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A FORMAL SYSTEM FOR THE ALTERNATIVE SET THEORY. A NON-EXTENSIONAL APPROACH *

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Abstract. *We presented a formal system which is intended to capture the essence of the set theory in the alternative version given by Vopěnka. Our purpose starts from Sochor's remark that in the formalization presented as AST, the notions of set and class correspond respectively to element of the universe of sets and to object from the extended universe.*

The fundamentals ideas of the Alternative Set Theory are explained in the 1979 text by P. Vopěnka (cf. [V]) and we refer the reader to that for any explanation or suggestions and definitions about the theory. In a series of papers (cf. [S1], [S2], [S3]), A. Sochor presents a formalization for the properties of «some kind» of objects treated by Vopěnka. In this paper we present a new non-extensional formal system for Alternative Set Theory different from Sochor's AST. In order to distinguish between the two, we call ours TAI (for the Italian *Teoria Alternativa degli Insiemi*). The formal axiom system we give is redundant, but we sacrifice minimality to follow the mathematical development in Vopěnka's book. A non-extensional theory is suggested from Vopěnka's words: «Each property of objects can be considered as an object. A property of objects understood as an object is said to be a *class.* Classes are further specific objects of our study. The fact that an object X is a class is denoted by $Cls(X)$ ». ([V], pag. 27). Assuming extensionality the axiomatic presentation can be shortened.

Parts of this research and a first attempt for this formalization have been presented in [M1] and [M2] and in a short communication to the Meeting of GNSAGA in Catania, October 1986.

1. THE LANGUAGE OF TAI

The language of TAI is a one-sorted, first order language (without identity!) with the following special symbols: a constant $\emptyset$, two two-place predicates $\in$ and $\div$, three one-place predicates V. Set, Cls and an *abstraction operator* $\{\ldots|\ldots\}$ which accepts a variable to the left of the stroke and a formula to the right of it. The variable appearing left is *bound* in the resulting term.

The definitions for terms and formulas are given in the usual way by using a double induction (cf. [B]).

Variables are denoted by capital Greek letters, and intuitively «represent» the idea of *objects* in Vopěnka terminology. In the presentation of the formal system we distinguish between

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axioms and postulates (presentated as axioms in [V]), reserving this name to some propositions provable from other axioms.

2. SET AXIOMS OF TAI

A1. $(\forall\Phi)(\Phi \div \Phi)$;

A2. $(\forall\Phi)(\forall\Psi)(Set(\Phi) \wedge Set(\Psi) \rightarrow (\Phi \div \Psi \equiv (\forall\Theta)(\Theta \in \Phi \div \Theta \in \Psi)))$;

A3. $V(\emptyset) \wedge (\forall\Phi)(\Phi \notin \emptyset)$.

Remark 1. From Axiom 2, we have that our equality predicate is reflexive, symmetric and transitive for Sets.

Small Latin letters are used to denote variables relativised to predicate V; e.g. $\varphi(x)$ means $\varphi(\Phi) \wedge V(\Phi)$. Objects of this kind will be informally called *sets from the universe of sets*, or V-sets.

Formulas in which only small Latin letters, the constant $\emptyset$ and predicates $\in$ and $\div$ are present, are called a *set-formulas.* For this kind of sets we assume the postulate, expressing extensionality for V-sets.

P1. $(\forall x)(\forall y)(x \div y \equiv (\forall z)(z \in x \equiv z \in y))$.

Remark 2. Similarly as stated in previous remark, the equality predicate $\div$ for V-sets is reflexive, symmetric and transitive.

P2. $(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \equiv (u \in x \vee u \div y))$.

For any given x and y, the set z determined by Postulate 2 is unique with that property by Postulate 1. Denote it functionally by $x\%y$. Postulate 2 can be rewritten as

P2. $(\forall x)(\forall y)(\forall u)(u \in x\%y \equiv (u \in x \vee u \div y))$.

