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(DF)-SPACES OF TYPE $CB(X,E)$ AND $C\overline{V}(X,E)$

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *Some locally convex properties of the spaces $CB(X,E)$ of the bounded continuous functions on a completely regular Hausdorff space X with values in a (DF)-space E are studied and applied to the (DF)-spaces of type $C\overline{V}(X,E)$ (e.g., see [5]).*
The following are our main results:

1. *$CB(X,E)$ is a (DF)-space if and only if E is a (DF)-space.*
2. *For a (DF)-space E, $CB(X,E)$ is quasibarrelled if and only if*

either (i) X is pseudocompact and E is quasibarrelled

or (ii) X is not pseudocompact and the bounded subsets of E are metrizable.

3. *If $\mathcal{V} \subset C(X)$ and if each $\overline{v} \in \overline{V}$ is dominated by some $\tilde{v} \in \overline{V} \cap C(X)$, then*

$C\overline{V}(X,E)$ (resp., $C\overline{V}(X) \otimes_\varepsilon E$) is a (DF)-space if and only if E is a (DF)-space.

4. *Let X be a locally compact and σ-compact space, $\mathcal{V} \subset C(X)$ and E a (DF)-space.*

Then $C\overline{V}(X,E)$ is quasibarrelled if and only if

(i) E is quasibarrelled and $\mathcal{V}$ satisfies condition $(M,\mathcal{K})$ or

(ii) the bounded subsets of E are metrizable and $\mathcal{V}$ satisfies condition (D).

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INTRODUCTION

The class of (DF)-spaces (containing all strong duals of Fréchet spaces, but also all (LB)-spaces) was introduced by A. Grothendieck around 1954. Quite recently, there has been a renewed interest in this class; e.g., J. Taskinen [19], [20] gave a negative solution to Grothendieck's «problème des topologies» (for tensor products of Fréchet spaces) and to some related problems on (DF)-spaces.

In [4], it is pointed out that the (DF)-spaces are the «right» setting for the so-called «dual density conditions», dual reformulations of S. Heinrich's density condition (cf. [3]) which are closely related to the important property that all bounded subsets are metrizable. A large part of [4] is devoted to the study of the locally convex properties of certain vector-valued sequence spaces; viz., the space $l_\infty(E)$ of all bounded sequences in a (DF)-space E and, more generally, (DF)-valued co-echelon spaces $K_\infty(E)$ (or, equivalently, spaces $\mathscr{L}_b(\lambda_1, E)$ of the continuous linear mappings from a Köthe echelon space λ_1 into a (DF)-space E).

Since some of the techniques used in [4] also work in the more general context of spaces of continuous vector-valued functions, we now set out to extend several theorems of [4] to this context. We start by investigating the space $CB(X,E)$ of the bounded continuous functions on a completely regular Hausdorff space X with values in a (DF)-space E and later apply our results to the more general spaces of type $C\overline{V}(X,E)$ i.e., to the «projective hulls» of the weighted inductive limits $\mathcal{V}C(X,E)$ (cf. [5] - [9]).

In the *first* of the four sections of the present article, we derive a «decomposition» lemma (Lemma 2) for certain absolutely convex subsets of $CB(X,E)$ which will permit direct «estimates» with the 0-neighborhoods and bounded sets in this space. The proof of this lemma is based on the «density argument» 3, involving locally finite continuous partitions of unity on X. In *Section 2*, we use Lemma 2 to prove (Theorem 5) that $CB(X,E)$ is a (DF)-space if and only if E is a (DF)-space. For a (DF)-space E, the quasibarrelled spaces $CB(X,E)$ are then characterized in Theorem 6, and there are two distinct cases: If X is pseudocompact, $CB(X,E)$ is quasibarrelled if and only if E is, but if X is not pseudocompact, then the quasibarrelledness of $CB(X,E)$ is equivalent to the metrizability of the bounded subsets of E (i.e., to the dual density condition (DDC) for E, cf. [4]).

In an *appendix* to Section 2, we compare $CB(X,E_b')$ and $\mathscr{L}_b(E,CB(X))$, where E is a quasibarrelled locally convex space, as well as the corresponding spaces $H^\infty(X,E_b')$ and $\mathscr{L}_b(E,H^\infty(X))$, where $H^\infty(X)$ denotes the space of the bounded *holomorphic* functions on an open subset X of $\mathbb{C}^N (N \geq 1)$. In particular, we obtain an example of a space of bounded holomorphic functions on the open unit disk D of the complex plane with values in the strong dual of a reflexive Fréchet space which does not have the (DF)-property. (In view of the fact that $H^\infty(D)$ is not a $\mathscr{L}_\infty$-space, this is a consequence of a recent result of Defant-Floret-Taskinen [21]).

In *Section 3*, we first list several characterizations of the type «$CB(X,E)$ satisfies property (P) if and only if E does». (They follow quite easily from Lemma 2.) We then proceed to make some remarks on quasibarrelled and barrelled $CB(X,E)$-spaces, where E is no longer assumed to be a (DF)-space.

Section 4 turns to the spaces $C\overline{V}(X,E)$ with values in a (DF)-space E; here $\overline{V} = \overline{V}(\mathcal{V})$ is associated with a decreasing sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive continuous weights on X. Generalizing Theorem 5, we establish the (DF)-property of $C\overline{V}(X,E)$ and of $C\overline{V}(X) \otimes_\varepsilon E$ (Theorem 13 and Corollary 14) if E is a (DF)-space and if each $\bar{v} \in \overline{V}$ is dominated by some $\tilde{v} \in \overline{V} \cap C(X)$. (The last hypothesis clearly holds if X is locally compact and σ-compact). If E is a (DF)-space and X is locally compact and σ-compact, we finally characterize the quasibarrelledness of $C\overline{V}(X,E)$ (Theorem 16) and again obtain a dichotomy: The space $C\overline{V}(X,E)$ is quasibarrelled if and only if (i) E is quasibarrelled and $\mathcal{V}$ satisfies condition $(M,\mathcal{K})$ or (ii) the bounded subsets of E are metrizable and $\mathcal{V}$ satisfies condition (D).

Notation. Our notation concerning locally convex spaces (which are always assumed to be different from $\{0\}$ and Hausdorff) is quite standard; e.g., see [13] and [14]. For a locally convex space $E, cs(E)$ denotes the system of all the continuous seminorms on E. If two locally convex spaces E and F are topologically isomorphic, we will sometimes abbreviate this by saying that E is isomorphic to F.

In the sequel, X will always denote a completely regular Hausdorff space and E a locally convex space. $l_\infty(E)$ denotes the space of all the bounded sequences in E with the uniform topology. $CB(X,E)$ is the space of all the bounded continuous functions from X into E, endowed with the topology of uniform convergence on X. (In particular, if $X = \mathbf{N}$ with the discrete topology, we clearly have $CB(X,E) = l_\infty(E)$.) If A is a subset of E, we put

$$CB(X,A) := \{f \in CB(X,E); f(X) \subset A\}.$$

We will also let $C(X,E)$ denote the space of all the continuous functions $f : X \to E$. If E is the field of real or complex scalars, we drop it from our notation and write $CB(X)$ and $C(X)$ (as well as $C\overline{V}(X)$), but we will also use the notation $CB(X,[0,1])$ in this case.

1. GENERALITIES, BASIC LEMMA

In this section, some facts of a more technical nature are established which we will need in the proof of our main results in Section 2.

Remark 1. *(a) $CB(X)$ and E are always topologically isomorphic to complemented subspaces of $CB(X,E)$.*

(b) If X is not pseudocompact, then $l_\infty(E)$ is also topologically isomorphic to a complemented subspace of $CB(X,E)$.

