and E' its topological dual. $\mathscr{U}_0(E)$ will be the filter of all 0-neighbourhoods in E and $\mathscr{B}(E)$ will be the collection of all absolutely convex bounded subsets in E. $E^\#$ will stand for the set of all locally bounded linear forms on E, i.e.,

$$E^\# := \{u \in E^*; u(B) \text{ is bounded } \forall B \in \mathscr{B}(E)\}.$$

Recall that for a metrizable locally convex space E the bornological topology associated with $\beta(E', E)$ coincides with $\beta(E', E")$.

Observe that for a Fréchet space E one has $(E'_b)' = (E'_b)^\#$ if and only if $(E', \mu(E', E"))$ is barrelled or, equivalently, bornological. This property is inherited by complemented subspaces and countable products.

In the original example of Komura, the topology $\beta(E', E)$ is strictly coarser than $\mu(E', E")$ which coincides with $\beta(E', E")$, whereas in the Köthe-Grothendieck example $\mu(E', E")$ is strictly coarser then $\beta(E', E")$.

For the structure theory of locally convex spaces we refer to Köthe [10].

First, we concentrate on Fréchet spaces of Moscatelli type. They were introduced and studied by Bonet and S. Dierolf [6]. Since we are interested in non distinguished spaces, the most relevant case occurs when $L = \ell^1$, i.e., when $L' = L^x = \ell^\infty$. Let us recall the definition.

Let $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ be two sequences of Banach spaces with unit balls A_k and B_k, respectively, let $f_k : Y_k \to X_k (k \in \mathbb{N})$ be a linear map with dense range such that $f_k(B_k) \subset A_k$. The Fréchet spaces of Moscatelli type associated with ℓ_1, $f_k : Y_k \to X_k (k \in \mathbb{N})$ is defined by $F := \operatorname{proj} \ell_1((Y_k)_{k<n}, (X_k)_{k\geq n})$, where the linking maps are determined by $f_n : Y_n \to X_n$ and the identity in the other coordinates. According to [6], F is distinguished if and only if there is $n \in \mathbb{N}$ with f_k surjective for $k \geq n$.

We first characterize those spaces presenting the same pathology as the Köthe-Grothendieck example. We give a general lemma whose proof is based on the original argument by Grothendieck. This result shows that Grothendieck's proof for $\lambda_1(A)$ is optimal.

Lemma 1. *Let E be a quasibarrelled l.c.s. The following statements are equivalent:*
(i) $E" \not\subseteq (E')^\#$,
(ii) there is a filter $\mathscr{F}$ in E such that
(a) $\forall U \in \mathscr{U}_0(E), \exists \rho_u > 0 : \rho_u U \in \mathscr{F}$
(b) $\forall B \in \mathscr{B}(E)$ $\exists V_B$ *closed 0-neighbourhood in* $(E, \sigma(E, E'))$ *such that* $E \setminus (B + V_B) \in \mathscr{F}$.

Proof. (i) $\Rightarrow$ (ii) Let $\varphi \in (E')^\# \setminus E"$ be given. We put $\mathscr{G} := \{E \cap (\varphi + V^{\sigma(E'^\#, E')}); V$ is an absolutely convex zero neighbourhood in $(E, \sigma(E, E'))\}$. Since $V^{\sigma(E'^\#, E')} + \varphi$ is a neighbourhood of φ in $(E'^\#, \sigma(E'^\#, E'))$ and E is dense in this space, $\mathscr{G}$ is a filter basis on E.

Now let $B \in \mathscr{B}(E)$ be given. Since $\varphi \notin \overline{B}^{\sigma(E'^\#, E')}$, there is $W \in \mathscr{U}_0(E'^\#, \sigma(E'^\#, E'))$ such that

$$(\varphi + 2W) \cap B = \emptyset.$$

Thus $(\varphi + W) \cap (B + W) = \emptyset$, which proves that

$$(\varphi + W) \cap E \subset E \setminus (B + W)$$

and we may conclude that $E \setminus (B + W) \in \mathscr{G}$.

Now let $U \in \mathscr{U}_0(E)$ be given. Since φ is bounded on U°, there is $\rho_U > 0$ with $|\varphi(u)| < \rho_U$ for all $u \in U^\circ$.

We claim that for any absolutely convex and $\sigma(E, E')$-closed elements $U_1, \dots, U_n$ of $\mathscr{U}_0(E)$ and for any $V \in \mathscr{G}$ we have

$$\left(\bigcap_{k=1}^{n} 2\rho_{U_k} U_k\right) \cap V \neq \emptyset .$$

Clearly V can be written as $V = (\varphi + W) \cap E$ with W in $\mathscr{U}_0((E'^{\#}, \sigma(E'^{\#}, E')))$, and we may assume without loss of generality that W is $\sigma(E'^{\#}, E')$-open. Hence $\varphi + W$ is a neighbourhood of φ in $(E'^{\#}, \sigma(E'^{\#}, E'))$. On the other hand, $2\rho_{U_j}\overline{U_j}^{\sigma(E'^{\#}, E')}$ is a neighbourhood of φ in $(E'^{\#}, \beta(E'^{\#}, E'))$. Therefore,

$$(\varphi + W) \cap \left(\bigcap_{1 \le j \le n} 2\rho_{U_j}\overline{U_j}^{\sigma(E'^{\#}, E')}\right) \neq \emptyset ,$$

and this set is a neighbourhood of φ in $(E'^{\#}, \beta(E'^{\#}, E'))$. Assume that

$$(\varphi + W) \cap E \cap \left(\bigcap_{1 \le j \le n} 2\rho_{U_j} U_j\right) = \emptyset .$$

$(\varphi + W) \cap E$ is a convex, open set in E with the relative $\sigma(E'^{\#}, E')$-topology, i.e., in $(E, \sigma(E, E'))$ and $\cap_{1 \le j \le n} 2\rho_{U_j} U_j$ is convex and $\sigma(E, E')$-closed. Using Hahn-Banach's separation theorem, we find f in E' such that

$$|f_{|\bigcap_{1 \le j \le n} 2\rho_{U_j} U_j}| \le 1$$

$$|f_{|(\varphi + W) \cap E}| > 1$$

We extend f by density to a continuous linear form $\widetilde{f} : (E'^{\#}, \sigma(E'^{\#}, E')) \to \mathbb{K}$, hence

$$|\widetilde{f}_{\Big|\left(\bigcap_{1\leq j\leq n} \rho_{U_j} U_j\right)^{ooE'^{\#}}}| \leq \frac{1}{2},$$

$$|\widetilde{f}_{\Big|\overline{(\varphi + W) \cap E}^{\sigma(E'^{\#}, E')}}| \geq 1.$$

