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THE EXISTENCE OF GENERALIZED SOLUTIONS FOR A CLASS OF LINEAR AND NONLINEAR EQUATIONS OF MIXED TYPE

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *In this paper we deal with the question of existence and uniqueness of the generalized solutions for a class of linear and nonlinear equations of the mixed type. In particular we consider* $L(u) \equiv k(y)u_{xx} + u_{yy} = f(x,y,u)$ *in a simply connected region* G*, where* $k(y) \geq 0$ *for* $y \geq 0$ *and* $k(y) < 0$ *for* $y < 0$. G *is bounded by the curves* $\Gamma_0, \Gamma_1, \Gamma_2$. Γ_0 *is a piecewise smooth curve lying in the half plane* $y > 0$ *which intersects the line* $y = 0$ *at the points* $P(-1,0)$, $Q(0,0)$; Γ_1 *is a piecewise smooth curve through* P *in* $y < 0$ *which meets the characteristics of the above operators issued from* Q *at the point* R *and* Γ_2 *consists of the portion* RQ *of the characteristic through* Q. *We assume that* Γ_1 *either lies in the characteristic triangle formed by the characteristics through* P *and* Q *(Frankl Problem) or coincides with the characteristics through* P *(Tricomi Problem).*

We seek sufficient conditions for the existence and uniqueness of generalized solutions of the boundary problem

$$L(u) = f(x,y,u) \text{ in } G, u_{\Gamma_0 \cup \Gamma_1} = 0.$$

1. INTRODUCTION

We consider the operator

$$L[u] = k(y)u_{xx} + u_y + r(x,y)u = f(x,y,u) \tag{1.1}$$

in a simply connected region G, where $k(y) \geq 0$ for $y \geq 0$ and $k(y) < 0$ for $y < 0$. G is bounded by the curves Γ_0, Γ_1 and $\Gamma_0\Gamma_2$. Γ_0 is a piecewise smooth curve lying in the half-plane $y > 0$ which intersects the line $y = 0$ in the points $P(-1,0)$ and $Q(0,0)$. Γ_1 is a piecewise smooth curve through P in $y < 0$ which meets the characteristic of (1.1) issued from Q at the point R and Γ_2 consists of the portion RQ of the characteristic through Q. Γ_1 either lies in the characteristic triangle formed by the characteristics through P and Q (Frankl Problem) or coincides with the characteristic through P (Tricomi Problem). We seek sufficient conditions for the existence and uniqueness of generalized solutions of the problem

$$L[u] = f(x,y,u) \text{ in } G, u|_{\Gamma_0 \cup \Gamma_1} = 0. \tag{1.2}$$

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Remark 1.3. We restrict our attention to equation (1.1). The same method applies equally well to more general equations of the form

$$L[u] = k(y)u_{xx} + u_{yy} + [a(x,y)u]_x + [b(x,y)u]_y + c(x,y,u)u = f.$$

To begin with we consider the linear Tricomi and Frankl problems (i.e. $f(x,y,u) = f(x,y)$ in (1.1)). Using Lemma 3.1 we obtain the a priori estimate (4.2) in Theorem 4.1.

It should be mentioned that a priori estimates in weighted Sobolev spaces were used earlier by V. P. Didenko in [4] and [5] to prove existence theorems for the Tricomi problem in case $k(y) = y, r(x,y) = 0$ and for a special region. The statement of Lemma 3.1 was used by A. G. Podgaev in [8] and later by Aziz and Schneider in [2] to prove the existence of solutions to non linear equations of mixed type by an approximation procedure of Galerkin type.

From the a priori estimate (4.2) we derive the existence of generalized solutions for the Tricomi and Frankl problems in weighted Sobolev space $H_1(\mathrm{d}, k)$ (see (2.6)). In general these solutions are not uniquely determined; but, by the method used in the proof of Theorem 5.1, we select one generalized solution u_0 in $H_1(b\mathrm{d}, k)$, where $\| u_0 \|_{H_1(b\mathrm{d},k)} \leq \frac{1}{C1} \| f_1 \|_0$ out of all possible solutions.

We thus have the uniqueness of the solutions of both problems with respect to the constructive method used in Theorem 5.1.

In the case of the Tricomi problem we have in addition to (4.2) (under additional assumptions on the coefficient $r(x,y)$ and the domain) the a priori estimate (4.8) and thus we not only obtain the existence but also the uniqueness of the solution in the space $H_1(b\mathrm{d}, k)$.

For the nonlinear Tricomi and Frankl problems we then use in connection with Theorem 5.1 and Theorem 5.8 a «linearization technique» to establish (using Schauder's fixed point theorem) the existence of solutions in $H_1(b\mathrm{d}, k)$. (For the uniqueness of these solutions see Lemmas 6.10 and 6.14).

2. NOTATIONS

In this paper we use tha same notations as in [2]. Moreover in the sequel we shall make use of some of the results (e.g., Lemma 3.1) in this paper.

If we use Pfaffian forms ([6], [9]) and introduce the operator

$$\mathrm{d}_n u = k(y)u_x \mathrm{d}y - u_y \mathrm{d}x,$$

equation (1.1) may be written as

$$L[u] = [\mathrm{d}, \mathrm{d}_n u] + r(x,y)u[\mathrm{d}x, \mathrm{d}y] = f(x,y,u)[\mathrm{d}x, \mathrm{d}y], \tag{2.1}$$

with the boundary condition

$$u|_{\Gamma_0 \cup \Gamma_1} = 0.$$

The adjoint boundary conditions ([9], pp. 248) are

$$v|_{\Gamma_0\cup\Gamma_1} = 0 \quad \text{for the Tricomi problem}$$
$$v|_{\Gamma_0\cup\Gamma_1\cup\Gamma_2} = 0 \quad \text{for the Frankl problem}. \tag{2.2}$$

We introduce the function spaces

$$U = \left\{u|u(x,y) \in C^\infty(\overline{G}),\ u|_{\Gamma_0\cup\Gamma_1} = 0\right\},$$
$$V_T = \left\{v|v(x,y) \in C^\infty(\overline{G}),\ v|_{\Gamma_0\cup\Gamma_2} = 0\right\}, \tag{2.3}$$
$$V_F = \left\{v|v(x,y) \in C^\infty(\overline{G}),\ v|_{\Gamma_0\cup\Gamma_1\cup\Gamma_2} = 0\right\}.$$