Proof. It is only necessary to show (b). Since X is not pseudocompact, it is easy to construct a sequence $(x_n)_{n\in\mathbf{N}} \subset X$ and a sequence $(\varphi_n)_{n\in\mathbf{N}} \subset CB(X, [0,1])$ with $\varphi_n(x_n) = 1$ for every $n \in \mathbf{N}$ such that the supports of the functions φ_n are mutually disjoint and locally finite (cf. [17], II. 11.9). Then one can check directly that

$$(Pf)(x) := \sum_{n=1}^{\infty} \varphi_n(x) f(x_n) \quad \text{for} \quad f \in CB(X,E) \quad \text{and} \quad x \in X$$

defines a continuous linear projection P on $CB(X,E)$ with range isomorphic to $l_\infty(E)$. ∎

Recall that, by a result of M. and S. Krein (cf. [17], IV.1.1), $CB(X)$ is separable if and only if X is compact and metrizable. Thus clearly, for a compact metrizable X, l_∞ cannot be isomorphic to a (complemented) subspace of $CB(X)$.

Lemma 2. *For an arbitrary absolutely convex 0-neighborhood U in E, for a finite number of absolutely convex bounded subsets $B_1, \dots, B_n (n \in \mathbf{N})$ of E and for any given $\varepsilon > 0$, we have*

$$CB\left(X, U + \sum_{k=1}^{n} B_k\right) \subset (1+\varepsilon) CB(X,U) + \sum_{k=1}^{n} CB(X, B_k).$$

This is our main lemma for the proof of which we first need an auxiliary result. The proof of 3 is rather standard (e.g., cf. M. Rome [16], Théorème 3.5).

Density argument 3. *For every $f \in CB(X,E), p \in cs(E)$ and $\varepsilon > 0$, there are a locally finite continuous partition of unity $(\varphi_j)_{j\in J}$ on X and a subset $(x_j)_{j\in J}$ of X such that*

$$\sup_{x\in X} p\left(f(x) - \sum_{j\in J} \varphi_j(x) f(x_j)\right) < \varepsilon.$$

Proof of 3.

$$x \sim y \iff p(f(x) - f(y)) = 0$$

clearly defines an equivalence relation on X, and $d(\tilde{x}, \tilde{y}) := p(f(x) - f(y))$ yields a metric on $X/\sim$ which makes the canonical mapping $\pi : X \to (X/\sim, d)$ continuous. Now $(X/\sim, d)$ is paracompact, and thus we can find a locally finite continuous partition of unity $(\psi_j)_{j\in J}$ on $(X/\sim, d)$ which is subordinate to the open cover

$$\{\eta \in X/\sim; d(\xi,\eta) < \varepsilon/3\}, \qquad \xi \in X/\sim.$$

We can assume that each ψ_j is different from zero, and for every $j \in J$, we can choose $x_j \in X$ with $\tilde{x}_j \in \operatorname{supp} \psi_j$. To conclude, it is now enough to note that

$$\sup_{x\in X} p\left(f(x) - \sum_{j\in J} (\psi_j \circ \pi)(x) f(x_j)\right) \leq 2\varepsilon/3$$

since $(\psi_j \circ \pi)_{j\in J}$ obviously is a locally finite continuous partition of unity on X. ∎

We remark that for a given $f \in CB(X,E)$, the function $g : X \to E$, $g(x) := \sum_{j\in J} \varphi_j(x) f(x_j)$ for $x \in X$, which appears in the previous result, will in general not be an element of the tensor product $CB(X) \otimes E$ (nor an element of its closure in $CB(X,E)$).

Proof of Lemma 2. We denote by p the Minkowski functional of U. For $\varepsilon > 0$ and a fixed $f \in CB(X, U+\sum_{k=1}^n B_k)$, we apply the density argument 3 to find a locally finite continuous partition of unity $(\varphi_j)_{j\in J}$ on X and a subset $(x_j)_{j\in J}$ of X such that

$$(*) \qquad \sup_{x\in X} p\left(f(x) - \sum_{j\in J} \varphi_j(x) f(x_j)\right) < \varepsilon.$$

Since $f \in CB(X,E)$, the function $g : x \to \sum_{j\in J} \varphi_j(x) f(x_j)$ also belongs to $CB(X,E)$, and (*) implies $f - g \in \varepsilon CB(X,U)$. By hypothesis we have $f(x) \in U + \sum_{k=1}^n B_k$ for each $j \in J$, and hence $f(x_j) = u_j + \sum_{k=1}^n b_{k,j}$ with $u_j \in U$ and $b_{k,j} \in B_k$, $1 \le k \le n$. If we put $g_0(x) := \sum_{j\in J} \varphi_j(x) u_j$ and $g_k(x) := \sum_{j\in J} \varphi_j(x) b_{k,j}$ for all $x \in X$, then clearly $g_0 \in C(X,E), g_k \in CB(X,E)$, $k = 1,\dots,n$, and $g = \sum_{k=0}^n g_k$ whence also $g_0 \in CB(X,E)$. Thus we obtain $g_0 \in CB(X,U)$ and $g_k \in CB(X,B_k)$, $1 \le k \le n$, and finally

$$f = (f-g) + \sum_{k=0}^n g_k = g_0 + (f-g) + \sum_{k=1}^n g_k \in (1+\varepsilon)CB(X,U) + \sum_{k=1}^n CB(X,B_k).$$

∎

In Remark 4 (below) we list some other «formulae» which can be derived in a similar way. But, to do this, we first introduce the following additional *notation:* For any subset A of E, let

$$PCB(X,A) := \{g \in CB(X,E); g(x) = \sum_{j\in J} \varphi_j(x) a_j \quad \text{for all} \quad x \in X,$$

where $(a_j)_{j\in J} \subset A$ and $(\varphi_j)_{j\in J}$ is a locally finite continuous partition of unity on $X\}$.

Remark 4. *The following assertions are true:*

(a) If $D_1,\dots,D_n \subset E$ are absolutely convex sets which are all bounded except for at most one, then

$$PCB\left(X, \sum_{k=1}^n D_k\right) = \sum_{k=1}^n PCB(X,D_k).$$

(b) For every absolutely convex bounded subset B of E,

(i) $PCB(X,B)$ is a bounded subset of $CB(X,E)$,

(ii) $\overline{PCB(X,B)} = \overline{PCB(X,\overline{B})}$ and

(iii) $\overline{CB(X,B)} = \overline{CB(X,\overline{B})} = CB(X,\overline{B}) = \overline{PCB(X,B)}$.

(c) If $B \subset E$ is absolutely convex bounded and U an absolutely convex neighborhood of 0 in E, then

$$CB(X,\overline{U+B}) = \overline{CB(X,U)+CB(X,B)}.$$

2. (DF)-SPACES OF TYPE $CB(X,E)$

Our first theorem is one of the main results of this article. It provides an affirmative answer to the question [14], 13.8.3 (and generalizes a theorem of S. Dierolf [12] for $l_\infty(E)$).

Theorem 5. *$CB(X,E)$ is a (DF)-space (resp., a (gDF)-space) if and only if E is a (DF)- (resp., (gDF)-) space.*

Proof. The necessity is obvious because E is isomorphic to a complemented subspace of $CB(X,E)$. To prove that the condition is also sufficient, we fix a fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of closed absolutely convex bounded subsets of E. It is easy to see that $(CB(X,B_n))_{n\in\mathbb{N}}$ forms a fundamental sequence of bounded sets in $CB(X,E)$.

We first suppose that E is a (DF)-space and also fix a sequence $(V_n)_{n\in\mathbb{N}}$ of closed absolutely convex 0-neighborhoods in $CB(X,E)$ such that $V := \bigcap_{n\in\mathbb{N}} V_n$ is bornivorous. Then for every $n\in\mathbb{N}$, there is $\varepsilon_n > 0$ with $\varepsilon_n CB(X,B_n) \subset 2^{-n-1}V$, and there also exists an absolutely convex 0-neighborhood U_n in E with $CB(X,U_n) \subset 2^{-2}V_n$. We set $W_n := U_n + \sum_{k=1}^{n} \varepsilon_k B_k$, $n = 1,2,\ldots$. Clearly, each W_n is an absolutely convex 0-neighborhood in E, and it easily follows from the definition of W_n that $W := \bigcap_{n\in\mathbb{N}} W_n$ is bornivorous. Since E is a (DF)-space, W must be a 0-neighborhood in E. From Lemma 2, we now obtain

$$CB(X,W_n) \subset 2CB(X,U_n) + \sum_{k=1}^{n} \varepsilon_k CB(X,B_k) \subset 2^{-1}V_n + \sum_{k=1}^{n} 2^{-k-1}V \subset V_n$$

and consequently

$$CB(X,W) \subset \bigcap_{n\in\mathbb{N}} CB(X,W_n) \subset \bigcap_{n\in\mathbb{N}} V_n = V.$$

Thus, $CB(X,E)$ is a (DF)-space.