Since E is $\sigma(E'^{\#}, E')$-dense and W is $\sigma(E'^{\#}, E')$-open, $\overline{(\varphi + W) \cap E}^{\sigma(E'^{\#}, E')} = (\varphi + W)$. On the other hand, as all the U_j^o are $\sigma(E', E)$-compact,

$$\bigcap_{1\leq j\leq n} \rho_{U_j} \overline{U_j}^{\sigma(E'^{\#}, E')} = \bigcap_{1\leq j\leq n} \rho_{U_j} U_j^{ooE'^{\#}} = \left(\bigcup_{1\leq j\leq n} \frac{1}{\rho_{U_j}} U_j^o\right)^{oE'^{\#}} =$$

$$= \left(\overline{\Gamma\left(\bigcup_{1\leq j\leq n} \frac{1}{\rho_{U_j}} U_j^o\right)}^{\sigma(E', E)}\right)^{oE'^{\#}} = \left(\bigcap_{1\leq j\leq n} \rho_{U_j} U_j\right)^{ooE'^{\#}}.$$

Therefore

$$|\widetilde{f}|_{\bigcap_{1\leq j\leq n} \rho_{U_j} \overline{U_j}^{\sigma(E'^{\#}, E')}}| \leq \frac{1}{2},$$

$$|\widetilde{f}|_{(W + \varphi)} \geq 1$$

which contradicts the fact that

$$(\varphi + W) \cap \left(\bigcap_{1\leq j\leq n} \rho_{U_j} \overline{U_j}^{\sigma(E'^{\#}, E')}\right) \neq \emptyset.$$

Thus the claim is proved. Now we put

$$\mathscr{F} := \left\{ V \cap \left(\bigcap_{1\leq j\leq n} 2\rho_{U_j} U_j\right) : V \in \mathscr{G}, U_j \in \mathscr{U}_0(E), 1 \leq j \leq n, n \in \mathbb{N} \right\}.$$

Then $\mathscr{F}$ is a filter basis on E satisfying (a) and (b).

(ii) $\Rightarrow$ (i) Let $\mathscr{W}$ be an ultrafilter finer than $\mathscr{F}$. Since every $f \in E'$ belongs to the polar of some $U \in \mathscr{U}_0(E)$, it follows that $\{f(u) : u \in \rho_U U\}$ is bounded in $\mathbb{K}$ and, consequently, $\mathscr{W}(f)$ converges in $\mathbb{K}$. Therefore $\mathscr{W}$ converges in $\sigma(E'^*, E')$ to some $\varphi \in E'^*$. Since $|\varphi|_{U^\circ}| \leq \rho_U$ for all $U \in \mathscr{U}_0(E)$ and E is quasibarrelled, the linear form φ is in $E'^{\#}$.

On the other hand $\varphi \notin E''$. In fact, assume φ is in E''. Then there exists $B \in \mathscr{B}(E)$ such that $\varphi \in \overline{B}^{\sigma(E'',E')}$. Whence

$$\left(\varphi + \frac{1}{2}V_B^{\infty}\right) \cap (E \setminus (B + V_B)) \neq \emptyset$$

and $\varphi \in B + \frac{1}{2}V_B^{\infty}$. Let $x \in E \setminus (B + V_B)$, $x \in \varphi + \frac{1}{2}V_B^{\infty}$ be given. Then

$$x \in \left(B + \frac{1}{2}V_B^{\infty} + \frac{1}{2}V_B^{\infty}\right) \cap E \subset (B + V_B^{\infty}) \cap E.$$

Therefore $x \in (B + V_B^{\infty}) \cap E = B + V_B$, a contradiction. ∎

Proposition 2. *Let F be the Fréchet space of Moscatelli type associated with ℓ_1, $f_k : Y_k :\to X_k (k \in \mathbb{N})$.*

Suppose that for all $k \in \mathbb{N}$ there exists a sequence $(u_{kj})_{j\in\mathbb{N}}$ in X_k' such that $r_k'(u_{kj}) = 1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} s_k'(f_k^t(u_{kj})) = 0$ (i.e., F is not distinguished). Moreover, we assume that for every $n \in \mathbb{N}$

$$\{x \in Y_k | r_k(f_k(x)) \leq 1,\ |u_{kj}(f_k(x))| \geq 1,\ 1 \leq j \leq n\} \neq \emptyset.$$

Then $F'' \not\subset F'^{\#}$.

Proof. It is enough to construct a filter basis on F satisfying (a), (b) of Lemma 1. Given $n \in \mathbb{N}$ and $(j_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ we put $M(n, (j_k)_{k\geq n}) := \cup_{k\geq n}\{(x\delta_{k\ell})_{\ell\in\mathbb{N}} ; x \in Y_k, r_k(f_k(x)) = 1$ and $|u_{kj}(f_k(x))| \geq 1$, $1 \leq j \leq j_k\}$. Then

$$\{M(n, (j_k)_{k\geq n}) : n \in \mathbb{N}, (j_k)_{k\geq n} \in \mathbb{N}^{\mathbb{N}}\}$$

is a filter basis $\mathscr{F}$ on F. Next we will check that $\mathscr{F}$ satisfies (a) and (b).

(a) It is enough to see that

$$\forall \sigma > 0\ \forall n \in \mathbb{N}\ \left(\prod_{k<n} \sigma B_k \times \prod_{k\geq n} Y_k\right) \cap F \text{ and}$$

$$F \cap \left\{(y_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} Y_k : \sum_{k\in\mathbb{N}} r_k(f_k(y_k)) \leq 1\right\}$$

belong to $\mathscr{F}$.

For the first one, we observe that given $m \geq n$ and $(y_k)_{k\in\mathbb{N}} \in M(m,(j_k)_{k\geq m})$, $y_k = 0$, $1 \leq k < n$, hence

$$(y_k)_{k\in\mathbb{N}} \in \left(\prod_{k<n} \sigma B \times \prod_{k\geq n} Y_k\right) \cap F.$$

For the second type of zero-neighbourhoods, we take $(j_k)_{k\in\mathbb{N}}$ arbitrary. Given y in $M(1,(j_k)_{k\in\mathbb{N}})$, we find $k \in \mathbb{N}$ such that $y = (x\delta_{k\ell})_{\ell\in\mathbb{N}}$ with $r_k(f_k(x)) \leq 1$, hence $y \in F \cap \{(z_k)_{k\in\mathbb{N}} : \sum_{k\in\mathbb{N}} r_k(f_k(z_k)) \leq 1\}$.

To check condition (b), we have to show that for every $B \in \mathscr{B}(E)$ there is a closed zero-neighbourhood V_B in $\sigma(F,F')$ such that $F \setminus (B + V_B)$ is in $\mathscr{F}$.