For $u \in U$ and $v \in V_T$ (or V_F), a formal application of Green's theorem to (2.1) yields

$$\begin{aligned} \mathcal{B}[u,v] &:= \int\int_G vL[u]\mathrm{d}x\mathrm{d}y \\ &= \int_{\partial G} v\mathrm{d}_n u - \int\int_G \left\{k(y)u_x v_x + u_y v_y - ruv\right\}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.4}$$

From (2.3) we conclude

$$\int_{\partial G} v\mathrm{d}_n u = 0 \quad \text{for} \quad u \in U \quad \text{and} \quad v \in V_T \text{ or } V_F.$$

Thus for the problem (2.1) we formally have the identity

$$\mathcal{B}[u,v] = -\int\int_G \left\{k(y)u_x v_x + u_y v_y - ruv\right\}\mathrm{d}x\mathrm{d}y = \int\int_G f(x,y,u)v\mathrm{d}x\mathrm{d}y$$

for all $u \in U$, $v \in V_T$ or $v \in V_F$.

Now we introduce the spaces $H_1(bd,k)$ and $H_1(bd^*,k)$, which are obtained by the completion of the function spaces (2.3) with respect to a weighted norm involving the function $k(y)$. More precisely we denote

(2.6)

$$H_1(bd,k) := \overline{\{u|u \in U\}} \text{ with } \|u\|_{1,k} := \int\int_G \left\{|k|(u_x)^2 + (u_y)^2 + u^2\right\}\mathrm{d}x\mathrm{d}y^{1/2},$$

$$H_1(bd^*,k) := \overline{\{v|v \in V_T\}} \text{ with } \|v\|_{1,k} := \int\int_G \left\{|k|(v_x)^2 + (v_y)^2 + v^2\right\}\mathrm{d}x\mathrm{d}y^{1/2}.$$

We note that for the Tricomi problem $H_1(bd^*,k)$ is the completion of the space V_T while for the Frankl problem it is the completion of V_F. However, in the sequel we call the space $H_1(bd^*,k)$ the space adjoint to the space $H_1(bd,k)$.

Definition 2.7. *A function $u \in H_1(\text{bd}, k)$ is called a generalized solution of (1.2) if*

$$\mathcal{B}[u, v] = \int\int_G f(x, y, u) v \, dx \, dy \quad \text{for all} \quad v \in H_1(\text{bd}^*, k). \tag{2.8}$$

Using Holder's inequality we have

Lemma 2.9. *If $k(y), r(x, y) \in C^0(\overline{G})$, then there exists a constant $c_0 > 0$ such that*

$$|\mathcal{B}[u, v]| \leq c_0 \| u \|_{H_1(\text{bd}, k)} \| v \|_{H_1(\text{bd}^*, k)} \quad \text{for all} \quad u \in H_1(\text{bd}, k), v \in H_1(\text{bd}^*, k).$$

Now we introduce the negative spaces which are adjoint to the corresponding positive spaces $H_1 := H_1(\text{bd}, k), H_1^* := H_1(\text{bd}^*, k)$,

$$\begin{aligned} H_{-1}(\text{bd}, k) &= \overline{\{w | w \in L^2(G)\}}, \quad \| w \|_{H_1} = \sup_{\substack{u \in H_1 \\ u \neq 0}} \frac{|(w, u)_0|}{\| u \|_{H_1}} \\ H_{-1}(\text{bd}^*, k) &= \overline{\{w | w \in L^2(G)\}}, \quad \| w \|_{H_1^*} = \sup_{\substack{v \in H_1^* \\ v \neq 0}} \frac{|(w, v)_0|}{\| v \|_{H_1^*}} \end{aligned} \tag{2.10}$$

From Lemma 2.9 we have

$$\| L[u] \|_{H_{-1}(\text{bd}^*, k)} = \sup_{\substack{v \in H_1^* \\ v \neq 0}} \frac{|\mathcal{B}[u, v]|}{\| v \|_{H_1^*}} \leq c_0 \| u \|_{H_1(\text{bd}, k)}, \tag{2.11}$$

for all $u \in H_1(\text{bd}, k)$.

$$\| L[v] \|_{H_{-1}(\text{bd}, k)} = \sup_{\substack{u \in H_1^* \\ u \neq 0}} \frac{|\mathcal{B}[u, v]|}{\| u \|_{H_1}} \leq c_0 \| v \|_{H_1(\text{bd}^*, k)} \tag{2.12}$$

for all $v \in H_1(\text{bd}^*, k)$.

3. A FUNDAMENTAL LEMMA

Let

$$G^+ = G \cap \{y > 0\}; G- = G \cap \{y < 0\}.$$

From [2] (Lemma 3.1) or [8] we have the following result:

Lemma 3.1. *If*

(i) $k(y) \in C^0(\overline{G}) \cap C^1(G^+)$; *sign* $k(y) =$ *sign* y; $yk'(y) \geq k(y)$ *for* $y \geq 0$,

(ii) $k(y)n_1^2 + n_2^2|_{\Gamma_1} \geq 0, n_1|_{\Gamma_1} > 0; xn_1 + \alpha y n_2|_{\Gamma_0} \leq 0$, *where* (n_1, n_2) *is the inward normal vector and* $\alpha \in (1/2, 1)$,

(iii) $v \in V_T$,

then there exists a solution $u \in H_1(bd, k) \cap L_\nu(G)$ *for all* $\nu \geq 2$ *of the boundary value problem*

$$\ell(u) = \alpha^1 u_x + \alpha^2 u_y = v, u|_{\Gamma_0 \cup \Gamma_1} = 0, \tag{3.2}$$

where

$$\alpha^1 = x, \alpha^2 = \begin{cases} \alpha y & \text{for } y \geq 0 \\ 0 & \text{for } y < 0. \end{cases} \tag{3.3}$$