Suppose next that E is a (gDF)-space and that the sequence $(B_n)_{n\in\mathbb{N}}$ satisfies

$2B_n \subset B_{n+1}$ for each $n \in \mathbb{N}$. In view of [13], 12.3.1, if we want to prove that $CB(X,E)$ is a (gDF)-space, it is enough to show the following: Given any sequence $(U_n)_{n\in\mathbb{N}}$ of absolutely convex 0-neighborhoods in E, the set

$$V := \bigcap_{n\in\mathbb{N}} \left(CB(X,B_n) + CB(X,U_n)\right)$$

is again a neighborhood of 0 in $CB(X,E)$. Take $(U_n)_{n\in\mathbb{N}}$ as above and set

$$U := \bigcap_{n\in\mathbb{N}} (B_n + 2^{-1}U_n);$$

then U is a 0-neighborhood in the (gDF)-space E. By Lemma 2, we get

$$CB(X,U) \subset \bigcap_{n\in\mathbb{N}} CB(X, B_n + 2^{-1}U_n) \subset \bigcap_{n\in\mathbb{N}} \left(CB(X,B_n) + CB(X,U_n)\right) = V,$$

and the proof is complete. ∎

The characterization of the quasibarrelled (DF)-spaces of type $CB(X,E)$ leads to an interesting dichotomy.

Theorem 6. *Let E denote a (DF)-space. Then $CB(X,E)$ is quasibarrelled if and only if one of the following conditions (i) and (ii) is satisfied:*

(i) X is pseudocompact and E is quasibarrelled,

(ii) X is not pseudocompact and every bounded subset of E is metrizable.

Proof. Let $CB(X,E)$ be a quasibarrelled. Then E must be quasibarrelled because it is isomorphic to a complemented subspace of $CB(X,E)$. Moreover, if X is not pseudocompact, we deduce from Remark 1.(b) that $l_\infty(E)$ must be quasibarrelled too, which by [4], Theorem 1.5.(a) is equivalent to the condition that every bounded subset of E is metrizable.

Conversely, each of the conditions (i) and (ii) is sufficient. First, if (ii) is satisfied, it is a direct matter to check that also every bounded subset of $CB(X,E)$ is metrizable. But $CB(X,E)$ is a (DF)-space by Theorem 5. Thus, the conclusion follows from a classical result of Grothendieck (e.g., see [14], 8.3.13.(ii)).

It remains to show that (i) implies the quasibarrelledness of $CB(X,E)$. We need the following

Lemma 7. *Let $f(X)$ be precompact in E for each $f \in CB(X,E)$. (This hypothesis certainly holds if X is pseudocompact, but it is also satisfied if each bounded subset of E is precompact.)*

Then for any bounded set $B \subset CB(X,E)$, there is a bounded set $C \subset CB(X) \otimes E$ (with the induced topology) with $B \subset \overline{C}$.

Proof. 1. We fix $p \in cs(E)$ and $\varepsilon > 0$. For given $f \in CB(X, E)$, the set $f(X)$ is precompact in E. Hence there exists a finite subset $\{x_1, \ldots, x_J\}$ of X such that

$$f(X) \subset \bigcup_{j=1}^{J} \{e \in E; p(f(x_j) - e) < \varepsilon\}.$$

For $j = 1, \ldots, J$ and $x \in X$, we set $\psi_j(x) := \sup\{\varepsilon - p(f(x_j) - f(x)), 0\}$ and observe that, obviously, $\sum_{j=1}^{J} \psi_j(x) > 0$ for each $x \in X$. Thus, the functions

$$\varphi_j := \frac{\psi_j}{\sum_{k=1}^{J} \psi_k}, \qquad j = 1, \ldots, J,$$

form a finite continuous partition of unity on X with

$$\sup_{x \in X} p\left(f(x) - \sum_{j=1}^{J} \varphi_j(x) f(x_j) \right) \leq \sup_{x \in X} \sum_{j=1}^{J} \varphi_j(x) p(f(x) - f(x_j)) \leq \varepsilon.$$

2. In this way, with every $f \in CB(X, E), p \in cs(E)$ and $\varepsilon > 0$, we have associated a finite continuous partition of unity $\{\varphi_1, \ldots, \varphi_J\}$ on X and a subset $\{x_1, \ldots, x_J\}$ of X such that $\sup_{x \in X} p(f(x) - \sum_{j=1}^{J} (\varphi_j \otimes f(x_j))(x)) \leq \varepsilon$. The conclusion now follows from the remark that for every bounded set B in $CB(X, E)$,

$$C := \{\sum_{j=1}^{J} \varphi_j \otimes f(x_j); \{\varphi_1, \ldots, \varphi_J\} \text{ is a finite continuous partition of unity on } X, \{x_1, \ldots, x_J\} \subset X \text{ and } f \in B\}$$

is a bounded subset of $CB(X, E)$ with $B \subset \overline{C}$. ∎

We can now finish the *proof of Theorem 6.* If (i) is satisfied, we may apply Lemma 7 from which it follows, in particular, that $CB(X) \otimes E$ is dense in $CB(X, E)$. (Actually, this last fact is well-kquasibarrelledn and can e.g. be found in [2].) Moreover, $CB(X, E)$ induces the ε-tensor product topology on $CB(X) \otimes E$ (again, see [2]). But $CB(X)$ is a $\mathscr{L}_\infty$-space, and we deduce from [14], 11.5.10 that $CB(X) \otimes_\varepsilon E$ is quasibarrelled. Hence $CB(X, E)$ must also be quasibarrelled (cf. [13], 11.3.1.(e)). ∎

For every quasibarrelled (DF)-space E and every compact space K, $CB(K,E)$ ($= C(K,E)$ with the uniform topology) is quasibarrelled by a result of Mendoza (cf. [18], IV. 6.8). On the other hand, according to [3] (also see [4]), there exist even strong duals E of reflexive Fréchet spaces with the property that $l_\infty(E)$ is not quasibarrelled. If E denotes such a space, then certainly $l_\infty(E)$ cannot be isomorphic to a complemented subspace of $CB(K,E)$.

In [4], Theorem 1.5.(b), the bornological (DF)-spaces $l_\infty(E)$ had also been characterized by the «strong dual density condition (SDDC)» for E. But a characterization of the bornological spaces $CB(X,E), E$ a (DF)-space, similar to the one in Theorem 6 does not seem to be available at this moment: Clearly, for a (DF)-space $E, CB(X,E)$ bornological implies E bornological, and if X is not pseudocompact, then E must satisfy (SDDC) in view of Remark 1.(b) and [4], Theorem 1.5.(b). In the converse direction, we can only conclude the following: If E is a quasicomplete bornological (DF)-space such that for every absolutely convex compact subset K of E, there is a disk $B \subset E$ such that K is contained and compact in E_B and if each function $f \in CB(X,E)$ has precompact range, then [2] implies $CB(X,E) = CB(X)\varepsilon E$, and this space is bornological by [14], Proposition 11.5.13. However, using our present methods which involve the density argument 3, and hence closures of absolutely convex sets, we are not able to prove (along the lines of the direct proof of (1) $\Rightarrow$ (3') in Theorem 1.4 of [3]) that $CB(X,E)$ must be bornological for any (DF)-space E with (SDDC).