Given B bounded in F, we find a sequence of positive real numbers, $(\lambda_k)_{k\in\mathbb{N}}$, such that

$$\sup\{s_k(y_k) : y = (y_k)_{k\in\mathbb{N}} \in B\} \leq \lambda_k \text{ for all } k \in \mathbb{N}.$$

Since $\lim_{j\to\infty} s'_k(f_k^t(u_{kj})) = 0$ for all $k \in \mathbb{N}$, there is $j_k \in \mathbb{N}$ such that

$$s'_k(f_k^t(u_{kj})) < \frac{1}{4\lambda_k} \text{ for all } j \geq j_k \text{ and all } k \in \mathbb{N}.$$

We define $v := (2u_{kj_k})_{k\in\mathbb{N}} \in \ell_\infty((X'_k, r'_k)_{k\in\mathbb{N}}) \subset F'$, and we put $V_B = \{v\}^\circ$. Then V_B is a closed zero-neighbourhood in $\sigma(F,F')$. We will see that $F \setminus (B + V_B) \in \mathscr{F}$. Let $y \in M(1,(j_k)_{k\in\mathbb{N}})$ be given. Then, there is $k \in \mathbb{N}$ such that $y = (\delta_{k\ell}x)_{\ell\in\mathbb{N}}$, where $x \in Y_k$, $r_k(f_k(x)) = 1$ and $|u_{kj}(f_k(x))| \geq 1$, $j \leq j \leq j_k$. If we assume $y \in B + V_B$, then $y = b + w$, with $b \in B$, $w \in V_B$, and we may suppose $b_\ell = 0$, $w_\ell = 0$ for all $\ell \neq k$. Then

$$2 \leq |v_k(f_k(x))| = |v_k(f_k(b_k + w_k))| \leq$$
$$\leq 2|u_{kj_k}(f_k(b_k))| + |v_k(f_k(w_k))| \leq 1 + 2\frac{2}{4\lambda_k}\lambda_k = \frac{3}{2},$$

a contradiction. Therefore

$$M(1,(j_k)_{k\in\mathbb{N}}) \cap (B + V_B) = \emptyset$$

and we are done. ∎

Corollary 3. *Let F be a Fréchet space of Moscatelli type associated with ℓ_1, $f_k : Y_k \to X_k (k \in \mathbb{N})$. In each of the following cases there is a linear form $f : F'_b \to \mathbb{K}$ bounded on the bounded sets but not continuous:*

(1) $Y_k = \ell_1(a_k)$, $X_k = \ell_1$, $\lim_{i\to\infty} a_k(i) = \infty$, $f_k : Y_k \to X_k$ the canonical injection,

(2) $Y_k = c_o(a_k)$, $X_k = \ell_1$, $\sum_{i\in\mathbb{N}} a_k(i)^{-1} \leq 1$, $f_k : Y_k \to X_k$ the canonical injection,

(3) $Y_k = L^p[0,1]$, $X_k = L^1[0,1]$, $1 < p < \infty$, $f_k : Y_k \to X_k$ the canonical injection.

Moreover, $E = C(\mathbb{R}) \cap L^1(\mathbb{R})$ is also of the same type because it contains a complemented subspace of type (2) (cf. [11]).

Proof. It is enough to see that the hypothesis in Proposition 2 holds:

(a) We have $(Y_k, s_k) = \ell_1(a_k)$, $(X_k, r_k) = \ell_1$, with $\lim_{i\to\infty} a_k(i) = \infty$. Then $(X'_k, r'_k) = \ell_\infty$, $(Y'_k, s'_k) = \ell_\infty(v_k)$, where $v_k = a_k^{-1}$. We put $u_{kj} = ((0)_{i<j}), (1)_{i\geq j}) \in \ell_\infty = X'_k$. Hence $r'_k(u_{kj}) = 1 \forall j \in \mathbb{N}$ and $\lim_{j\to\infty} s'_k(u_{kj}) = 0$. Moreover, for all $n \in \mathbb{N}$

$$\{x \in Y_k : r_k(x) = 1, |u_{kj}(x)| \geq 1, 1 \leq j \leq n\} \supset \{e_i : i > n\}$$

(b) We may take u_{kj} as in (a).

(c) Now, we have $(Y_k, s_k) = L^p[0,1]$, $(X_k, r_k) = L^1[0,1]$. Then $X'_k = L^\infty[0,1]$, $Y'_k = L^q[0,1] (p^{-1} + q^{-1} = 1)$.

We select $\alpha_k > 0 (k \in \mathbb{N})$ with $\sum_{\ell=1}^\infty \alpha_\ell < 1$, and a sequence of intervals $\{J_\ell\}_{\ell\in\mathbb{N}}$ such that $J_\ell \subset [0,1]$, $J_\ell \cap J_{\ell'} = \emptyset$ if $\ell \neq \ell'$ and $\mu(J_\ell) = \alpha_\ell$. We put

$$u_{kj} = \sum_{\ell=1}^\infty \chi_{J_\ell} \in L^\infty[0,1], \text{ and } r'_k(u_{kj}) = \text{ess sup} \, |\sum_{\ell=j}^\infty \chi_{J_\ell}| = 1$$

but

$$s'_k\left(u_{kj}\right) = \left[\int_0^1 |u_{kj}|^q\right]^{1/q} = \left(\sum_{\ell=j}^\infty \int_0^1 \chi_{J_\ell}^q\right)^{1/q} = \left(\sum_{\ell=j}^\infty \alpha_\ell\right)^{1/q}$$

and this tends to zero. Hence

$$\{x \in L^p[0,1]; r_k(x) = 1, |u_{kj}(x)| \geq 1, 1 \leq j \leq n\} \supset \{\alpha_\ell^{-1}\chi_{J_\ell}; \ell > n\}.$$

■

Next, we give examples of non-distinguished Fréchet spaces of Moscatelli type with the same behaviour as Komura's example. Before we go on, let us introduce some notations. Let f be the Fréchet space of Moscatelli type with respect to ℓ_1, $f_k : (Y_k, s_k) \to (X_k, r_k) (k \in \mathbb{N})$, where f_k has dense range. Then for all $k \in \mathbb{N}$, the transpose $f_k^t : (X'_k, r'_k) \to (Y'_k, s'_k)$ is injective and maps the unit ball A'_k of (X'_k, r'_k) into the unit ball B'_k of (Y'_k, s'_k). Thus we may form the (LB)-space of Moscatelli type with respect to ℓ_∞, $(Y'_k, s'_k)_{k\in\mathbb{N}}$. $(X'_k, r'_k)_{k\in\mathbb{N}}$, $E := \text{ind } E_n$, where $E_n := \ell_\infty((Y'_k)_{k<n}, (X'_k)_{k\geq n})$ and we may also consider the projective limit $\check{E}$ associated to E. We recall from [3] the definition of the projective hull $\check{E}$ of the inductive limit $E = \text{ind } \ell_\infty((Y'_k)_{k<n}, (X'_k)_{k\geq n})$. Given $\delta > 0$ and $\varepsilon_k > 0 (k \in \mathbb{N})$, the