By a transformation of the coordinates we obtain from Lemma 3.1

Lemma 3.4. *(Tricomi Problem)*
If

(i) $k(y) \in C^0(\overline{G}) \cap C^1(G^+); k(y) \gtreqless 0; yk'(y) \geq k(y)$ *for* $y \geq 0$;

(ii) $dx + (-k)^{1/2} dy|_{\Gamma_1} = 0, (x+1)n_1 + \tilde{\alpha} y n_2|_{\Gamma_0} \leq 0$, *where* (n_1, n_2) *is inward normal vector and* $\tilde{\alpha} \in (1/2, 1)$;

(iii) $u \in U$,

then there existss a solution $v \in H_1(bd^*, k) \cap L_\nu(G)$ *for all* $\nu \geq 2$ *of the boundary value problem*

$$\tilde{\ell}(v) = \tilde{\alpha}^1 v_x + \tilde{\alpha}^2 v_y = u, v|_{\Gamma_0 \cup \Gamma_2} = 0 \tag{3.5}$$

where

$$\tilde{\alpha}^1 = (x+1), \tilde{\alpha}^2 = \begin{cases} \tilde{\alpha} y & \text{for} \geq 0 \\ 0 & \text{for } y < 0. \end{cases} \tag{3.6}$$

We observe that by the assumption (ii) of Lemma 3.1 we have a starlike condition on G^+ with respect to Q and by assumption (ii) of Lemma 3.4 a starlike condition on G^+ with respect to A.

It can be seen (in Lemma 3.4) that for the Frankl problem and a function $u \in U$ there exissts no trivial solution $v \in H_1(bd^*, k) \cap L_\nu(G)$ in $G-$ of the boundary value problem (3.5) with the boundary condition $v|_{\Gamma_1 \cup \Gamma_2} = 0$.

4. A PRIORI ESTIMATES

From Lemma 3.1 we have

Theorem 4.1. *(Tricomi or Frankl Problem)*
If the assumptions (i), (ii) of Lemma 3.1 hold and

$$r(x,y) \in C^1(\overline{G}), r(x,y)|_{\Gamma_2} \leq 0;$$

(iii)

$$[\alpha^1 r]_x + [\alpha^2 r]_y \leq 0 \ in \ G, (\alpha^1, \alpha^2 \ \ as \ in \ \ (3.3))$$

then there exists a constant $C_1 > 0$ *such that*

$$C_1 \| \, |k|^{1/2} v \, \|_{L^2(G)} \leq \| \, L[v] \, \|_{H_{-1}(bd, k)} = \sup_{\substack{u \in H_1 \\ u \neq 0}} \frac{|\mathcal{B}[u,v]|}{\| \, u \, \|_{H_1}}, \tag{4.2}$$

for all $v \in H_1(bd^*, k)$.

Proof. For fixed $v \in V_{T(F)}$, there exists by Lemma 3.1 a solution $u \in H_1(bd, k) \cap L_\nu(G), \nu \geq 2$ of the boundary value problem

$$\ell(u) = \alpha^1 u_x + \alpha_2 u_y = v, u|_{\Gamma_1 \cup \Gamma_2} = 0. \tag{4.3}$$

From (2.4) and the Green's theorem we have

$$\begin{aligned} &2\mathcal{B}[u, \ell(u)] = 2(L[u], \ell(u))_0 = \\ &= \int_{\partial G} \Big\{ \Big[\alpha^1 k(y)(u_x)^2 - \alpha^1 (u_y)^2 + 2\alpha^2 k u_x u_y + \alpha^1 r u^2 \Big] \, \mathrm{d}y + \\ &+ \Big[\alpha^2 k(y)(u_x)^2 - \alpha^2 (u_y)^2 - 2\alpha^1 u_x u_y - \alpha^2 r u^2 \Big] \, \mathrm{d}x \Big\} - \\ &- \int\int_G \Big[A(u_x)^2 + 2Bu_x u_y + C(u_y)^2 + Du^2 \Big] \, \mathrm{d}x \mathrm{d}y, \end{aligned}$$

where

$$\begin{aligned} A &= k(y)\left(\alpha_x^1 - \alpha_y^2\right) - \alpha^2 k', \\ B &= k(y)\alpha_x^2 + \alpha_y^1, \\ C &= -\left(\alpha_x^1 - \alpha_y^2\right), \\ D &= \left(\alpha^1 r\right)_x + \left(\alpha^2 r\right)_y. \end{aligned} \tag{4.5}$$

Letting:

$$2m_0 = \min\left\{\frac{2\alpha-1}{2}, \frac{1-\alpha}{2}, \frac{1}{2}\right\}$$

and using the boundary condition $u|_{\Gamma_0\cup\Gamma_1} = 0$ we have

$$B[u, \ell(u)] \geq m_0 \int\int \left\{|k|(u_x)^2 + (u_y)^2\right\} dx dy.$$

Since $u|_{\Gamma_0\cup\Gamma_1} = 0$, using Friedrich's inequality, we obtain

$$m_0 \; \| u \|^2_{H_1(bd,k)} \leq B[u, \ell(u)]. \tag{4.6}$$

From (4.3) we have

$$|k|^{1/2} v = |k|^{1/2}\left(\alpha^1 u_x + \alpha^2 u_y\right),$$

thus

$$\| \, |k|^{1/2} v \, \|_{L^2(G)} \leq \widetilde{C}_1 \; \| u \|_{H_1(bd,k)} \, .$$

Hence from (4.6) we obtain

$$C_1 \; \| \, |k|^{1/2} v \, \|_{L^2(G)} \leq \frac{|\mathcal{B}[u,v]|}{\| u \|_{H_1}} \leq \sup_{\substack{u\in H_1(bd,k) \\ u\neq 0^1}} \frac{|\mathcal{B}[u,v]|}{\| u \|_{H_1}}.$$

Since v was arbitrary we have (4.2) by completion. ∎

Using the same procedure we obtain from Lemma 3.4

Theorem 4.7. *(Tricomi Problem)*
If the assumptions (i), (ii) of Lemma 3.4 hold and

$$r(x,y) \in C^1(\overline{G}), r(x,y)|_{\Gamma_1} \leq 0,$$

(iii)

$$[\widetilde{\alpha}^1 r]_x + [\widetilde{\alpha}^2 r]_y \leq 0 \;\; in \; G, (\widetilde{\alpha}^1, \widetilde{\alpha}^2 \;\; as \; in \;\; (3.6))$$

then there exists a constant $C_2 > 0$ such that

$$C_2 \; \| \, |k|^{1/2} u \, \|_{L^2(G)} \leq \| \, L[u] \, \|_{H_1(bd^*,k)} = \sup_{\substack{v\in H_1^* \\ v\neq 0}} \frac{|\mathcal{B}[u,v]|}{\| v \|_{H_1^*}} \tag{4.8}$$

for all $u \in H_1(bd, k)$.

