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ON JUST INSEPARABLE FINITE GROUPS

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Abstract. *A finite group is called «just inseparable» if every proper subgroup of G has a complement if and only if it isn't normal. We show that a group is just inseparable if and only if it is a cyclic group of prime power order or is isomorphic to the quaternion group.*

Definition 1. *A finite group is inseparable if no normal proper subgroup of G has a complement in G.*

Let **B** be the class of finite inseparable groups.

Denote by **A** the class of finite groups in which every non-normal subgroup has a complement.

Definition 2. *A finite group G is «just inseparable» if $G \in \mathbf{A} \cap \mathbf{B}$, that is if has the following property: a proper subgroup of G has a complement iff it isn't normal.*

Denote with **E** the class $\mathbf{A} \cap \mathbf{B}$ of just inseparable finite groups.

In this Note we characterize the just inseparable finite groups. Precisely, we prove that a finite group is just inseparable iff it is a cyclic p-group or it is isomorphic to the quaternion group.

All groups considered in this Note are supposed finite.

We have immediately:

If $G \in \mathbf{A}$, every subgroup of G and every epimorphic image of G is in **A** too.

Lemma 1. *Every just inseparable group G is a p-group with $\Omega_1(G) \leq Z(G) \cap \Phi(G)$.*

Proof. Let p the smallest prime divisor of $|G|$ and P a Sylow p-subgroup of G. Let X be a subgroup of G or order p. If X is not normal, then $G = XC$ with C subgroup of G and $X \cap C = \langle 1 \rangle$. So C has index p in G, hence it is normal, and has X as complement, contradicting the hypothesis. Therefore X is normal in G. Since $|X| = p$ with p the smallest prime divisor of G, then we have $X \leq Z(G)$. But X has no complement because it is normal, so $X \leq \Phi(G)$. From that it follows $X \leq Z(G) \cap \Phi(G)$, so $\Omega_1(P) \leq Z(G) \cap \Phi(G)$. Let $\overline{P}$ be a conjugate of P. Since P is normal, then we have $\Omega_1(\overline{P}) = \Omega_1(P)$, hence $\Omega_1(P)$ contains every subgroup of order p of G.

Now we distinguish two cases:

a) P is normal in G. If $P \neq G$, then because of the Schur-Zassenhaus theorem, P has a complement, contradicting the hypothesis. So $P = G$ and lemma 1 follows.

b) P is not normal in G. Then, by hypothesis exists a subgroup H of G such that $PH = G$ and $P \cap H = \langle 1 \rangle$.

So H is a Hall p'-subgroup of G. If $\Omega_1(P)$ is normal in $G, \Omega_1(P)H$ is a subgroup of G. If $\Omega_1(P)H$ has a complement $D \neq \langle 1 \rangle$, then D is a p-group and contains a subgroup of order p, necessarily contained in $\Omega_1(P)$, but that is impossible, because $D \cap \Omega_1(P)H = \langle 1 \rangle$.

So $\Omega_1(P)H$ has no complement, hence it is normal in G.

Because of $\Omega_1(P) \leq Z(G)$, we have $\Omega_1(P)H = \Omega_1(P) \times H$ and so H is characteristic in $\Omega_1(P)H$. Moreover, since $\Omega_1(P)H$, is normal in G, then H must be normal in G. If $H \neq \langle 1 \rangle$, then H has P as a complement, contradicting the hypothesis. Therefore $H = \langle 1 \rangle$, that is $G = P$, and $\Omega_1(G) = \Omega_1(P) \leq Z(G) \cap \Phi(G)$.

Lemma 2. *If G is not cyclic and just inseparable then $G = \langle s, c \rangle$ with $o(s) = p^a, o(c) = p^b, a \geq 2, b \geq 2, G' = \langle [s,c] \rangle$ and $|G'| = p$.*

If G is not isomorphic to the quaternion group, then $\langle s \rangle \cap \langle c \rangle = \langle 1 \rangle$.

Proof. By lemma 1, G is a p-group and $\Omega_1(G) \leq Z(G) \cap \Phi(G)$.

In particular every subgroup of order p of G is normal. If every subgroup of G is normal, then G is isomorphic to the quaternion group hence lemma 2 is true for $p = 2, a = b = 2$. In the opposite case let H be maximal among the non normal subgroups of G. Then, by hypothesis, H has a complement C in G. So we have $G = HC, H \cap C = \langle 1 \rangle$. Also C is not normal in G, because it has H as a complement. Therefore, there exists a cyclic subgroup $Y \subseteq C$ that is not normal in G. So we have $G = XY$ with $X \cap Y = \langle 1 \rangle$, and X subgroup of G. So $C = Y(C \cap X)$. Let now $Y_0 = \Omega_1(Y)$. Then Y_0 is normal in G, because it has order p, and HY_0 is a subgroup of G, which is necessarly normal by the maximality condition on H. Since $HY = (HY_0)Y$ with HY_0 normal, then HY is a subgroup of G, which is necessarily normal by the same maximality condition. Then we have $G = HC = HY(C \cap X)$. Furthermore we have $|G| = |H||C| = |H||Y||C \cap X| = |HY||C \cap X|$ and so $(HY) \cap (C \cap X) = (1)$. Since HY is normal in G, then $G = HY$, that is $Y = C$, and C is cyclic. Let M be maximal in H with $M \geq \Omega_1(H)$. Since every subgroup of order p of H is in $\Omega_1(H)$, then M has no complement in H, and therefore neither in G.

So M is normal in G. Since H is not normal in G, then M is the only maximal subgroup of G including $\Omega_1(H)$. So $H/\Omega_1(H)$ is cyclic, and moreover H is abelian, because of $\Omega_1(H) \leq Z(H)$.