Minkowski functional of $\varepsilon_k B'_k + \delta A'_k$ is denoted by $p_{\varepsilon_k,\delta}$ and it is a norm on Y'_k equivalent to s'_k. Then $\check{E}$ is the projective limit

$$\check{E} := \bigcap_{\delta,(\varepsilon_k)} \ell_\infty((X_k, p_{\varepsilon_k,\delta})_{k\in\mathbb{N}}).$$

The (LB)-space F is continuously injected in $\check{E}$ and $\check{E}$ is a complete (DF)-space. Since in this case E is regular, E and $\check{E}$ coincide algebraically. A basis of 0-neighbourhoods of $\check{E}$ is given by the sets

$$\left(\prod_{k\in\mathbb{N}} \varepsilon_k B'_k + \delta \mathscr{B}_1\right) \cap \check{E} \varepsilon_k > 0, \delta > 0$$

where $\mathscr{B}_1$ is the unit ball of E_1.

Proposition 4. *Let F, E and $\check{E}$ be as before. The following conditions are equivalent:*
(i) $E' = (\check{E})'$,
(ii) $\forall g \in E'_1$, $g|_{\oplus_{k\in\mathbb{N}} Y'_k} = 0$, $\exists \varepsilon_k > 0 (k \in \mathbb{N})$ *such that g is bounded on*

$$\left(\prod_{k\in\mathbb{N}} \varepsilon_k B'_k\right) \cap E_1.$$

Proof. (ii)$\Rightarrow$ (i). Fix $f \in E'$. Since E and $\check{E}$ induce the same topology on $\oplus_{k\in\mathbb{N}} Y'_k$ (cf. [5, 3.]), $f|_{\oplus_{k\in\mathbb{N}} Y'_k}$ is continuous for the topology induced by $\check{E}$. Applying Hahn-Banach's extension theorem, there is $h : \check{E} \to \mathbb{K}$ linear and continuous such that $h|_{\oplus_{k\in\mathbb{N}} Y'_k} = f|_{\oplus_{k\in\mathbb{N}} Y'_k}$. Now, $f = (f-h) + h$. We have $g := (f-h)|_{E_1} \subset E'_1$, $g|_{\oplus_{k\in\mathbb{N}} Y'_k} = 0$. By condition (ii), there are $\varepsilon_k > 0 (k \in \mathbb{N})$ such that $|g(x)| \le 2^{-1}$ for every $x \in (\prod_{k\in\mathbb{N}} \varepsilon_k B'_k) \cap E_1$. On the other hand, there is $\delta > 0$ such that $|g(x)| \le 2^{-1}$ if $x \in \delta \prod_{k\in\mathbb{N}} A'_k$.

Now $U := (\prod_{k\in\mathbb{N}} \varepsilon_k B'_k + \delta \prod_{k\in\mathbb{N}} A'_k) \cap E$ is a zero-neighbourhood in $\check{E}$. Moreover, if $x \in U$, there is $n \in \mathbb{N}$ with $x \in E_n$, hence $x = y + z$, $y \in \oplus_{k=1}^{n-1} Y'_k$, $z \in E_1$, $z_k = 0$, $1 \le k < n$. Then

$$(f-h)(x) = (f-h)(y+z) = (f-h)(y) + (f-h)(z) = g(z).$$

But $z_k = 0$, $1 \le k < n$, $z_k \in \varepsilon_k B'_k + \delta A'_k$, $k \ge n$. Thus $|g(z)| \le 1$. Consequently, $|(f-h)(x)| \le 1$, hence, $f - h$ is also $\check{E}$-continuous and we obtain $f \in \check{E}'$.

(i) $\Rightarrow$ (ii) Given $g \in E_1'$ with $g|_{\oplus_{k\in\mathbb{N}} Y_k'} = 0$, we define $f : E \to \mathbb{K}$ by $f(x+y) = g(x)$, $\forall x \in E_1$, $\forall y \in \oplus_{k\in\mathbb{N}} Y_k'$. f is well defined. In fact, if $x' + y' = x + y$, $x, x' \in E_1$, $y, y' \in \oplus_{k\in\mathbb{N}} Y_k'$, it follows $g(x) = g(x')$. Certainly, f is linear. Since $f|_{\oplus_{k\in\mathbb{N}} Y_k'} = 0$, $f|_{E_1} \in E_1' \in E_1'$, and E has the quotient topology with respect to the mapping

$$\psi : E_1 \times \oplus_{k\in\mathbb{N}} Y_k' \to E,\ \psi(x,y) = x + y,$$

it follows that $f \in E'$. By (i), there is $\varepsilon_k > 0 (k \in \mathbf{N})$ and there is $\delta > 0$ such that f is bounded on $(\prod_{k\in\mathbb{N}} \varepsilon_k B_k' + \delta \prod_{k\in\mathbb{N}} A_k') \cap E$. Therefore, f is bounded on $(\prod_{k\in\mathbb{N}} \varepsilon_k B_k') \cap E_1$. But $f|_{E_1} = g$. Thus, g is bounded on $(\prod_{k\in\mathbb{N}} \varepsilon_k B_k') \cap E_1$. ∎

Remark 5. Given $1 < p < \infty$, and $0 < a \leq 1$, there are b, c, $0 < b < 1$, $0 < c < 1$, such that $b^p + c^p = 1$ and $b + ac > 1$.

We take $b = 1/(1+a^q)^{1/p}$ $(p^{-1}+q^{-1} = 1)$ and $c = a^{1/p-1}/(1+a^q)^{1/p}$. Hence $b^p+c^p = 1$ and $b + ca = (1 + a^q)/(1 + a^q)^p = (1 + a^q)^{1-1/p} > 1$.

Next, we give a lemma inspired by the original example of Komura.

Lemma 6. *Let $(Y_k, s_k)_{k\in\mathbb{N}}$ be a sequence of Banach spaces. Suppose there is $1 < p < \infty$ such that for each $k \in \mathbb{N}$ there exists a net $(P_i^k)_{i\in I_k}$ of projections on Y_k with norm not greater than 1 and finite dimensional range such that $(P_i^k y)_{i\in I_k}$ tends to y for all $y \in Y_k$, and satisfying $s_k(y)^p \leq s_k(P_i^k y)^p + s_k((I - P_i^k)y)^p$ $\forall y \in Y_k$, $\forall i \in I_k$.*
If $f \in \ell_\infty((Y_k, s_k)_{k\in\mathbb{N}})'$, $f \neq 0$, vanishes on $\oplus_{k\in\mathbb{N}} Y_k$, then for all $k \in \mathbb{N}$ there is $i(k) \in I_k$ such that $f((y_k)_{k\in\mathbb{N}}) = f((P_{i(k)}^k y_k)_{k\in\mathbb{N}})$ $\forall y = (y_k)_{k\in\mathbb{N}} \in \ell_\infty((Y_k, s_k)_{k\in\mathbb{N}})$.