Remark 4.9. From the a priori estimate (4.2) we obtain the existence of generalized solutions of Tricomi and Frankl problems (Theorem 5.1). If in addition, in case of the Tricomi problem (4.8) holds, then this solution is unique.

5. THE LINEAR EQUATION $L[u] = f(x,y)$

We now give an existence theorem for the linear problem (1.2).

Theorem 5.1. *(Tricomi and Frankl Problems) if*

(i) $k(y) \in C^0(\overline{G}) \cap C^1(G^+)$; $\operatorname{sign} k(y) = \operatorname{sign} y$; $yk'(y) \geq k(y)$ *for* $y \geq 0$;
(ii) $k(y)(n_1)^2 + (n_2)^2|_{\Gamma_1} \geq 0, n_1|_{\Gamma_1} > 0; (xn_1 + \alpha y n_2)|_{\Gamma_0} \lesseqgtr 0$,
where (n_1, n_2) *is the inward normal and* $\alpha \in (1/2, 1)$,
(iii) $r(x,y) \in C^1(\overline{G}), r|_{\Gamma_2} \leq 0; [\alpha^1 r]_x + [\alpha^2 r]_y \leq 0$ *in* G; $(\alpha^1, \alpha^2$ *as in (3.4))*
(iv) $f_1(x,y) = |k|^{-1/2} f(x,y) \in L^2(G)$,
then there exists a generalized solution of the boundary value

$$L[u] = k(y)u_{xx} + u_{yy} + r(x,y)u = f(x,y), \quad u|_{\Gamma_0 \cup |\Gamma_1} = 0,$$

i.e., there exists a function $u_0 \in H_1(\mathrm{bd}, k)$, *such that*

$$(5.2) \qquad \begin{aligned} \mathcal{B}[u_0, u] &= -\int\int_G \left\{k(y)(u_0)_x v_x + (u_0)_y v_y - r u_0 v\right\} \mathrm{d}x \mathrm{d}y = \\ &= \int\int_G f(x,y) v \mathrm{d}x \mathrm{d}y \quad \textit{for all } v \textit{ in } H_1(\mathrm{bd}^*, k), \textit{ and} \end{aligned}$$

$$(5.3) \qquad \| u_0 \|_{H_1(\mathrm{bd},k)} \leq \frac{1}{C_1} \| f_1 \|_{L^2(G)}$$

We note that C_1 *is the same constant as in (4.2).*

Proof. For fixed $v \in H_1(\mathrm{bd}^*, k), \psi_v(u) := \mathcal{B}[u,v]$ is a linear bounded functional on $H_1(bd, k)$ with

$$\| \psi_v(u) \|_{H_1} = \sup_{\substack{u \in H_1 \\ u \neq 0}} \frac{|\psi_v(u)|}{\| u \| H_1} \leq C_0 \| v \|_{H_1^*} .$$

Thus we have

$$\psi_v(u) = \mathcal{B}[u,v] = (u,w)_{H_1}, \quad \text{where } w \in H_1(bd, k),$$

and

$$\| w \|_{H_1} \le C_0 \| v \|_{H_1^*} .$$

Thus a linear operator S is defined:

$$S : H_1(\mathrm{bd}^*, k) \to H_1(\mathrm{bd}, k)$$

such that

$$\mathcal{B}[u, v] = (u, Sv)_{H_1}, \quad \text{for all } u \in H_1(bd, k), v \in H_1(bd^*, k).$$

Using (4.2) we have

$$(5.4) \qquad C_1 \| |k|^{1/2} v \|_{L^2(G)} \le \| Sv \|_{H_1(\mathrm{bd}, k)} \le C_0 \| v \|_{H_1^*} .$$

For a function $v \in H_1(bd^*, k)$ and

$$(5.5) \qquad \ell(Sv) := (f, v)_0 = (|k|^{1/2} f_1, v)_0$$

we have

$$|\ell(Sv)| \le \| f_1 \|_0 \| |K|^{1/2} v \|_0 \le \frac{1}{C_1} \| f_1 \|_0 \sup_{\substack{u \in H_1 \\ u \neq 0}} \frac{|\mathcal{B}[u, v]|}{\| u \|_{H_1}} \le \frac{1}{C_1} |f_1 \|_0 \| Sv \|_{H_1} .$$

Thus (we observe that S is injective, see (5.4))

$$\ell(\widetilde{v}) = \ell(Sv) = (|k|^{1/2} f_1, v)_0 (\widetilde{v} = Sv \in H_1(bd, k))$$

is a bounded linear operator on $S(H_1) \subset H_1$, which can be extended by Hahn-Banach theorem to $H_1(bd, k)$ preserving the norm. There exists a unique function $u_0 \in H_1(bd, k)$ such that

$$(5.6) \qquad \begin{aligned} \widetilde{\ell}(w) &= (u_0, w)_{H_1(\mathrm{bd}, k)} \quad \text{for all } w \in H_1(bd, k), \\ \| u_0 \|_{H_1(\mathrm{bd}, k)} &= \sup_{\substack{w \in H_1 \\ w \neq 0}} \frac{|\widetilde{\ell}(w)|}{\| w \|_{H_1}} \le \frac{1}{C_1} \| f_1 \|_0 . \end{aligned}$$

For each $v \in H_1(bd^*, k)$ we have

$$(5.7) \qquad \ell(Sv) = (u_0, Sv)_{H_1} = \mathcal{B}[u_0, v] = (f, v)_0,$$

i.e., u_0 is a generalized solution.