APPENDIX

As S. Dierolf [12], proof of Proposition 5.14 showed, the spaces $l_\infty(I,E'_b)$ and $\mathscr{L}_b(E,l_\infty(I))$ are topologically isomorphic whenever E is a quasibarrelled locally convex space and I an arbitrary index set. The canonical topological isomorphism Θ : $\mathscr{L}_b(E,l_\infty(I)) \to l_\infty(I,E'_b)$ is given by

$$\Theta(T) := (\delta_i \circ T)_{i\in I} \quad \text{for} \quad T \in \mathscr{L}_b(E,l_\infty(I)),$$

where $\delta_i : f \to f(i)$ denotes the projection of $l_\infty(I)$ onto its i-th coordinate; the image of an element $F = (F(i))_{i\in I}$ of $l_\infty(I,E'_b)$ under $\Theta^{-1} : l_\infty(I,E'_b) \to \mathscr{L}_b(E,l_\infty(I))$ has the form

$$T = \Theta^{-1}(F),\ T(e) = ((F(i))(e))_{i\in I} \quad \text{for} \quad e \in E.$$

(A generalization of this result to vector-valued co-echelon spaces $K_\infty(I,E'_b)$ was established in [4], Proposition 2.13.(a).)

The corresponding topological isomorphism with «l_∞» replaced by «CB» is not true in general; i.e., it is not possible to generalize the full extent of S. Dierolf's result from bounded sequences to bounded continuous functions. However, a part remains valid in our case.

Proposition A. *If E is quasibarrelled, $CB(X, E_b')$ is topologically isomorphic to a subspace of $\mathscr{L}_b(E, CB(X))$. The canonical isomorphism Λ of $CB(X, E_b')$ into $\mathscr{L}_b(E, CB(X))$ is defined by*

$$[\Lambda(F)](e) : x \to (F(x))(e) \quad for \quad e \in E,$$

but it is not surjective in general (even if X is compact).

Proof. Λ is the restriction of $\Theta^{-1} : l_\infty(X, E_b') \to \mathscr{L}_b(E, l_\infty(X))$ to the topological subspace $CB(X, E_b')$ of $l_\infty(X, E_b')$. It only takes a short glance to verify that $\Lambda(CB(X, E_b'))$ is contained in the topological subspace $\mathscr{L}_b(E, CB(X))$ of $\mathscr{L}_b(E, l_\infty(X))$; hence the first (positive) part of the proposition.

To see that $\Lambda : CB(X, E_b') \to \mathscr{L}_b(E, CB(X))$ is not surjective in general, we take $E = CB(X)$ and note that the image of the identity (inclusion) $\in \mathscr{L}_b(CB(X), l_\infty(X))$ under Θ is nothing but $(\delta_x)_{x \in X} \in l_\infty(X, CB(X)_b')$. Thus, the only possible preimage of $\mathrm{id}_{CB(X)} \in \mathscr{L}(CB(X), CB(X))$ under Λ is the mapping

$$\Delta : x \to \delta_x, \ \delta_x(f) = f(x) \quad \text{for} \quad f \in CB(X),$$

of X into $CB(X)_b'$. But $\Delta : X \to CB(X)_b'$ is continuous if and only if the unit ball $CB(X)_1$ of $CB(X)$ is equicontinuous on X, and clearly, the last property does not hold for many compact spaces X, e.g. $X = [0, 1]$. ∎

On the other hand, if we replace the bounded continuous functions by the bounded holomorphic functions; i.e., $CB(X)$ by $H^\infty(X)$ (where X is an open subset of $\mathbb{C}^N$) and $CB(X, E_b')$ by $H^\infty(X, E_b')$, then the situation improves again, and we end up with the full topological isomorphism as in the l_∞-case.

Proposition B. *Let X denote an open subset of $\mathbb{C}^N (N \geq 1)$ and E a quasicomplete (quasi-) barrelled space. Then $H^\infty(X, E_b')$ is topologically isomorphic to $\mathscr{L}_b(E, H^\infty(X))$.*

Proof. For Λ as in the proof of the preceding proposition, it is obvious from the definition of the vector-valued (weakly) holomorphic functions that Λ maps $H^\infty(X, E_b') \subset CB(X, E_b')$ into the topological subspace $\mathscr{L}_b(E, H^\infty(X))$ of $\mathscr{L}_b(E, CB(X))$. It remains to show the equality $\Lambda(H^\infty(X, E_b')) = \mathscr{L}(E, H^\infty(X))$, and it clearly suffices to prove $\Theta(\mathscr{L}(E, H^\infty(X))) \subset H^\infty(X, E_b')$.

Fixing $T \in \mathscr{L}(E, H^\infty(X))$, we have $\Theta(T) : z \to \delta_z \circ T$. Since for arbitrary $e \in E$ the function $z \to (\Theta(T)(z))(e) = (T(e))(z)$ belongs to $H^\infty(X)$, the mapping $\Theta(T)$ is (weakly) holomorphic into (say) E_τ'; i.e., E' with the Mackey topology $\tau = \tau(E', E)$. But since E is quasicomplete, E_τ' and E_b' have the same bounded sets and hence the same holomorphic functions (e.g., see Grothendieck [22], p. 40). We conclude that $\Theta(T)$ is holomorphic from X into E_b' and thus an element of $H^\infty(X,E_b')$. ∎

From the point of view of Banach space theory, there is a big difference between (the $C(K)$-space) $CB(X) = C(\beta X)$ and, say, $H^\infty(D)$, where D denotes the open unit disk in the complex plane: Viz., $H^\infty(D)$ is not even a $\mathscr{L}_\infty$-space (see Pelczynski [23]). Interestingly enough, this difference is reflected in the locally convex properties of $CB(X,E)$ and $H^\infty(D,E)$ for (DF)-spaces E (or even for strong duals E of Fréchet spaces).

Proposition C. *For some reflexive Fréchet space E, $H^\infty(D,E_b') = \mathscr{L}_b(E,H^\infty(D))$ is not a (DF)-space.*

Proof. (The result is a consequence of a recent theorem in [21]; the authors thank Andreas Defant and Klaus Floret for conversations on this subject).

It is well-kquasibarrelledn that $H^\infty(D)$ is a dual Banach space; let F denote a Banach space such that $F_b' = H^\infty(D)$. Since $H^\infty(D)$ is not a $\mathscr{L}_\infty$-space, it follows that F cannot be a $\mathscr{L}_1$-space. Thus, by Defant-Floret-Taskinen [21], 1.3 Corollary, there exists a reflexive Fréchet space E such that $\mathscr{L}_b(F,E_b')$ is not a (DF)-space. But transposition of mappings induces a topological isomorphism of $\mathscr{L}_b(F,E_b')$ and $\mathscr{L}_b(E,H^\infty(D))$. (This last isomorphism can be proved in a direct way. Alternatively, see the discussion in Jarchow [13], Section 15.3 and use $E_b' = \operatorname{ind}_n E'_{U_n^\circ}$, where $(U_n)_{n\in\mathbb{N}}$ denotes a decreasing basis of 0-neighborhoods in E, to deduce the algebraic equality $\mathscr{L}(F,E_b') = \mathscr{L}(F,E'_{cont})$ and hence $\mathscr{L}(F,E_b') = \mathscr{L}(E,H^\infty(D))$. Then the topological isomorphism $\mathscr{L}_b(F,E_b') = \mathscr{L}_b(E,H^\infty(D))$ is obvious.) ∎

We finish this appendix by pointing out that Proposition B can also serve to demonstrate that the space $H^\infty(X,E_b')$ of vector-valued functions is quite different from the corresponding ε-product $H^\infty(X)\varepsilon(E_b')$ in many cases: Viz., if E is a Banach space, then $H^\infty(X)\varepsilon(E_b')$ is isometrically isomorphic to the space $C(E,H^\infty(X))$ of all compact operators from E into $H^\infty(X)$ (under the operator norm).

3. SOME ADDITIONAL REMARKS ON RELATED PROPERTIES

At the beginning of this section, we list some easy characterizations of the following type: «$CB(X,E)$ satisfies property (P) if and only if E does».