Proof. We may assume without loss of generality that $\| f \| = 1$. For each $r \in \mathbf{N}$ we determine $u^r = (u_k^r)_{k\in\mathbb{N}} \in \ell^\infty((Y_k, s_k)_{k\in\mathbb{N}})$ with $\| u^r \| = \sup_{k\in\mathbb{N}} s_k(u_k^r) \leq 1$ and $f(u^r) > 1 - \frac{1}{r}$. For each $k \in \mathbb{N}$ we determine $i(k) \in I_k$ such that for $i \in I_k$, $i \geq i(k)$

$$s_k(u_k^r - P_i^k u_k^r) < \frac{1}{k},\ \text{for } r = 1, 2, \ldots, k;$$

in particular,

$$s_k(u_k^r - P_{i(k)}^r u_k^r) < \frac{1}{k},\ \text{for } r = 1, 2, \ldots, k.$$

Suppose there is $v = (v_k)_{k\in\mathbb{N}} \in \ell^\infty((Y_k, s_k)_{k\in\mathbb{N}})$ such that $\sup_{k\in\mathbb{N}} s_k(v_k) = 1$, $P_{i(k)}^k v_k = 0$, $\forall k \in \mathbf{N}$, and $a := f(v) > 0$. Certainly, $0 < a \leq 1$. We define for every $r \in \mathbb{N}$

$w^r := (P_{i(k)}^k u_k^r)_{k\in\mathbb{N}}$, $y^r := ((0)_{k\leq r}, (u_k^r)_{k>r})$

$z^r := ((0)_{k\leq r}, (w_k^r)_{k>r})$.

Since $s_k(w_k^r) = s_k(P_{i(k)}^k u_k^r) \leq \| P_{i(k)}^k \| \, s_k(u_k^r) \leq 1$ for all $k \in \mathbb{N}$, it follows that w^r, y^r, z^r belong to the unit ball of $\ell^\infty((Y_k, s_k)_{k\in\mathbb{N}})$ for all $r \in \mathbb{N}$. Moreover, since f vanishes on $\oplus_{k\in\mathbb{N}} Y_k$ we have

$$|f(u^r - w^r)| = |f(y^r - z^r)| \leq \| z^r - y^r \| = \sup_{k>r} s_k(u_k^r - w_k^r) =$$

$$= \sup_{k>r} s_k(u_k^r - P_{i(k)}^k u_k^r) \leq \frac{1}{r+1} < \frac{1}{r}.$$

Consequently, $1 \geq f(w^r) \geq f(u^r) - |f(w^r - u^r)| \geq 1 - \frac{2}{r}$. Whence $\lim_{r\to\infty} f(w^r) = 1$.

Now, given $a = f(v) > 0$, $0 < a \leq 1$, we find $0 < b < 1$, $0 < c < 1$ with $b + ca > 1$, $b^p + c^p = 1$. Then $\lim_{r\to\infty} f(bw^r + cv) = b + ac > 1$. But $s_k(bw_k^r + cv_k)^p \leq s_k(P_{i(k)}^k(w_k^r b + cv_k))^p + s_k((I - P_{i(k)}^k)(bw_k^r + cv_k))^p = b^p s_k(w_k^r)^p + c^p s_k(v_k)^p \leq b^p + c^p \leq 1$, a contradiction.

Proposition 7. *Let $(Y_k, s_k)_{k\in\mathbb{N}}$, $(X_k, r_k)_{k\in\mathbb{N}}$ be two sequences of Banach spaces such that*
(a) $Y_k \subset X_k, B_k := \{y \in Y_k; s_k(y) \leq 1\} \subset \{x \in X_k; r_k(x) \leq 1\} =: A_k$, $\forall k \in \mathbb{N}$,
(b) $\exists p > 1 \, \forall k \in \mathbb{N}$ there exists a net $(P_i^k)_{i\in I_k}$ of continuous projections of norm not greater than 1 on (Y_k, s_k) such that
(b.1) $(P_i^k y)_{i\in I_k}$ converges to y for all $y \in Y_k$,
(b.2) $s_k(y)^p \leq s_k(P_i^k y)^p + s_k((I - P_i^k)y)^p$, $\forall y \in Y_k$, $\forall i \in I_k$,
(b.3) $P_i^k : (Y_k, r_k) \to (Y_k, r_k)$ is continuous $\forall i \in I_k$,
(c) Y_k is not a topological subspace of X_k, for all $k \in \mathbb{N}$.

Let $E_n := \ell^\infty((X_k, r_k)_{k<n}, (Y_k, s_k)_{k\geq n})$, $E = \operatorname{ind} E_n$ be the (LB)-space of Moscatelli type associated with ℓ^∞, $(X_k, r_k)_{k\in\mathbb{N}}$ and $(Y_k, s_k)_{k\in\mathbb{N}}$, and let $\check{E}$ denote the projective hull of E. Then E and $\check{E}$ do not coincide topologically, but they have the same dual.

Proof. It is known by [5] that E and $\check{E}$ do not coincide topologically. According to Proposition 4, it is enough to show that for every continuous linear form g on E_1', vanishing on $\oplus_{k\in\mathbb{N}} Y_k$, there is $\varepsilon_k > 0 (k \in \mathbb{N})$ such that g is bounded on $(\prod_{k\in\mathbb{N}} \varepsilon_k A_k) \cap E_1$. Now, given such a g, we may apply Lemma 6 to obtain a sequence $(i(k))_{k\in\mathbb{N}}$ such that $g((y_k)_{k\in\mathbb{N}}) = g((P_{i(k)}^k y_k)_{k\in\mathbb{N}})$, $\forall (y_k)_{k\in\mathbb{N}} \in \ell^\infty((Y_k, s_k)_{k\in\mathbb{N}})$. Since $P_{i(k)}^k(Y_k)$ is a finite dimensional subspace of Y_k, we can find $M_k > 0$ such that $s_k(P_{i(k)}^k y) \leq M_k r_k(P_{i(k)}^k y)$ for all $y \in Y_k (k \in \mathbb{N})$. Using (b.3), we find $C_k > 0$ $(k \in \mathbb{N})$ such that $r_k(P_{i(k)}^k y) \leq C_k r_k(y)$ for all $y \in Y_k$. We put $\varepsilon_k := 1/C_k M_k$ $(k \in \mathbb{N})$. Given $x \in (\prod_{k\in\mathbb{N}} \varepsilon_k A_k) \cap E_1$, we have that

$$s_k(P_{i(k)}^k x_k) \leq M_k r_k(P_{i(k)}^k x_k) \leq M_k C_k r_k(x_k) \leq 1,$$