We observe that the solution $u_0 \in H_1(bd, k)$ obtained in Theorem 5.1 (see 5.6) is uniquely determined.

Furthermore we have the following result: let $u_0, \widetilde{u}_0 \in H_1(bd, k)$ be the unique solutions obtained in Theorem 5.1 such that

$$\mathcal{B}[u_0, v] = (f, v)_0, \mathcal{B}[\widetilde{u}_0, v] = (\widetilde{f}, v)_0, \text{ for all } v \in H_1(bd^*, k).$$

Then from (5.6) we have

$$\ell(w) = (u_0, w)_{H_1(bd,k)} = (|k|^{1/2} f_1, w)_0,$$
$$\widetilde{\ell}(w) = (|\widetilde{u}_0, w)_{H_1(bd,k)} = (|k|^{1/2} \widetilde{f}_1, w)_0 \text{ for all } w \in H_1(bd, k),$$

and so $u_0 - \widetilde{u}_0 \in H_1(bd, k)$ is the unique solution of the problem

$$\mathcal{B}[u_0 - \widetilde{u}_0, v] = (f - \widetilde{f}, v)_0;$$

Furthermore

$$(5.8) \qquad \| u_0 - \widetilde{u}_0 \|_{H_1(bd,k)} \leq \frac{1}{C_1} \| f_1 - \widetilde{f}_1 \|_0 .$$

By the construction used in the proof of Theorem 5.1 we have a linear solution operator

$$T_0 : f_1 = |k|^{-1/2} f \in L^2(G) \to u_0 = T_0 f \in H_1(bd, k).$$

The uniqueness of the solution in Theorem 5.1 and the linearity of the solution operator T_0 is essential for proving an existence theorem for the nonlinear Tricomi and Frankl problems by Schauder's fixed point theorem (see Theorem 6.1).

If in addition the a priori estimate (4.8) holds, then the uniqueness of the generalized solution follows. Thus we have

Theorem 5.8. *(Tricomi Problem) if*

(i) $k(y) \in C^0(\overline{G}) \cap C^1(G^+), sign\, k(y) = sign\, y, yk'(y) \geq k(y)$ *for* $y \geq 0$ *;*

(ii) $dx + (-k)^{1/2} dy|_{\Gamma_1} = 0, xn_1 + \alpha y n_2 \leq 0, (x+1)n_1 + \widetilde{\alpha} y n_2|_{\Gamma_0} \leq 0$*, where* (n_1, n_2) *denotes the inward normal vector and* α, $\hat{\alpha} \in (1/2, 1)$ *;*

(iii) $r(x,y) \in C^1(\overline{G}), r|_{\Gamma_1 \cup \Gamma_2} \leq 0, [\alpha^1 r]_x + [\alpha^2 r]_y \leq 0, [\widetilde{\alpha}^1 r]_x + [\widetilde{\alpha}^2 r]_y \leq 0$*, where* α^1, α^2 *are given by (3.3) and* $\widetilde{\alpha}^1, \widetilde{\alpha}^2$ *are given by (3.6).*

(iv) $f_1(x,y) := |k|^{-1/2} f(x,y) \in L^2(G)$;
then there exists one and only one generalized solution $u_0 \in H_1(\mathrm{bd}, k)$, *i.e.,*

$$\mathcal{B}[u_0, v] = (f, v)_0 \quad \textit{for all} \quad v \in H_1(\mathrm{bd}^*, k),$$

where u_0 satisfies the inequality

$$\| u_0 \|_{H_1} < \frac{1}{C_1} \| f_1 \|_0 . \tag{5.9}$$

Remark 5.10. The last two assumptions in (ii) above impose a starlike condition on $G(^+)$ with respect to the points A and B.

Definition 5.11. *(See [4]). A generalized solution* $u_0 \in H_1(\mathrm{bd}, k)$ *of the Tricomi or Frankl problem is called a strong solution, if a sequence of smooth functions* $\{u_n\} \subset \mathrm{U}$ *exists such that*

$$\lim_{n\to\infty} \| u_0 - u_n \|_{H_1(\mathrm{bd},k)} = 0 , \lim_{n\to\infty} \| L[u_n] - f \|_{H_{-1}(\mathrm{bd}^*,k)} = 0 .$$

From Theorem 5.1 we have

Lemma 5.12. *A generalized solution* $u_0 \in H_1(\mathrm{bd}, k)$ *of the Tricomi (Frankl) problem (Theorem 5.1) is a strong solution in the sense of Definition 5.11.*

Proof. For $u_0 \in H_1(\mathrm{bd}, k)$ there exists by (2.6) a sequence $\{u_n\} \subset \mathrm{U}$ such that $\lim_{n\to\infty} \| u_0 - u_n \|_{H_1} = 0$. If $f_n := L[u_n]$, from (2.12) we have the existence of an element $\tilde{f} \in H_{-1}(\mathrm{bd}^*, k)$ with $\lim_{n\to\infty} \| \tilde{f} - f_n \|_{H_{-1}(\mathrm{bd}^*,k)} = 0$. For $u_n \in U, f_n = L[u_n]$ we have

$$\mathcal{B}[u_n, v] = (f_n, v)_0 \quad \textit{for all} \quad v \in H_1(\mathrm{bd}^*, k),$$

thus

$$|(f_n - f, v)| = |\mathcal{B}[u_n - u_0, v]| \leq \| u_n - u_0 \|_{H_1} \| v \|_{H_1^*} .$$

For fixed $v \in H_1(\mathrm{bd}^*, k)$, from the above inequality we get $(\tilde{f} - f, v)_0 = 0$. Since $v \in H_1(\mathrm{bd}^*, k)$ is arbitrary, we obtain $(f - \tilde{f}, v) = 0$ for all $v \in H_1(\mathrm{bd}^*, k)$, which implies $f = \tilde{f}$ and $\lim_{n\to\infty} \| f - L[u_n] \|_{H_{-1}(\mathrm{bd}^*,k)} = 0$. In [4] and [5] it is shown that the estimates (4.2) and (4.8) imply inequalities of the form

$$\begin{aligned} &\| u \|_{H_1(\mathrm{bd},k)} \leq C \| L[u] \|_{L^2(G)} \quad \textit{for all} \quad u \in H_1(\mathrm{bd}, k) \cap W^{2,2}(G), \\ &\| v \|_{H_1(\mathrm{bd}^*,k)} \leq C \| L[V] \|_{L^2(G)} \quad \textit{for all} \quad v \in H_1(\mathrm{bd}^*, k) \cap W^{2,2}(G). \end{aligned} \tag{5.13}$$

∎

6. THE NONLINEAR TRICOMI AND FRANKL PROBLEMS

Using Theorem 5.1 we prove the following existence theorem.