E.g., this trivially holds for normability, metrizability and (quasi-, sequential, local) completeness. Next, it is easily checked that such an equivalence is also true for the property that the bounded subsets are metrizable (see the proof of Theorem 6), for the existence of a fundamental sequence of bounded sets, for the *countable neighborhood property* (i.e., for every sequence $(p_n)_{n\in\mathbb{N}} \subset cs(E)$, there are $p \in cs(E)$ and positive numbers λ_n, for

$n = 1, 2, \ldots$, such that $p_n \leq C_n p$ for each n) and for the *countable boundedness property* (i.e., for every sequence $(B_n)_{n\in\mathbb{N}}$ of bounded sets in E, there are $\lambda_n > 0, n = 1, 2, \ldots$, with $\cup_{n\in\mathbb{N}} \lambda_n B_n$ bounded in E). Moreover, using the M. and S. Krein characterization of the separable $CB(X)$-spaces, one can verify directly that $CB(X, E)$ is separable (resp., seminorm separable) if and only if X is compact and metrizable and E is separable (resp., seminorm separable).

As an example of how easily our basic Lemma 2 applies to deduce another such characterization, we state:

Proposition 8. *The space $CB(X, E)$ is quasinormable if and only if E is quasinormable.*

Proof. By 1.(a), only the sufficiency needs a proof. Given any 0-neighborhood W in $CB(X, E)$, there is an absolutely convex 0-neighborhood U in E with $CB(X, U) \subset W$. By hypothesis, we find an absolutely convex 0-neighborhood V in E such that for every $\varepsilon > 0$, there is an absolutely convex bounded set B in E with $V \subset B + 2^{-1}\varepsilon U$. From Lemma 2, we quasibarrelled get

$$CB(X, V) \subset CB(X, B + 2^{-1}\varepsilon U) \subset \varepsilon CB(X, U) + CB(X, B) \subset \varepsilon W + CB(X, B),$$

which completes the proof. ∎

In the rest of this section, we add some remarks on quasibarrelled and barrelled spaces of type $CB(X, E)$, where E is an arbitrary locally convex space (i.e., not necessarily a (DF)-space).

Proposition 9. *We assume that every $f \in CB(X, E)$ has precompact range in E. If X is infinite, then $CB(X, E)$ is quasibarrelled if and only if E is quasibarrelled and its strong dual E'_b satisfies property (B) of Pietsch [15].*

Proof. By Lemma 7 (and [14], 8.3.24), $CB(X, E)$ is quasibarrelled if and only if the tensor product $CB(X) \otimes_\varepsilon E$ is quasibarrelled. The conclusion quasibarrelled follows from [14], 11.5.10. ∎

Note that if the space $CB(X, E)$ is quasibarrelled, then $CB(K, E) = C(K, E)$ must also be quasibarrelled for every compact $K \subset X$. Indeed, if K is an arbitrary compact subset of X, the restriction $f \to f_{|K}$ defines a linear map $R_K : CB(X, E) \to C(K, E)$ which is open and has dense range (cf. [10]).

Let us quasibarrelled turn to the barrelled $CB(X, E)$-spaces. We first recall that, according to a result of Mendoza (cf. [18], IV.7.9), for any compact and infinite space $X = K$, the space $CB(X, E) = C(K, E)$ is barrelled if and only if E is barrelled and E'_b has Pietsch's property (B) (see [15]). There are essentially two ways to obtain results on the barrelledness of $CB(X,E)$: On one hand, observe that $CB(X,E)$ is locally complete for every locally complete space E and that every locally complete quasibarrelled space is barrelled. Hence Theorem 6 and Proposition 9 have obvious consequences for barrelled spaces of type $CB(X,E)$ if E is, in addition, assumed to be locally complete. On the other hand, one can also generalize a trick of Defant-Govaerts [11] to obtain a partial result without any completeness assumption.

Lemma 10. *If $(F_n)_{n\in\mathbb{N}}$ is an increasing closed cover of X, then for every barrel T in $CB(X,E)$ and every bounded subset B of E, there exists a natural number m_0 such that the set $\{f \in CB(X,E); f \equiv 0$ on F_{m_0} and $f(X) \subset B\}$ is absorbed by T.*

Proof. If this is not the case, then for every $n \in \mathbb{N}$, there is $f_n \in CB(X,E)$ with $f_n \equiv 0$ on F_n and $f_n(X) \subset B$, but $f_n \notin 2^{2n}T$. quasibarrelled the series $\sum_{n=1}^{\infty} c_n f_n$ is Cauchy in $CB(X,E)$ for every $c = (c_n)_{n\in\mathbb{N}}$ in the unit ball of l_1, and $\sum_{n=1}^{\infty} c_n f_n(x)$ reduces to a finite sum for every $x \in X$. At this point, clearly, $(2^{-n}f_n)_n$ must be a null sequence in the space $CB(X,E)$, and the set $C := \{\sum_{n=1}^{\infty} 2^{-n}c_n f_n; c \in l_1$ with $\|c\|_1 \leq 1\}$ is absolutely convex and compact in $CB(X,E)$. Therefore, C is absorbed by T. As a consequence, there is $r > 0$ with $C \subset rT$ and thus $2^{-n}f_n \in rT$ for each $n \in \mathbb{N}$. This is the desired contradiction. ∎

Proposition 11. *Let X be a metrizable locally compact and σ-compact space. Then $CB(X,E)$ is barrelled if and only if $CB(X,E)$ is quasibarrelled and E is barrelled.*

Proof. If X is compact, the proposition is a consequence of the results of Mendoza (quoted above). If X is not compact, we assume that E is barrelled and $CB(X,E)$ is quasibarrelled, and we fix a barrel T in $CB(X,E)$. It is enough to prove that T must be bornivorous. Thus, we also fix an absolutely convex bounded subset B of E, and we want to show that T absorbs $CB(X,B)$.

By hypothesis on X, there is an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X with $K_n \subset K_{n+1}^0$ for every n and $X = \cup_{n\in\mathbb{N}} K_n$. By Lemma 10, we can pick $n_0 \in \mathbb{N}$ such that T absorbs $\{f \in CB(X,E); f \equiv 0$ on K_{n_0} and $f(X) \subset B\}$.

At this point, let us denote by ρ the restriction map from X to K_{n_0}. Since K_{n_0} is a closed subset of the metrizable space X, the Arens-Borsuk-Dugundji theorem yields a continuous linear extension map $A : C(K_{n_0}, E) \to C(X,E)$ with $(Af)(X) \subset \Gamma(f(K_{n_0}))$ for every $f \in C(K_{n_0}, E)$. From $\rho \circ A = \mathrm{id}$, we obtain $CB(X,E) = A(C(K_{n_0}, E)) \oplus \ker \rho$ algebraically and topologically. quasibarrelled $F := A(C(K_{n_0}, E))$ is a (complemented) subspace of $CB(X,E)$ isomorphic to $C(K_{n_0}, E)$. But the last space is barrelled by the result of Mendoza; therefore the barrel $T \cap F \subset F$ is a 0-neighborhood in F and hence absorbs the bounded set $(A \circ \rho)(CB(X,B))$. On the other hand, we note that for every $f \in CB(X,B)$, the function $(A \circ \rho)(f)$ takes its values in the set $\Gamma(f(K_{n_0})) \subset B$ so that

$$(\mathrm{id} - A \circ \rho)(CB(X,B)) \subset 2\{f \in CB(X,E); f \equiv 0 \text{ on } K_{n_0} \text{ and } f(X) \subset B\}.$$

It follows that T absorbs $(\mathrm{id} - A \circ \rho)(CB(X,B))$, and since we have seen before that T absorbs $(A \circ \rho)(CB(X,B))$, too, the conclusion is immediate. ∎

From Remark 1.(b) and from our considerations in Section 2, it should be clear that the space $l_\infty(E)$ is very important in the study of the locally convex properties of $CB(X,E)$. Similarly, it is quite obvious that the space $c_0(E)$ is essential for the study of $C(X,E)$ with the compact-open topology. We close this section with a remark on the interplay between the quasibarrelledness of the spaces $l_\infty(E)$ and $c_0(E)$.