hence $(P^k_{i(k)} x_k)_{k\in\mathbb{N}} \in (\prod_{k\in\mathbb{N}} B_k) \cap E_1$, and therefore there is $M > 0$ such that

$$|g((P^k_{i(k)} x_k)_{k\in\mathbb{N}})| \leq M.$$

Finally, we apply Lemma 6 to obtain

$$|g((x_k)_{k\in\mathbb{N}})| = |g((P^k_{i(k)} x_k)_{k\in\mathbb{N}})| \leq M$$

which finishes the proof. ∎

Corollary 8. *Let F be the Frechét space of Moscatelli type with respect to ℓ_1, $f_k : (Y_k, s_k) \to (X_k, r_k) (k \in \mathbb{N})$. Suppose that the sequences $(X'_k, r'_k)_{k\in\mathbb{N}}$, $(Y'_k, s'_k)_{k\in\mathbb{N}}$ satisfy conditions (a), (b), (c) in 7, identifying $f^t_k : X'_k \to Y'_k$ with the injection. Then F is not distinguished, but every locally bounded linear form on F'_b is continuous.*

Proof. Let E be the (LB)-space of Moscatelli type with respect to ℓ^∞, $(Y'_k, s'_k)_{k\in\mathbb{N}}$ and $(X'_k, r'_k)_{k\in\mathbb{N}}$, and let $\check{E}$ denote its projective hull. According to [6], F'_b and $\check{E}$ coincide topologically and E is the bornological space associated to F'_b. Then we may apply Proposition 7 to conclude that E and $\check{E}$ do not coincide topologically, but E and $\check{E}$ have the same dual. Consequently, F is not distinguished, but every locally bounded linear form on F'_b is continuous. ∎

Corollary 9. *Let F be the Fréchet space of Moscatelli type with respect to ℓ^1, $f_k : (Y_k, s_k) \to (X_k, r_k)$. If either*
(i) $Y_k = \ell_p$, $X_k = \ell_q$, $1 \leq p < q < \infty$ and f_k is the injection, or
(ii) $Y_k = \ell_p(a_k)$, $X_k = \ell_p$, $1 < p < \infty$, $\lim_{i\to\infty} a_k(i) = \infty$, and f_k is the injection, then F is not distinguished, but every linear form on F'_b, which is locally bounded, is continuous.

Now we deal with sequence spaces and function spaces. We first extend the original example of Komura to a wider setting of sequence spaces. For notations we refer to [4].

Proposition 10. *Let A be a Köthe matrix on an index set I such that $\lambda_1(A)$ is not distinguished. Then, there is a partition $(I_k)_{k\in\mathbb{N}}$ of I such that for every $1 < p < \infty$ the space*

$$\mu_p(A) := \left\{ (x_i)_{i\in I};\, p_n(x) := \sum_{k\in\mathbb{N}} \left(\sum_{i\in I_k} a_n(i)|x_i|^p \right)^{1/p} < \infty,\ \forall n \in \mathbb{N} \right\}$$

endowed with the topology defined by the seminorms $(p_n)_{n\in\mathbb{N}}$ is not distinguished and every locally bounded linear form on $\mu_p(A)'_b$ is continuous.

Proof. Without loss of generality, we may assume that a_1 is identically 1. Since $\lambda_1(A)$ is not distinguished, according to [1], there is a decreasing sequence $(J_k)_{k\in\mathbf{N}}$ of subsets of J, with $J_1 = I$, such that for all $k \in \mathbf{N}$

$$(i)\ \varepsilon_k := \inf_{i\in J_k} v_k(i) > 0; \qquad (ii)\ \inf_{i\in J_k} v_{k+1}(i) = 0,$$

where $v_k(i) := a_k(i)^{-1}$.

We put $I_k := J_k \setminus J_{k+1}$. Then $(I_k)_{k\in\mathbf{N}}$ is a partition of I and

$$\inf\{v_k(i); i \in I_k\} \geq \varepsilon_k > 0; \inf\{v_{k+1}(i); i \in I_k\} = 0$$

for all $k \in \mathbf{N}$. Now, we define a Köthe matrix on I, $B = (b_n)_{n\in\mathbf{N}}$, as follows

$$b_n(i) := \begin{cases} a_n(i) & \text{if } i \in I_k, k < n \\ 1 & \text{if } i \in I_k, k \geq n \end{cases}$$

and we will see that $\mu_p(A)$ is topologically isomorphic to $\mu_p(B)$, where

$$\mu_p(B) := \left\{ (x_i)_{i\in I}; q_n((x_i)_{i\in I}) := \sum_{k\in\mathbf{N}} \left(\sum_{i\in I_k} b_n(i)\,|x(i)|^p \right)^{1/p} < \infty,\ \forall n \in \mathbf{N} \right\}.$$

In fact, the two spaces coincide algebraically also. Since $b_n \leq a_n\ \forall n \in \mathbf{N}$, one obviously has $\mu_p(A) \subset \mu_p(B)$ continuously. On the other hand, given $x = (x_i)_{i\in I}$ in $\mu_p(B)$ and $n \in \mathbf{N}$, since

$$\sup\left\{ a_n(i) : i \in \bigcup_{k\geq n} I_k \right\} = \varepsilon_n^{-1} < \infty,$$

we have

$$\sum_{k=1}^{\infty} \left(\sum_{i\in I_k} a_n(i)|x_i|^p \right)^{1/p} = \sum_{k=1}^{n-1} \left(\sum_{i\in I_k} a_n(i)|x_i|^p \right)^{1/p} + \sum_{k=n}^{\infty} \left(\sum_{i\in I_k} a_n(i)|x_i|^p \right)^{1/p} \leq$$

$$\leq \sum_{k=1}^{n-1} \left(\sum_{i\in I_k} b_n(i)|x_i|^p \right)^{1/p} + \frac{1}{\varepsilon_n^{1/p}} \sum_{k=1}^{\infty} \left(\sum_{i\in I_k} |x_i|^p \right)^{1/p} \leq \max\left(1, \frac{1}{\varepsilon_n^{1/p}}\right) q_n(x) < \infty,$$

which proves that $\mu_p(B) \subset \mu_p(A)$ continuously.

Thus, we may work with $\mu_p(B)$ instead of $\mu_p(A)$. Its dual, $\mu_p(B)'$, coincides algebraically with its α-dual $\mu_p(B)^\times := \{(x_i)_{i\in I};\ \exists n \in \mathbb{N} : \sup_{k\in\mathbb{N}} \ (\sum_{i\in I_k} \frac{1}{b_n(i)}|x_i|^q)^{1/q} < \infty\}$ $(p^{-1} + q^{-1} = 1)$, and the bornological space associated with $\mu_p(B)'_b$ is the corresponding co-echelon space. The topology on $\mu_p(B)'_b$ is given by the sets

$$B_{\overline{v}} := \left\{ (x_i)_{i\in I} \in \mu_p(B)';\ \sup_k \left(\sum_{i\in I_k} \overline{v}(i)\,|x_i|^q \right)^{1/q} \leq 1 \right\},$$

where $\overline{v} \in \lambda_\infty(B)_+$, i.e., $\sup_{i\in I} \overline{v}(i)\,b_n(i) =: C_n < \infty$.