Theorem 6.1. *If*

(i) $k(y) \in C^0(\overline{G}) \cap C^1(G); k(y) = sign\, y; yk'(y) \geq k(y)$ *for* $y \geq 0$; $k'/(|k|^{1/2}) \in L^{2m/(2-m)}(G), m \in (1,2)$.

(ii) $k(y)(n_1)^{2+}|_{\Gamma_1} \geq 0, n_1|_{\Gamma_1} > 0; (xn_1 + \alpha y n_2)|_{\Gamma_0} \leq 0$, *where* (n_1, n_2) *is the inward normal vector and* $\alpha \in (1/2, 1)$.

(iii) $r(x,y) \in C^1(\overline{G}), r|_{\Gamma_2} \leq 0; [\alpha^1 r]_x + [\alpha^2 r]_y \leq 0$ *in* $G, (\alpha^1, \alpha^2$ *as in (3.3);*

(iv) $f_1(x,y,u) := |k|^{-1/2} f(x,y,u); |f_1(x,y,u_1) - f_1(x,y,u_2)| \leq L_0 \, \| k|^{1/2}(u_1 - u_2)|$

$f_1(x,y,u)| \leq \frac{1}{\sqrt{2}}(A_0 + [(C_1/k_0)^2 - \eta^2]^{1/2} \, \| k|^{1/2} u|)$ *for* $(x,y) \in G$,
where A_0 *is an arbitrary constant,* $C_1 > 0$ *is the constant appearing in (4.2),*

$$k_0 = \max_G |k(x,y)|, 0 < \eta^2 \leq [C_1/k_0]^2,$$

then there exists a generalized solution $u_0 \in H_1(bd, k)$ *of the problem*

$$L[u] = k(y)u_{xx} + u_{yy} + r(x,y)u = f(x,y,u), u|_{\Gamma_0 \cup \Gamma_1} = 0, i.e.,$$

$$\mathcal{B}[u_0, v] = (f(x,y,u_0), v)_0 \; forall \; v \in H_1(bd^*, k),$$

where

$$\| u_0 \|_{H_1(bd,k)} \leq \frac{1}{C_1} \| f_1(x,y,u_0) \|_0 \leq \frac{1}{k_0}(\frac{A_0^2 G_0}{\eta^2})^{1/2}, \tag{6.2}$$

and G_0 = *measure of* G.

Proof. We consider the closed ball

$$W = \left\{ w | w \in L^2(G), \| |k|^{1/2} w \|_0^2 \leq G_0 [A_0/\eta]^2 \right\}.$$

Then for all $w \in W$, we have

$$\| f_1(x,y,w) \|_0^2 \leq A_0^2 G_0 + \left[(C_1/k_0)^2 - \eta^2\right] \| |k|^{1/2} w \|_0^2 \leq G_0 \left[C_1 A_0 / k_0 \eta\right]^2. \tag{6.3}$$

From Theorem 5.1 we know, that for every $w \in W$ there exists a unique generalized solution $u \in H_1(bd, k)$ such that

$$\mathcal{B}[u, v] = (f(x, y, w), v)_0 \quad for\ all \quad v \in H_1(bd^*, k),$$

$$\| u \|_{H_1(bd,k)} \leq \frac{1}{C_1} \| f_1 \|_0 .$$

Thus we have

$$\| |k|^{1/2} u \|_0 < k_0 \| u \|_{H_1(bd,k)} \leq \frac{k_0}{C_1} \| f_1 \|_0 \leq \left[(A_0/\eta)^2 G_0 \right]^{1/2} . \tag{6.4}$$

Now we define an operator

$$T : w \in W \rightarrow u = Tw \in H_1(bd, k), T(W) \subset W. \tag{6.5}$$

From (5.7), (5.9) and the Lipschitz condition on f it follows that T is a continuous operator. To see this, let $w_n, w_0 \in W$ be such that

$$\lim_{n \to \infty} \| |k|^{1/2} (w_n - w_0) \|_0 = 0 .$$

Letting $u_n = Tw_n, u_0 = Tw_0$, we have

$$\mathcal{B}[u_n - u_0, v] = (f(x, y, w_n) - f(x, y, w_0), v) \text{ for all } v \in H_1(bd^*, k),$$

$$\| u_n - u_0 \|_{H_1(bd,k)} \leq \frac{1}{C_1} \| f_1(x, y, w_n) - f_1(x, y, w_0) \|_0,$$

$$\| |k|^{1/2} (u_n - u_0) \|_0 \leq k_0 \| u_n - u_0 \|_{H_1(bd,k)} \leq \frac{k_0}{C_1} \| f_1(x, y, w_n) - f_1(x, y, w_0) \|_0 \leq$$

$$\leq L_0 \frac{k_0}{C_1} \| |k|^{1/2} (w_n - w_0) \|_0 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

■

Lemma 6.6. *If* $k(y) \in C^1(G), k'(|k|^{-1/2}) \in L_{2m/(2-m)}(G), m \in [1, 2)$ *then* $T(W) \subset W^{1,m}(G)$.