Proposition 12. *If $l_\infty(E)$ is quasibarrelled, then $c_0(E)$ must be quasibarrelled, too, but the converse does not hold.*

Proof. $I_k(x) := (x_1, \ldots, x_k, 0, 0, \ldots)$ for every $x = (x_n)_{n\in\mathbf{N}} \in l_\infty(E)$ defines a continuous linear mapping $I_k : l_\infty(E) \to c_0(E)$ for every $k \in \mathbf{N}$. Fix a bornivorous barrel T in $c_0(E)$. Let us first show that $\tilde{T} := \{x \in l_\infty(E); I_k x \in T \text{ for each } k \in \mathbf{N}\}$ is a bornivorous barrel in $l_\infty(E)$.

Of course, $\tilde{T} = \bigcap_{k\in\mathbf{N}} I_k^{-1}(T)$ is a closed absolutely convex subset of $l_\infty(E)$. But $\tilde{T}$ is also *bornivorous*: Let C denote a bounded set in $l_\infty(E)$. Then there is an absolutely convex bounded set $B \subset E$ with

$$C \subset \{x = (x_n)_n \in l_\infty(E); x_n \in B \text{ for all } n \in \mathbf{N}\}.$$

We set

$$C_0 := \{x = (x_n)_n \in c_0(E); x_n \in B \text{ for all } n \in \mathbf{N}\}.$$

Since C_0 is a bounded subset of $c_0(E)$, we can find $r > 0$ with $C_0 \subset rT$, and it easily follows that $C \subset r\tilde{T}$.

From the hypothesis we quasibarrelled conclude that $\tilde{T}$ is a 0-neighborhood in $l_\infty(E)$, and hence there exists a 0-neighborhood U in E with

$$W := \{x = (x_n)_n \in l_\infty(E); x_n \in U \text{ for all } n \in \mathbf{N}\} \subset \tilde{T}.$$

At this point, it suffices to observe that

$$W_0 := \{x = (x_n)_n \in c_0(E); x_n \in U \text{ for all } n \in \mathbf{N}\}$$

is contained in T.

On the other hand, the examples in [3] and [4] (cf. also the remark after the end of the proof of Theorem 6 above) serve to show that the converse implication is false. ∎

4. APPLICATIONS TO WEIGHTED INDUCTIVE LIMITS OF SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

As it was mentioned in the introduction, one of the original aims of our investigations was to extend the results in [4], Section 2 from co-echelon sequence spaces to weighted inductive limits of spaces of continuous (DF)-valued functions and to their «projective hulls». To this purpose, Theorems 5 and 6 were needed.

In the present section, we fix a decreasing sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions on X. $\overline{V} = \overline{V}(\mathcal{V})$ denotes the *maximal Nachbin family* associated with $\mathcal{V}$; viz.,

$$\overline{V} := \{\overline{v} : X \to \mathbb{R}_+ \text{ upper semicontinuous; for each } n \in \mathbb{N}, \sup_{x\in X} \frac{\overline{v}(x)}{v_n(x)} < \infty\}$$

$= \{\overline{v} \geq 0$ upper semicontinuous; there are $\alpha_n > 0, n = 1, 2, \ldots,$ with $\overline{v} \leq \inf_n \alpha_n v_n$ on $X\}$.

For each $n \in \mathbb{N}$, we first put

$$Cv_n(X,E) := \{f \in C(X,E); (v_n f)(X) \text{ bounded in } E\},$$

$$C(v_n)_0(X,E) := \{f \in C(X,E); \text{ for each } p \in cs(E), p\circ(v_n f) \text{ vanishes at infinity on } X\},$$

both endowed with the locally convex topology generated by the system of seminorms

$$q_{v_n,p}(f) := \sup_{x\in X} v_n(x)p(f(x)), \quad f \in Cv_n(X,E) \text{ and } p \in cs(E).$$

Then the *weighted inductive limits* are defined by

$$\mathcal{V}C(X,E) := \operatorname*{ind}_n Cv_n(X,E) \text{ and } \mathcal{V}_0C(X,E) := \operatorname*{ind}_n C(v_n)_0(X,E).$$

Finally, the «*projective hulls*» of these inductive limits are the spaces

$$C\overline{V}(X,E) := \{f \in C(X,E); \text{ for each } \overline{v} \in \overline{V}, (\overline{v}f)(X) \text{ is bounded in } E\},$$

$$C\overline{V}_0(X,E) := \{f \in C(X,E); \text{ for each } \overline{v} \in \overline{V} \text{ and each } p \in cs(E), \text{ the function } p\circ(\overline{v}f) \text{ vanishes at infinity on } X\}$$

with the weighted topology given by all the seminorms $q_{\overline{v},p}$ for $\overline{v} \in \overline{V}$ and $p \in cs(E)$.

If E is the field of real or complex scalars, we again omit it from the notation and write $Cv_n(X), \mathcal{V}C(X), C\overline{V}(X)$ etc.

Clearly, each space $Cv_n(X,E)$, $n \in \mathbf{N}$, is canonically isomorphic to $CB(X,E)$. Hence our topological results in Sections 2 and 3 have direct corollaries (which we will not state here) for $Cv_n(X,E)$ as well as for $\mathcal{V}C(X,E) = \operatorname{ind}_n Cv_n(X,E)$. (For more consequences, mainly of Theorem 5, in this context compare already [6], Sections 1 and 3, where, quite often, the hypothesis was made that each $Cv_n(X,E)$ is a (DF)-space.) We directly turn to the spaces of type $C\overline{V}(X,E)$. According to [9], 1.6 and 1.10, if X is normal or if every weight $\overline{v} \in \overline{V}$ is dominated by some $\tilde{v} \in \overline{V} \cap C(X)$ (which happens, for instance, if X is locally compact and σ-compact), then $C\overline{V}(X,E)$ is a (DF)-space for every normed space E. quasibarrelled we extend the second case of this result to more general range spaces E by adapting the proof of [4], Proposition 2.3.(a).

Theorem 13. *If every $\overline{v} \in \overline{V}$ is dominated by a continuous weight $\tilde{v} \in \overline{V}$, then $C\overline{V}(X,E)$ is a (DF)-space (resp., a (gDF)-space) if and only if E is a (DF)-space (resp., a (gDF)-space).*

Proof. This time, it takes a moment's thought to check that E is isomorphic to a complemented subspace of $C\overline{V}(X,E)$. (Utilize $\mathcal{V} \subset C(X)$ and $\overline{V} > 0$). But then one direction of the equivalence is clear.

To show the converse, we assume first that E is a (DF)-space. By [8], Theorem 8, $C\overline{V}(X,E)$ coincides with $\mathcal{V}C(X,E)$ algebraically, the two spaces have the same bounded sets, and the inductive limit $\mathcal{V}C(X,E) = \operatorname{ind}_n Cv_n(X,E)$ is regular. Hence $C\overline{V}(X,E)$ clearly has a fundamental sequence of bounded sets. Thus, we fix a sequence $(W_k)_{k\in\mathbf{N}}$ of closed absolutely convex 0-neighborhood in $C\overline{V}(X,E)$ such that $W := \cap_{k\in\mathbf{N}} W_k$ is bornivorous and must then prove that W is a 0-neighborhood.

Since, obviously, $Cv_n(X,E)$ is isomorphic to $CB(X,E)$, we may apply Theorem 5 to conclude that $Cv_n(X,E)$ is a (DF)-space. Utilizing this fact, we can proceed as in [4], proof of Proposition 2.3.(a) to obtain $\lambda_k > 0, \mu_k > 0, \overline{w}_k \in \overline{V} \cap C(X), k = 1,2,\ldots,$ and $p \in cs(E)$ such that

$$(1) \quad \{f \in C\overline{V}(X,E); \sup_{x\in X} \mu_k \overline{w}_k(x) p(f(x)) \leq 1\} \subset 2^{-2} W_k,$$

$$(2) \quad \{f \in Cv_n(X,E); \sup_{x\in X} \lambda_n v_n(x) p(f(x)) \leq 1\} \subset 2^{-(n+3)} W.$$

At this point, setting

$$\overline{v}_k := \inf\{\mu_k \overline{w}_k, \lambda_1 v_1, \ldots, \lambda_k v_k\} \in \overline{V} \cap C(X) \text{ and}$$
$$U_k := \{f \in C\overline{V}(X,E); \sup_{x\in X} \overline{v}_k(x) p(f(x)) \leq 1\},$$

we *claim* that $U_k \subset W_k$, where $k \in \mathbf{N}$ is fixed.