We first show that $\mu_p(B)$ is not distinguished. If it were, the set

$$C := \bigcup_{n\in\mathbb{N}} \left\{ (x_i)_{i\in I} : \sup_k \left(\sum_{i\in I_k} \frac{1}{b_k(i)}|x_i|^q \right)^{1/q} \leq 1 \right\}$$

would contain a set of the form $B_{\overline{v}}$ for some $\overline{v} \in \lambda_\infty(B)_+$. Since $\inf\{\frac{1}{b_{k+1}(i)} : i \in I_k\} = 0$, we may find $i_k \in I_k$ such that $\overline{v}(i_k)$ is arbitrarily small. We may take $\widetilde{x}_{i_k} \in \mathbb{K}$ with $2 < \widetilde{x}_{i_k} < 3$ and $\overline{v}(i_k)\widetilde{x}_{i_k} < 1$. Now $x = (x_i)_{i\in I}$, defined by $x_i := \widetilde{x}_{i_k}$ if $i = i_k$ and 0 otherwise, clearly belongs to $B_{\overline{v}}$, but it does not belong to C, since $x_{i_k} > 2\,b_k(i_k)$ for all $k \in \mathbb{N}$.

Now we take a locally bounded linear form $f \not\equiv 0$ on $\mu_p(B)'_b$. Then, if e_i is the element of $\mu_p(B)^\times$ whose coordinates are all equal to zero except the i-th which is equal to one, we have that the family $(f(e_i))_{i\in I}$ belongs to $\mu_p(B)$, whence $(f(e_i))_{i\in I} \in (\mu_p(B)'_b)'$, and obviously

$$f = (f(e_i))_{i\in I} + f - (f(e_i))_{i\in I}.$$

Therefore, it is enough to show that $g := f - (f(e_i))_{i\in I}$ belongs to $(\mu_p(B)'_b)'$. Since g is continuous on the co-echelon space $\mu_p(B)^\times$ (provided with the inductive topology), and b_1 is identically 1, we have that the restriction of g to $\ell^\infty((\ell^q(I_k))_{k\in\mathbb{N}})$ is continuous, and it clearly vanishes on $\oplus_{k\in\mathbb{N}}\ell^q(I_k)$. Applying Lemma 6, we get $\forall k \in \mathbb{N}\,\exists X_k \subset I_k$ finite such that

$$g(((x_i)_{i\in I_k})_{k\in\mathbb{N}}) = g\left(\left(\sum_{i\in I_k} x_i e_i\right)_{k\in\mathbb{N}}\right) = g\left(\left(\sum_{i\in X_k} x_i e_i\right)_{k\in\mathbb{N}}\right) \quad \text{for all}$$

$(x_i)_{i\in I} = ((x_i)_{i\in I_k})_{k\in\mathbb{N}} \in \ell^\infty((\ell^q(I_k))_{k\in\mathbb{N}})$.

We will see that there is $\overline{v} \in \lambda_\infty(B)_+$ such that $\overline{v}(i) \geq 1$, $\forall i \in X_k (k \in \mathbb{N})$. We put

$$C_n := \sup\{b_n(i); i \in X_k, k \in \mathbb{N}\}.$$

Since $b_n(i) = 1$ for all $i \in I_k$ with $k \geq n$, we have that C_n is finite and $C_n \geq 1$. Then

$$\overline{v}(i) = \inf_{n \in \mathbb{N}} \frac{C_n}{b_n(i)} \geq 1$$

for all $i \in X_k (k \in \mathbb{N})$.

Next, we prove that g is bounded on $B_{\overline{v}}$. In fact, given $x = ((x_i)_{i \in I_k})_{k \in \mathbb{N}} \in B_{\overline{v}}$, there is $n \in \mathbb{N}$ such that

$$(((0)_{i \in I_k})_{k<n}, ((x_i)_{i \in I_k})_{k \geq n}) \in \ell^\infty((\ell^q(I_k))_{k \in \mathbb{N}}),$$

hence

$$g((((0)_{i \in I_k})_{k<n}, ((x_i)_{i \in I_k})_{k \geq n})) = g((((0)_{i \in I_k})_{k<n}), ((x_i)_{i \in X_k})_{k \geq n})),$$

but the restriction of g to $\ell^\infty((\ell^q(I_k))_{k \in \mathbb{N}})$ is continuous. Therefore, there is $M > 0$ with $|g(y)| \leq M$ for all y in the unit ball of $\ell^\infty((\ell^q(I_k))_{k \in \mathbb{N}})$. Moreover, since $\overline{v}(i) \geq 1 \forall i \in X_k$, $\forall k \in \mathbb{N}$, and since for $(x_i)_{i \in I} \in B_{\overline{v}}$, $(((0)_{i \in I_k})_{k<n}, ((x_i)_{i \in X_k})_{k \geq n})$ belongs to the unit ball of $\ell^\infty((\ell^q(I_k))_{k \in \mathbb{N}})$, we have

$$|g((x_i)_{i \in I})| \leq M.$$

∎

Now we prove that for the example of Bonet and Taskinen [7] mentioned in the introduction there is even a non-continuous locally bounded linear form on the strong dual.

Let Ω be a non-void open subset of $\mathbb{R}^m$. Given $\ell \in \mathbb{N}_0 \cup \{\infty\}$, $\ell \geq t$ we define $C^\ell(\Omega) \cap H^{t,1}(\Omega)$ as the Fréchet space intersection of the Fréchet space $C^\ell(\Omega)$ and the Banach Sobolev space $H^{t,1}(\Omega)$, endowed with the natural intersection topology. The completeness of $C^\ell(\Omega) \cap H^{t,1}(\Omega)$ easily follows from the fact that both spaces in the intersection are continuously injected in the Fréchet space $L^1_{loc}(\Omega)$. Since $\ell \geq t$ we have

$$C^\ell(\Omega) \cap H^{t,p}(\Omega) = \{f \in C^\ell(\Omega); f^{(\alpha)} \in L^1(\Omega)\ \forall |\alpha| \leq t\}$$

and its locally convex topology is defined by the sequence of seminorms

$$q_0(f) := \max_{|\alpha| \leq t} \int_\Omega |f^{(\alpha)}|,$$

$$q_k(f) := \max_{|\alpha| \leq \ell} \max_{x \in L_k} |f^{(\alpha)}(x)| \text{ if } \ell \in \mathbb{N}_0,$$

$$q_k(f) := \max_{|\alpha| \leq k+1} \max_{x \in L_k} |f^{(\alpha)}(x)| \text{ if } \ell = \infty,$$

where $(L_k)_{k \in \mathbb{N}}$ is a fundamental sequence of compact subsets in Ω with $L_k \subset L^\circ_{k+1}$.