A4. For each set-formula $\varphi(x)$,

$$(\varphi(\emptyset) \wedge (\forall x)(\forall y)(\varphi(x) \rightarrow \varphi(x\%y))) \rightarrow (\forall x)\varphi(x).$$

A5. For each set-formula $\varphi(x)$,

$$(\exists x)\varphi(x) \rightarrow (\exists x)(\varphi(x) \wedge (\forall y)(y \in x \rightarrow \neg\varphi(y))).$$

Remark 3. The axioms above are in [V], in the same form except for Axioms 1, 2 and 3, which are only presented in words. Here we interpret Vopěnka statement that $\emptyset$ is a set in Axiom 3, as a V-set, since from Axiom 7 below follows Set($\emptyset$). Following [V] the usual set-operations such as union, intersection, power-set, etc., can be recovered. Hence $x\%y$ can be written as $x \cup \{y\}$. Axioms 4 and 5 can be resumed both in

$$(\varphi(\emptyset) \wedge (\forall x)(\forall y)((\varphi(x) \wedge \varphi(y)) \rightarrow \varphi(x\%y))) \rightarrow (\forall x)\varphi(x),$$

but this does not appear in [V] and it is presented in [PS].

3. ABSTRACTION AND Λ -CLASSES

Our aim is now to capture the powerfull abstraction principle of Vopěnka, assuming that «properties» are identifiable by well-formed formulas of the language of TAI, and using the abstraction operator. With these tools the axiom can be stated in the form: for each formula $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n), Cls(\{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\})$. In [V] there is nothing more. In the search of a powerful, but consistent axiomatisation, we can enforce it requiring.

A6. For each formula $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)$:

$$Cls(\{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\}) \wedge (\forall\Theta)(\Theta \in \{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\} \rightarrow \varphi(\Theta, \Psi_1, \Psi_2, \ldots, \Psi_n)).$$

In Axiom 6 the converse implication

$$(\forall\Theta)(\varphi(\Theta, \Psi_1, \Psi_2, \ldots, \Psi_n) \rightarrow \Theta \in \{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\})$$

must be avoided in order to prevent Russel paradox. We state later this implication in some particular cases.

A special kind of classes are those whose elements are all from the universe of set. These object are called, in Vopěnka's terminology, *classes from the extended universe.* We use for these a special symbol: Λ , given, by definition: $\Lambda(\Phi)$ for $Cls(\phi) \wedge (\forall\Psi)(\Psi \in \Phi \rightarrow V(\Psi))$. Capital Latin letters in formulas and quantifier are reserved to variables which are relativised to predicate Λ , e.g. $\varphi(X)$ means $\varphi(\Phi) \wedge \Lambda(\Phi)$. Sometimes we refer to these classes as Λ -classes.

Remark 4. With this choice of symbols, we can derive the following property stated in [V]:

- for each formula $\varphi(x, \Psi_1, \Psi_2, \ldots, \Psi_n), \Lambda(\{x | \varphi(x, \Psi_1, \Psi_2, \ldots, \Psi_n)\})$;
- in particular, $\Lambda(V)$, where V is written for $\{\Phi | V(\Phi)\}$.

Classes of the extended universe, (general) sets and V -sets are strictly connected. Suppose we have a collection of V -sets, and suppose this class is a set in the general sense, i.e. a collection obtained from a list, so this set must be a set of the universe of sets and viceversa.

A7. $(\forall\Phi)(V(\Phi) \equiv Set(\Phi) \wedge \Lambda(\Phi))$.

Remark 5. This principle seems quite natural, but it is not presented in [V]. By the way we obtain,

$$(1) \qquad (\forall\Phi)(V(\Phi) \rightarrow Set(\Phi)),$$

in particular, by Axiom 3, Set $(\emptyset)$. Take care that Postulate 1 does not follow form Axiom 2 and (1) only, since we are not able to state that every object that is element of a V -set is a V -set. Moreover, if we add

$$(2) \qquad (\forall\Phi)(V(\Phi) \rightarrow \Lambda(\Phi)),$$

we deduce $(\forall\Phi,\Psi)(V(\Phi)\wedge\Psi\in\Phi\rightarrow V(\Psi))$. Hence Postulate 1 follows from Axiom 2, (1) and (2). The formula (2) can be viewed as a translation of the ambiguous request, stated in [V], that sets are classes. Another possible interpretation of the same request is the axiom

A8. $(\forall\Phi)(Set(\Phi)\rightarrow Cls(\Phi))$.

Remark 6. A choice can be made between the two ways of formalizing these ideas, but it is also possile to assume both.

Extensionality is assumed for Λ -classes in the form of the axiom

A9. $(\forall X)(\forall Z)(X\div Z\equiv(\forall x)(x\in X\equiv x\in Z))$.

Remark 7. Of course, Postulate 1 is a consequence of the Axiom 9 too, since every V -set is a Λ -class. From Axiom 9 the equality predicate for Λ -classes is reflexive, symmetric and transitive.