So we have $H = S \times L$, with S cyclic and L elementary abelian, hence fulfilling $L \leq \Omega_1(G)$. From that we find $H\Omega_1(G) = S\Omega_1(G)$.

Since they have no complement in H, then $S\Omega_1(G)$ and $C\Omega_1(G)$ are normal in G. Furthermore, since $H \cap C = \langle 1 \rangle$, we have $H\Omega_1(G) \cap C\Omega_1(G) = \Omega_1(G)$. From that we find: $\frac{G}{\Omega_1(G)} = \frac{H\Omega_1(G)}{\Omega_1(G)} \times \frac{C\Omega_1(G)}{\Omega_1(G)} = \frac{S\Omega_1(G)}{\Omega_1(G)} \times \frac{C\Omega_1(G)}{\Omega_1(G)}$ with $\frac{S\Omega_1(G)}{\Omega_1(G)}, \frac{C\Omega_1(G)}{\Omega_1(G)}$ cyclic groups.

Let $S = \langle s \rangle$ and $C = \langle c \rangle$. Then $s\Omega_1(G)$ and $c\Omega_1(G)$ are generators of $\frac{G}{\Omega_1(G)}$.

Now, by lemma 1, we find $\Omega_1(G) \leq \Phi(G)$. Hence $s\Phi(G)$ and $c\Phi(G)$ are generators in $\frac{G}{\Phi(G)}$, hence s and c are generators in G.

Since $\frac{S}{\Omega_1(G)}, \frac{C}{\Omega_1(G)}$ are normal and cyclic, then $G/\Omega_1(G)$ is abelian so we have $G' \leq \Omega_1(G) \leq Z(G)$.

From there we have $[s,c] \in Z(G)$, so $[s,c]$ is permutable with s and c, and $G' = \langle [s,c] \rangle$. Since $G' \leq \Omega_1(G)$, then $[s,c]$ has order p and $|G'| = p$.

Therefore the lemma is proved with p^a, p^b orders of s and c recpectively.

Proposition 3. *A group G is just inseparable iff it is a cyclic p-group or it is isomorphic to the quaternion group.*

Proof. Cyclic p-groups and the quaternion group have only normal subgroups, so they are inseparable; hence just inseparable.

To prove the converse, let G be a just inseparable group which is neither cyclic nor isomorphic to the quaternion group. Now, by lemma 2, we find: $G = \langle s,c \rangle$ with $s^{p^a} = c^{p^b} = 1; G' = \langle [s,c] \rangle$ and $|G'| = p$ $\langle s \rangle \cap \langle c \rangle = \langle 1 \rangle$. We suppose $a \geq b$ and set $S = \langle s \rangle$ and $C = \langle c \rangle$. Then SG' is a normal subgroup of G, and $SG'C = G$. So $SG' \cap C \neq \langle 1 \rangle$ because SG' cannot have a complement. Hence $c^{p^{b-1}} = s^m[s,c]^n$ with $0 \leq n < p$. Now $n \neq 0$ because of $S \cap C \neq 1$, therefore $1 \leq n < p$.

Since $[s,c]^n$ and $c^{p^{b-1}}$ have order p and are permutable, then also we find $(s^m)^p = 1$, hence $m = hp^{a-1}$, with $0 \leq h < p$. Therefore we have $[s,c]^n \in \langle s^{p^{a-1}}, c^{p^{b-1}} \rangle$ and moreover, since $[s,c]$ is a power of $[s,c]^n$, we have $[s,c] \in \langle s^{p^{a-1}}, c^{p^{b-1}} \rangle$, that is $[s,c] = s^{kp^{a-1}} c^{tp^{b-1}}$, with $0 \leq k < p, 0 \leq t < p$. From there we find $G = SC$.

Now we have $k \neq 0$, otherwise $[s,c] \in C$, and C is normal, though it has a complement, contradicting the hypothesis.

Similarly we have $t \neq 0$. We set $w = s^{kp^{a-b}} c^t$. Then $w^{p^{b-1}} = (s^{kp^{a-b}} c^t)^{p^{b-1}} = s^{kp^{a-1}} c^{tp^{b-1}} [s,c]^{-ktp^{a-b}\binom{p^{b-1}}{2}} = [s,c][s,c]^{-ktp^{a-b}\binom{p^{b-1}}{2}}$.

Since $b \geq 2$, then excluding the case $a = b = 2, p = 2, -ktp^{a-b}\binom{p^{b-1}}{2}$ is divisible for p. So $w^{p^{b-1}} = [s,c]$, that is, $W =: \langle w \rangle$ is normal in G. On the other hand, $\langle s,w \rangle$ contains s and contains $c^t = s^{-kp^{a-b}} w$.

Since $1 \leq t < p$, then c is a power of c^t, so $\langle s,w \rangle$ also contains c. Hence it follows $G = \langle s,w \rangle = SW$.

But $w^{p^{b-1}} = [s,c]$, cannot be in S, otherwise S is normal, although having a complement C. Hence it follows $S \cap W = \langle 1 \rangle$, with W normal in G, contradicting the hypothesis.

Therefore we have $p = 2$, $a = b = 2$, hence $[s,c] = w^2 = (s,c)^2 = s^2c^2[s,c]^{-1}$, hence $s^2c^2 = [s,c]^2 = 1$, that is, $s^2 = c^{-2}$, with $s^2 \neq 1$.

So $\langle s\rangle \cap \langle c\rangle \neq \langle 1\rangle$, contradicting the hypothesis. The proposition is proved.

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