Proposition 11. *There is a non-continuous locally bounded linear form on*

$$(C^{\ell}(\Omega) \cap H^{t,1}(\Omega))'_b.$$

Proof. We take $g \in \mathscr{D}([0,1]^m)$ with $\int_{\mathbf{R}^m} |D_1^t g| = 1$ and put $\varepsilon := [\max_{|\alpha| \le t} \int_{\mathbf{R}^m} |g^{(\alpha)}|]^{-1} > 0$, $D := 2\varepsilon^{-1} > 0$.

For each $k \in \mathbf{N}$, we select $(I_{k,n})_{n \in \mathbf{N}}$, where $I_{k,n}$ is a compact cube contained in $L_{k+1}^{o} \setminus L_k$ so that they are pairwise disjoint and $\mu(I_{k,n}) \le 1/(D2^{k+n}n)$.

For arbitrary $k, n \in \mathbf{N}$, let $\varphi_{k,n}(x) = a_{k,n}x + b_{k,n}$, $a_{k,n} \in \mathbf{R}$, $b_{k,n} \in \mathbf{R}^m$, be such that $\varphi_{k,m}([0,1]^m) = I_{k,m}$. Now, we define $f_{k,n} := C_{k,n} g \circ \varphi_{k,n}^{-1}$ with $C_{k,n} > 0$ such that $q_0(f_{k,n}) = 1$. For every $k, n \in \mathbf{N}$, there is a mesurable mapping $h_{k,n}$ on $I_{k,n}$ such that

$$D_1^t f_{k,n}(y) = h_{k,n}(y) |D_1^t f_{k,n}(y)| \text{ with } |h_{k,n}(y)| = 1.$$

Given $n \in \mathbf{N}$, $(n_j)_{j \in \mathbf{N}} \in \mathbf{N}^{\mathbf{N}}$, we put $M(k, (n_j)_{j \in \mathbf{N}}) := \{f_{\ell,r} : r \ge n_\ell, \ell \ge k\}$.

Then $\{M(k, (n_j)_{j \in \mathbf{N}}) : k \in \mathbf{N}, (n_j)_{j \in \mathbf{N}} \in \mathbf{N}^{\mathbf{N}}\}$ is a filter basis in $C^{\ell}(\Omega) \cap H^{t,1}(\Omega)$.

Since $q_0(f_{\ell,r}) = 1$, $\forall \ell, r \in \mathbf{N}$, it is clear that $M(k, (n_j)_{j \in \mathbf{N}})$ is contained in the semi-unit ball of q_0. Moreover, supp $f_{\ell,r} \subset L_{\ell+1}^{o} \setminus L_\ell$, hence $q_k(f_{\ell,r}) = 0$ for all $\ell \ge k$ and for all $r \in \mathbf{N}$. Hence, $M(k, (n_j)_{j \in \mathbf{N}})$ is contained in the semi-unit ball corresponding to q_k. Thus, condition (ii) (a) in Lemma 1 holds.

Now, we take a bounded set B in $C^{\ell}(\Omega) \cap H^{t,1}(\Omega)$. Then, we find $\lambda_k > 0$ such that $q_k(f) \le \lambda_k$ for all $f \in B$ and for all $k \in \mathbf{N}_0$. We choose $j_k \in \mathbf{N}$ with $j_k \ge \lambda_{k+1} (k \in \mathbf{N})$ and put

$$u(f) := D \sum_{k \in \mathbf{N}} \sum_{\ell \ge j_k} \int_{I_{k,\ell}} (D_1^t f) h_{k,\ell}.$$

Since $|u(f)| \le q_0(f)$, we have that $u \in (C^{\ell}(\Omega) \cap H^{t,1}(\Omega))'$; we will see that $u \in B^{o}$. In fact, given $f \in B$,

$$|u(f)| \le D \sum_{k \in \mathbf{N}} \sum_{\ell \ge j_k} \mu(I_{k,\ell}) \sup_{I_{k,\ell}} |D_1^t f(x)| \le D \sum_{k \in \mathbf{N}} \sum_{\ell \ge j_k} \frac{1}{D2^{\ell+k} j_k} \lambda_{k+1} \le 1,$$

whence $u \in B^{o}$.

We put $V_B = \frac{1}{2}\{u\}^{o} = \{2u\}^{o}$. We have $B + V_B \subset \{u\}^{o} + \frac{1}{2}\{u\}^{o} \subset \frac{3}{2}\{u\}^{o}$. We claim that

$$M(1, (j_k)_{k \in \mathbf{N}}) \subset F \setminus (B + V_B), \text{ where } F = C^{\ell}(\Omega) \cap H^{t,1}(\Omega).$$

In fact, given $f_{k,\ell}$ with $\ell \geq j_k$, $u(f_{k,\ell}) \geq 2$ because

$$u(f_{k,\ell}) = D\int_{I_{k,\ell}} (D_1^t f_{k,\ell}) h_{k,\ell} = D\int_{I_{k,\ell}} |D_1^t f_{k,\ell}| = \frac{DC_{k,\ell}|a_{k,\ell}|^{m-t}}{q_0(f_{k,\ell})} =$$

$$= \frac{DC_{k,\ell}|a_{k,\ell}|^{m-t}}{\max_{|\alpha|\leq t} C_{k,\ell}|\alpha_{k,\ell}|^{m-|\alpha|}\int_{\mathbf{R}^m}|g^{(\alpha)}|} \geq \frac{D}{\max_{|\alpha|\leq t}\int_{\mathbf{R}^m}|g^{(\alpha)}|} = D\varepsilon = 2\,.$$

Assume $M(1,(j_k)_{k\in\mathbf{N}})\cap(B+V_B) \neq \emptyset$. Then, there is $f_{k,\ell} \in M(1,(j_k)_{k\in\mathbf{N}})\cap\frac{3}{2}\{u\}^o$, hence there is $\ell \geq j_k$ such that $f_{k,\ell} \in \frac{3}{2}\{u\}^o$. Therefore,

$$2 \leq u(f_{k,\ell}) \leq \frac{3}{2}$$

a contradiction. This yields condition (ii) (b) of Lemma 1, and an application of this Lemma allows to conclude. ∎

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Added in proof. V. B. Moscatelli communicated to the authors that all the spaces $C^\ell(\Omega) \cap H^{t,1}(\Omega)$ which appear in Proposition 11 are of Moscatelli type.

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