To prepare the proof of this claim, we also fix $f \in U_k$ and note that

$$F_0 := \{x \in X; \mu_k \overline{w}_k(x) p(f(x)) \leq 2^{-2}\} \text{ and}$$
$$F_1 := \{x \in X; \mu_k \overline{w}_k(x) p(f(x)) \geq 2^{-1}\}$$

are disjoint zero zets of continuous functions on X; hence there is $\psi \in CB(X,[0,1])$ with $\psi_{|F_0} \equiv 0$ and $\psi_{|F_1} \equiv 1$. Next, we let $\Omega := X \setminus \overline{w}_k^{-1}(0)$ and introduce

$$Z_1^0 := \{x \in \Omega; \mu_k \overline{w}_k(x) \leq 3\overline{v}_k(x)/2\} \text{ and}$$
$$Z_1^j := \{x \in \Omega; \lambda_j v_j(x) \leq 3\overline{v}_k(x)/2\} \text{ for } j = 1, \dots, k,$$

as well as

$$Y_0 := \{x \in \Omega; \mu_k \overline{w}_k(x) < 2\overline{v}_k(x)\} \text{ and}$$
$$Y_j := \{x \in \Omega; \lambda_j v_j(x) < 2\overline{v}_k(x)\} \text{ for } j = 1, \dots, k$$

and finally $Z_2^j := \Omega \setminus Y_j$ for $j = 0, 1, \dots, k$. By definition, Z_1^j and Z_2^j $(j = 0, \dots, k)$ are zero sets of continuous functions on Ω with $\Omega = \cup_{j=0}^k Z_1^j$ and $Z_1^j \cap Z_2^j = \emptyset$ for each j. Therefore (see the proofs of Proposition 5.8 and of Corollary 6.6.(1) in [7]), we can construct $\varphi_j \in C(\Omega), j = 0, \dots, k$, with $0 \leq \varphi_j \leq 1, \varphi_{j|\Omega \setminus Y_j} \equiv 0$ and $\sum_{j=0}^k \varphi_j \equiv 1$ on Ω. For $j = 0, 1, \dots, k$, we quasibarrelled define ψ_j by $\psi_j(x) := \psi(x)\varphi_j(x)$ for $x \in \Omega$ and $\psi_j(x) := 0$ for $x \in X \setminus \Omega = \overline{w}_k^{-1}(0)$. As F_0 is a neighborhood of $\overline{w}_k^{-1}(0)$ in X, each ψ_j belongs to $C(X)$, and we have $\psi = \sum_{j=0}^k \psi_j$ on X.

At this point, it is a direct matter (proceeding as in the proof of [4], Proposition 2.3.(a)) to verify that by (1), $\psi_0 f$ (resp., $(1-\psi)f$) is an element of $2^{-1}W_k$ (resp., of $2^{-3}W_k$) and that $\psi_j f$ belongs to $Cv_j(X,E)$ and, because of (2), satisfies $\psi_j f \in 2^{-(j+2)}W \subset 2^{-(j+2)}W_k$ for $j = 1, \dots, k$. Utilizing the decomposition

$$f = \psi_0 f + \sum_{j=1}^k \psi_j f + (1-\psi) f,$$

we end up with $f \in W_k$ which proves our claim.

Once the inclusion $U_k \subset W_k$ is established for every k, we define $\overline{v} := \sup_{k \in \mathbf{N}} \overline{v}_k$ and check that $\overline{v} \in \overline{V}$. quasibarrelled,

$$U := \{f \in C\overline{V}(X,E); \sup_{x \in X} \overline{v}(x) p(f(x)) \leq 1\}$$

is a 0-neighborhood in $C\overline{V}(X,E)$. But we clearly have $U \subset U_k$ for each k, and hence the inclusion $U \subset \bigcap_{k\in\mathbb{N}} U_k \subset \bigcap_{k\in\mathbb{N}} W_k = W$ holds, and the proof for the (DF)-case is complete.

In the case of a (gDF)-space E, one can proceed in a similar way. ∎

Utilizing the canonical topological isomorphism of $C\overline{V}(X) \otimes_\varepsilon E$ with a subspace of $C\overline{V}(X,E)$ (cf. [2]) and repeating the proof of 13, except that the use of Theorem 5 is quasibarrelled replaced by an application of the result (e.g., see [14], 11.5.11) that $CB(X) \otimes_\varepsilon E$ has the (DF)-property (resp., the (gDF)-property) for each (DF)- (resp., (gDF)-) space E, we can also deduce:

Corollary 14. *If each $\overline{v} \in \overline{V}$ is dominated by some $\tilde{v} \in \overline{V} \cap C(X)$, then $C\overline{V}(X) \otimes_\varepsilon E$ is a (DF)-space (resp., a (gDF)-space) if and only if E is a (DF)-space (resp., a (gDF)-space).*

For the purpose of comparison, let us recall the following result from [8]:

If X is locally compact, then E (DF) (resp., (gDF)) implies that each of the spaces $\mathcal{V}_0C(X) \otimes_\varepsilon E, C\overline{V}_0(X) \otimes_\varepsilon E, \mathcal{V}_0C(X,E)$ and $C\overline{V}_0(X,E)$ has the same property.

We finish by analyzing the quasibarrelledness of the (DF)-spaces $C\overline{V}(X,E)$ and $C\overline{V}(X) \otimes_\varepsilon E$ in case X is locally compact and σ-compact, fixing an increasing sequence $\mathcal{K} = (K_m)_{m\in\mathbb{N}}$ of compact subsets of X with $\mathcal{K}_m \subset K^0_{m+1}$ for each m and $X = \bigcup_{m\in\mathbb{N}} K_m$. It would certainly be possible to formulate some partial results in a more general setting, but at this moment we present a full characterization under the above hypothesis on X.

Let us recall the following definition (cf. [7] or [5]).

Definition 15. The sequence $\mathcal{V} = (v_n)_n$ is said to satisfy *condition (D)* if there as an increasing sequence $J = (X_m)_{m\in\mathbb{N}}$ of subsets of X such that

(N,J) for each $m \in \mathbb{N}$, there is $n_m \geq m$ with $\inf_{x\in X_m} v_k(x)/v_{n_m}(x) > 0$ for $k = n_m+1, n_m+2, \ldots$ and

(M,J) for each $n \in \mathbb{N}$ and each subset Y of X with $Y \cap (X\setminus X_m) \neq \emptyset$ for all $m \in \mathbb{N}$, there exists $n' = n'(n,Y) > n$ with $\inf_{y\in Y} v'_n(y)/v_n(y) = 0$.

Obviously, condition $(N,\mathcal{K})$ is always satisfied, and hence condition (D) for the sequence $\mathcal{K} = (K_m)_{m\in\mathbb{N}}$ amounts to

$(M,\mathcal{K})$ for each $n \in \mathbb{N}$ and each subset Y of X which is not relatively compact, there exists $m > n$ with $\inf_{y\in Y} v_m(y)/v_n(y) = 0$.

Theorem 16. *Let X denote a locally compact and σ-compact space (with an increasing sequence $\mathcal{K} = (K_m)_{m\in\mathbb{N}}$ of compact subsets such that we have $K_m \subset K^0_{m+1}$ for each m and $X = \bigcup_{m\in\mathbb{N}} K_m$) and E a (DF)-space. Then $C\overline{V}(X,E)$ is quasibarrelled if and only if*

(i) E is quasibarrelled and $\mathcal{V} = (v_n)_n$ satisfies condition $(M,\mathcal{K})$ for the sequence $\mathcal{K} = (K_m)_m$, or

(ii) the bounded subsets of E are metrizable and $\mathcal{V} = (v_n)_n$ satisfies condition (D).