Proof. For $w \in W$ we have $u = Tw \in H_1(bd, k)$, $\| u \|_{H_1(bd,k)} \leq \frac{(G_0)}{k_0}^{1/2} [A_0/\eta]^2$. Let $\tilde{u} := |k|^{1/2} u$. For $m \in [1, 2)$, using Holder inequality we obtain

$$\| \tilde{u} \|^m_{W^{1,m}} = \int \int_G \left\{ |\tilde{u}_x|^m + |\tilde{u}_y|^m + |\tilde{u}|^m \right\} \mathrm{d}x \mathrm{d}y \leq$$

$$\leq C_3 \left(\int \int_G \left\{ |k| u_x^2 + u_y^2 + u^2 \right\} \mathrm{d}x \mathrm{d}y \right)^m /2 + C_4 \int \int_G |\frac{k'}{|k|^{1/2}}|^m |u|^m \mathrm{d}x \mathrm{d}y.$$

Using

$$\int\int_G |\frac{k'}{|k|^{1/2}}|^m |u|^m \mathrm{d}\,x \mathrm{d}\,y \leq \left(\int\int_G |\frac{k'}{|k|^{1/2}}|^{2m/(2-m)} \mathrm{d}\,x \mathrm{d}\,y\right)^{(2-m)/2m}$$

$$\left(\int\int_G |u|^2 \mathrm{d}\,x \mathrm{d}\,y\right)^{m/2}$$

We see that $\widetilde{u} = |k|^{1/2} u \in W^{1,m}(G)$ and

$$(6.7) \qquad \| \, |k|^{1/2} u \, \|_{W^{1,m}(G)} \leq C_5 \, \| \, u \, \|_{H_1(bd,k)} \; .$$

The Rellich-Kondrashov Theorem [1; pg. 144] implies that $W^{1,m}(G)$ is compact in $L^2(G)$ for $m \in (1,2)$.

Next we show that $T : W \to W$ is compact. To see this we take a sequence $w_n \subset W$, where we have $\| \, |k|^{1/2} w_n \, \|_0^2 \leq G_0 [A_0/\eta]^2$.

For $u_n = T w_n$ from (6.4) we have

$$\| \, u_n \, \|_{H_1(bd,k)} \leq \frac{1}{C_1} \, \| \, f_1 \, \|_0 \leq \frac{1}{k_0} \, [A_0/\eta]^2 \, (G_0)^{1/2} \; ;$$

thus there exists a subsequence (which we again denote by u_n) and an element $u_0 \in H_1(bd, k)$ such that

$$(6.8) \qquad u_n \to u_0 \text{ weakly in } H_1(bd, k).$$

From (6.7) it follows that

$$\| \, |k|^{1/2} u_n \, \|_{W^{1,m}(G)} \leq C_5 \, \| \, u_n \, \|_{H_1} \leq \frac{C_5}{K_0} \, [A_0/\eta] \, (G_0)^{1/2} \, .$$

$W^{1,m}(G)$, $m \in (1,2)$ is compact in $L^2(G)$, therefore there exists a subsequence (which we again denote by u_n) such that

$$(6.9) \qquad |k|^{1/2} u_n \to |k|^{1/2} u_0 \text{ strongly in } L^2(G).$$

Now from (6.4) we conclude that $|k|^{1/2} u_0 \in W$.

$W \in L^2(G)$ is a closed, bounded and convex subset and $T : W \to W$ a continuous compact operator. Then from Schauder fixed point theorem ([10] pg. 355) we infer that there exists at least one fixed point u_0, i.e., $u_0 = T u_0 \in H_1(bd, k)$. u_0 is a generalized solution of our nonlinear problem. ∎

We have the following uniqueness result.

Lemma 6.10. *If the assumptions of Theorem 6.1 hold and $L_0 k_0 / C_1 < 1$, then there exists one and only one generalized solution of the nonlinear Tricomi and Frankl problems.*

Proof. Let $u_1, u_2 \in H_1(bd, k)$ be two generalized solutions, then we have

$$\mathcal{B}[u_1 - u_2, v] = (f(x,y,u_1) - f(x,y,u_2), v)_0 \text{ for all } v \in H_1(bd^*, k),$$

using (5.8) we obtain

$$\| (u_1 - u_2) \|_{H_1} \leq \frac{1}{C_1} \| f_1(x,y,u_1) - f_1(x,y,u_2) \|_0 .$$

Thus we have

$$\| |k|^{1/2}(u_1 - u_2) \|_0 \leq k_0 \| u_1 - u_2 \|_{H_1} \leq \frac{k_0}{C_1} \| f_1(x,y,u_1) - f_1(x,y,u_2) \|_0 \leq$$

$$\leq \frac{k_0 L_0}{C_1} \| |k|^{1/2}(u_1 - u_2) \|_0 ,$$

which in view of $L_0 k_0 / C_1 < 1$ implies $u_1 = u_2$.

Using Theorem 5.8 we get the following result: ∎

Theorem 6.11. *(Tricomi Problem) if*

(i) $k(y) \in C^0(\overline{G}) \cap C^1(G)$, $signk = signy, yk'(y) \geq k(y)$ *for* $y \geq 0, k'(|k|^{-1/2}) \in L_{2m/(2-m)}(G)$, *where* $m \in (1,2)$.

(ii) $dx + (-k)^{1/2} dy|_{\Gamma_1} = 0$, $xn_1 + \alpha y n_2|_{\Gamma_0} \leq 0$, $(x+1)n_1 + \widetilde{\alpha} y n_2|_{\Gamma_0} \leq 0$, *where* (n_1, n_2) *is the inward normal vector and* $\alpha, \widetilde{\alpha} \in (1/2, 1)$.

(iii) $r(x,y) \in C^1(\overline{G})$, $r|_{\Gamma_1 \cup \Gamma_2} \leq 0$, $[\alpha^1 r]_x + [\alpha^2 r]_y \leq 0$, $[\widetilde{\alpha}^1 r]_x + [\widetilde{\alpha}^2 r]_y \leq 0$($\alpha^1, \alpha^2$, *and* $\widetilde{\alpha}^1, \widetilde{\alpha}^2$ *are as in (3.3) and (3.6) respectively in* G.

(iv) $f_1(x,y,u) = |k|^{-1/2} f(x,y,u), |f_1(x,y,u_1) - f_1(x,y,u_2)| \leq L_0 |u_1 - u_2|$.

$$(6.12) \qquad |f_1(x,y,u)| \leq \frac{1}{\sqrt{2}} \left(A_0 + \left([C_1 C_2 / C_0]^2 - \eta^2 \right)^{1/2} |k|^{1/2} |u| \right),$$

where $A_0 > 0$ *is an arbitrary constant and* C_0, C_1 *and* C_2 *are the constants appearing in (2.12), (4.2) and (4.8) respectively, and* $0 < \eta^2 < [C_1 C_2 / C_0]^2$. *Then there exists a generalized solution* $u_0 \in H_1(bd, k)$ *of the nonlinear Tricomi problem*

$$L[u] = k(y) u_{xx} + u_{yy} + r(x,y) u = f(x,y,u), u|_{\Gamma_0 \cup \Gamma_1} = 0$$

i.e.,

$$\mathcal{B}[u_0, v] = (f(x, y, u_0), v)_0 \text{ for all } v \in H_1(bd^*, k),$$

where

$$\| u_0 \|_{H_1(bd,k)} \leq \frac{1}{C_1} \| f_1(x, y, u_0) \|_0 \leq [G_0]^{1/2} [A_0 C_2 / C_1 \eta]$$

$$G_0 = \text{Measure of } G.$$

Proof. As in Theorem 6.1 we consider the closed ball

$$W = \left\{ w \in W | L^2(G), \| |k|^{1/2} w \|_0^2 \leq G_0 (A_0/\eta)^2 \right\}.$$

Then for all $w \in W$ we have

$$\| f_1(x, y, w) \|_0^2 \leq A_0^2 G_0 + [(C_1 C_2 / C_0)^2 - \eta^2] \| |k|^{1/2} w \|_0^2 \leq G_0 [A_0 C_1 C_2 / C_0 \eta]^2.$$

From Theorem 5.8 we know that for every $w \in W$ there exists a generalized solution $u \in H_1(bd, k)$ such that

$$\mathcal{B}[u, v] = (f(x, y, w), v)_0 \text{ for all } v \in H_1(bd^*, k),$$

$$\| u \|_{H_1} \leq \frac{1}{C_1} \| f_1(x, y, w) \|_0 .$$

Let

$$T : w \in W \to u = Tw \in H_1(bd, k).$$

From estimate (4.8) it follows that

$$\| |k|^{1/2} u \|_0 \leq \frac{C_0}{C_2} \| u \|_{H_1(bd,k)} \leq \frac{C_0}{C_1 C_2} \| f_1(x, y, w) \|_0 \leq (G_0)^{1/2} [A_0/\eta]^2,$$

i.e., $T(W) \subset W$.

From (4.8) and the Lipschitz condition on f it follows that T is continuous. To see this, let $w_n, w_0 \in W$ be such that $\| |k|^{1/2} (w_n - w_0) \|_0 \to 0$ as $n \to \infty$.

Then for $u_n = Tw_n, u_0 = Tw_0$, using (4.8) we obtain

$$C_2 \| |k|^{1/2} (u_0 - u_n) \|_0 \leq \sup_{\substack{v \in H_1^* \\ v \neq 0}} \frac{|\mathcal{B}[u_n - u_0, v]|}{\| v \|_{H_1^*}} =$$

$$= \sup_{\substack{v \in H_1^* \\ v \neq 0}} \frac{|(f(x, y, w_n) - f(x, y, w_0), v)_0|}{\| v \|_{H_1^*}} \leq$$

$$\leq L_0 \| |k|^{1/2} (w_n - w_0) \|_0 \to 0 \text{ as } n \to \infty.$$

Using Lemma 6.6 it follows as in the proof of Theorem 6.1 that $T : W \to W$ is a compact operator. Then from Schauder's fixed point theorem we conclude the existence of at least one generalized solution u_0 of the nonlinear Tricomi problem. ∎

Remark 6.13. We observe that in Theorem 6.11, using estimate (4.8) we get the continuity of T under a weaker Lipschitz condition on f (see (6.12)).

Now we give a uniqueness result for the nonlinear Tricomi problem.

Lemma 6.14. *If the hypotheses of Theorem 6.11 hold, and $L_0 < C_2$, then there exists one and only one generalized solution of the nonlinear Tricomi problem.*

Proof. Suppose $u_1, u_2 \in H_1(bd, k)$ are two generalized solutions of the nonlinear Tricomi problem, then

$$\mathcal{B}[u_1 - u_2, v] = (f(x,y,u_1) - f(x,y,u_2), v)_0 \text{ for all } v \in H_1(bd^*, k).$$

From estimate (4.8) we have

$$C_2 \, \| \, |k|^{1/2}(u_1 - u_2) \, \|_0 \leq \sup_{\substack{v \in H_1^* \\ v \neq 0}} \frac{|\mathcal{B}[u_1 - u_2, v]|}{\| v \|_{H_1^*}} =$$

$$= \sup_{\substack{v \in H_1^* \\ v \neq 0}} \frac{|(f(x,y,u_1) - f(x,y,u_2), v)_0|}{\| v \|_{H_1^*}} \leq$$

$$\leq \| \, |k|^{1/2}[(f(x,y,u_1) - f(x,y,u_2)] \, \|_0 \leq L_0 \, \| \, |k|^{1/2}(u_1 - u_2) \, \|_0 \, .$$

Since $L_0 < C_2$, the above inequality implies $u_1 = u_2$. ∎

emark 6.15. In theorems 6.1 and 6.11 we have the essential condition $[k'/|k|^{-1/2}] \in L_{2m/(2-m)}(G)$, which is needed in Lemma 6.6. For the special case $k(y) = signy|y^q$, $q > 1$, the case $q = 1$ is not allowed. However we observe that in Lemma 3.1 and Lemma 3.4 we have solutions $u \in H_1(bd, k) \cap L_{\nu'}(G)$, $\nu \geq 2$ and $v \in H_1(bd^*, k) \cap L_{\nu'}(G)$, $\nu' \geq 2$. We can show that the estimate (4.2) holds for the space $H_1(bd, k) \cap L_\nu(G)$ and similarly the estimate (4.8) holds for the space $H_1(bd^*, k) \cap L_{\nu'}(G)$. Using the same method we obtain in Lemma 6.6 the weaker condition $[k'/|k|^{-1/2}] \in L\nu_{m/(\nu-m)}(G)$, $m \in (1,2)$, $\nu \geq 2$ (see [2] pg. 441), but this implies that the case $q = 1$ can be included in our results.

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