Proof. 1. Sufficiency. We first assume that (i) is satisfied. Condition $(M,\mathcal{K})$ clearly implies that $C\overline{V}(X,E)$ and $C\overline{V}_0(X,E)$ coincide algebraically (and topologically), cf. [7], Proposition 5.3. But from [8], we obtain that $C\overline{V}_0(X,E)$ is quasibarrelled for every quasibarrelled (DF)-space E.

Suppose quasibarrelled that (ii) is satisfied. Without loss of generality, we may assume that each X_m is closed in X (just replace $J = (X_m)_m$ by $(\overline{X_m})_m$ and use the continuity of the functions v_n) and that any function $f : X \to \mathbb{R}_+$ with $f_{|X_m}$ continuous for all $m \in \mathbb{N}$ must already be continuous on X (replace $(X_m)_{m\in\mathbb{N}}$ by $(X_m \cup K_m)_{m\in\mathbb{N}}$, if necessary). Then, by [5], Theorem 4.6, $C\overline{V}(X,E) = \mathcal{V}C(X,E)$ holds topologically. Since every «step» $Cv_n(X,E)$ of $\mathcal{V}C(X,E)$ is isomorphic to $CB(X,E)$, we quasibarrelled apply Theorem 6 to obtain the quasibarrelledness of $C\overline{V}(X,E)$.

2. Conversely, if $C\overline{V}(X,E)$ is quasibarrelled, then both $C\overline{V}(X)$ and E must be quasibarrelled since they are isomorphic to complemented subspaces of $C\overline{V}(X,E)$. By [5], Corollary 4.10, $C\overline{V}(X)$ must then even be bornological, and we apply Bastin [1] to conclude that $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ satisfies condition (D) for some increasing sequence $J = (X_m)_{m\in\mathbb{N}}$ of (closed) subsets of X, where we can take $n_m = m$ in condition (N,J) for $m = 1,2,\ldots$. From [5], Theorem 5.7, it quasibarrelled follows that the bounded subsets of $C\overline{V}(X)$ are metrizable, and a look at the proof of Proposition 5.5.(a) in [5] shows that we may assume without loss of generality that the sequence $X = (X_m)_{m\in\mathbb{N}}$ with $X = \cup_{m\in\mathbb{N}} X_m$ satisfies $X_m \subset X^0_{m+1}$ for each $m \in \mathbb{N}$.

If X_m is compact for each m, $(X_m)_m$ forms a basis of the compact sets in X, and thus (M,J) implies $(M,\mathcal{K})$, whence (i) holds. It remains to consider the case that from some index $m \geq 2$ on, the sets X_{m-1} fail to be compact, and we fix m for the rest of the proof. Renumbering the sequence $(K_k)_{k\in\mathbb{N}}$ if necessary (and proceeding by induction), we can then find points $y_i \in X_{m-1} \cap (K^0_{i+1} \setminus K_i)$, and since $X_{m-1} \subset X^0_m$, we can also choose functions $\varphi_i \in CB(X,[0,1])$ with $\varphi_i(y_i) = 1$ and

$$\text{supp } \varphi_i \subset X_m \cap (K^0_{i+1} \setminus K_i) \cap \{x \in X; v_m(x) < 2v_m(y_i)\}, \quad i = 1,2,\ldots$$

At this point, we define $w : \mathbb{N} \to \mathbb{R}_+ \setminus \{0\}$ by $w(i) := v_m(y_i)$ for each $i \in \mathbb{N}$ and let

$$l_\infty(w,E) := \{e = (e_i)_{i\in\mathbb{N}} \in E^{\mathbb{N}}; (w(i)e_i)_i \text{ is bounded in } E\},$$

endowed with the topology given by the seminorms $q_p(e) := \sup_{i\in\mathbb{N}} w(i)p(e_i)$ for each $e = (e_i)_{i\in\mathbb{N}}$, where $p \in cs(E)$. We next define the mapping $P : C\overline{V}(X,E) \to l_\infty(w,E)$

by $[P(f)](i) := f(y_i)$ for all $i \in \mathbb{N}$ and $f \in C\overline{V}(X,E)$; P is indeed a well-defined, linear and continuous mapping since it is possible to choose $\overline{v} \in \overline{V}$ with $v_m \leq \overline{v}$ on X_m (cf. [7], Remark 3.9), and hence for each $p \in cs(E)$ and $f \in C\overline{V}(X,E)$,

$$q_p(P(f)) = \sup_{i \in I} w(i) p([P(f)](i)) =$$
$$= \sup_{i \in I} v_m(y_i) p(f(y_i)) \leq \sup_{i \in I} \overline{v}(y_i) p(f(y_i)) \leq q_{\overline{v},p}(f).$$

We also let $Q : l_\infty(w,E) \to C\overline{V}(X,E), [Q(e)](x) := \sum_{i\in\mathbb{N}} \varphi_i(x) e_i$ for an arbitrary $e = (e_i)_{i\in\mathbb{N}} \in l_\infty(w,E)$. In fact, by definition the functions φ_i have supports $\subset K_{i+1}^0 \setminus K_i, i = 1,2,\ldots$, and hence $Q(e)$ clearly belongs to $C(X,E)$ for every $e \in l_\infty(w,E)$. We *claim* that Q takes its values in $C\overline{V}(X,E)$ and is a continuous linear mapping of $l_\infty(w,E)$ into $C\overline{V}(X,E)$.

To establish this claim, fix $\overline{v} \in \overline{V}$ and $p \in cs(E)$. There is $C > 0$ with $\overline{v} \leq C v_m$. For $e = (e_i)_{i\in\mathbb{N}} \in l_\infty(w,E)$ and any $x \in X, \overline{v}(x)p([Q(e)](x))$ is 0 or coincides with $\overline{v}(x)\varphi_i(x)p(e_i)$ for some $i \in \mathbb{N}$, where $x \in \operatorname{supp} \varphi_i$. In this second case we get

$$\overline{v}(x)p([Q(e)](x)) \leq C v_m(x) p(e_i) \leq 2 C v_m(y_i) p(e_i) = 2 C w(i) p(e_i),$$

whence $q_{\overline{v},p}(Q(e)) \leq 2 C q_p(e)$.

But observe that $P \circ Q$ is the identity on $l_\infty(w,E)$ and that the last space is isomorphic to $l_\infty(E)$. Therefore $l_\infty(E)$ is isomorphic to a complemented subspace of $C\overline{V}(X,E)$, and hence must be quasibarrelled, too. From [4], Theorem 1.5.(a), we now conclude that the bounded subsets of E are metrizable; i.e., (ii) holds. ∎

Theorem 16 should be compared with the following obvious consequence of [5], Corollary 4.7, Bastin [1] and [14], 11.5.10.

Proposition 17. *Let X be a locally compact and σ-compact space and E a (DF)-space. Then $C\overline{V}(X) \otimes_\varepsilon E$ is quasibarrelled if and only if $\mathcal{V}$ satisfies condition (D) and E is quasibarrelled.*

Finally, we note that it is possible to extend Theorem 16 to the case of a locally compact and paracompact space X (since each such space is the topological sum of locally compact and σ-compact spaces). We leave the details to the reader.

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Added in proof: Under some (mild) technical assumptions, F. Bastin recently characterized when a weighted space $CV(X)$ with a fundamental sequence of bounded subsets has the (DF)-resp. (gDF)-property and deduced that $CV(X,E)$ is a (DF)-resp. (gDF)-space if and only if both $CV(X)$ and E are (DF)-resp. (gDF)-spaces. In this way, she was able to prove some of the results in our Section 4 in a more general context. See F. Bastin, *Weighted spaces of continuous functions*, Bull. Soc. Roy. Sci. Liège **59** (1990), 3-82.

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