Other interesting features, obtained by Remark 4, are:

- each V -set is equal to a Λ -class, in the form, $(\forall x)(x\div\{y\in x\})$, since $\Lambda(x)$ and $\Lambda(\{y|y\in x\})$;
- each Λ -class $X\div\{y|y\in X\}$, since $\Lambda(\{y|y\in X\})$.

4. THREE ALTERNATIVE AXIOMS OF TAI

The alternative principles of existence of semisets, prolongation, extensional coding and two-cardinals are recovered in four axioms. First we present the formal version of only some of these: the extensional coding Axiom will be presented further on in section 6. It can be noted that the existence of a semiset is a direct consequence of the prolongation Axiom, but, as already announced, we stick to Vopěnka's presentation.

P3. $(\exists X)(\exists y)(X\subseteq y\wedge\neg V(X))$.

Remark 8. The statement above corresponds to Vopěnka's infinity principle. We use predicate V, instead of Set. But for a Λ -class $X,\neg V(X)\equiv\neg Set(X)$, by Axiom 7.

We assume the same definitions of finite, countable and uncountable classes as presented in [V] and denoted by the symbols: $Fin(X), Count(X), Uncount(X)$ respectively. Latin letters $F,G,f,\ldots,$ will denote functions. The symbol $\simeq$ is reserved to equivalence: $X\simeq Z$ means that there exists a function F which is injective and such that $dom(F)\div X$ and $rng(F)\div Z$.

A10. $(\forall F)(Count(F)\rightarrow(\exists f)(F\subseteq f))$;

A11. $(\forall X)(\forall Z)((Uncount(X)\wedge Uncount(Z))\rightarrow X\simeq Z)$.

Remark 9. In Postulate 3 and Axiom 10, the intended meaning of $\Phi\subseteq\Psi$ is $(\forall y)(y\in\Phi\rightarrow y\in\Psi)$, and not $(\forall\Theta)(\Theta\in\Phi\rightarrow\Theta\in\Psi)$. The second is a new inclusion relation.

Definition. *Let* Φ,Ψ *be objects, then* $\Phi\Subset\Psi$ *means* $(\forall\Theta)(\Theta\in\Phi\rightarrow\Theta\in\Psi)$.

As a consequence of definition of Λ -classes, these two different interpretations of inclusion coincide for Λ -classes and V -sets.

Proposition 10. *(i)* $(\Lambda(\Phi) \wedge \Psi \Subset \Phi) \rightarrow A(\Psi)$
(ii) $(\Lambda(\Phi) \wedge \Lambda(\Psi)) \rightarrow (\Psi \Subset \Phi \equiv \Phi \subseteq \Psi)$;
(iv) $(Set(\Phi) \wedge \Phi \Subset V) \rightarrow V(\Phi)$.

Proof. Trivial by definition of Λ -classes, Remark 5 and Axiom 7.

Another natural property is $(Cls(\Phi) \wedge \Psi \Subset \Phi) \rightarrow Cls(\Psi)$, but it must be added as specific axioms.

5. JUSTIFICATION OF ADDITIONAL AXIOMS OF TAI

Till Axiom 11 our presentation follows Vopěnka's textbook, except for some details and the extensional coding Axiom. But that axiom presents more subtle difficulties, since Vopěnka's formulation widely uses objects such as classes of classes. The difficulty is to ward off the danger of a type-stratification. In [S1] Sochor introduces a new membership relation denoted by η, but we think this way of approach is unsatisfactory, since one cannot recover Vopěnka's dictum that each property of objects can be considered as an object and is to be a class.

Another kind of difficulty is presented by the concepts of list and set of generic objects obtained from a list. Also in this case the same danger rises: the type-stratification. By inspection of the mathematical techniques used in [V], it seems possible to reduce the request about the existence to sets (in the sense of Set) such as the «successor».

A12. $(\forall\Phi)(\forall\Psi)(Set(\Phi) \rightarrow (\exists\Sigma)(Set(\Sigma) \wedge (\forall\Theta)(\Theta \in \Sigma \equiv (\Theta \in \Phi \vee \Theta \div \Psi))))$.

Remark 11. By Axiom 12, starting from a Set, we get a Set too. In particular, starting with V-sets x and y, we obtain a Set that is also a Λ -class, since, by Axiom 7, it is a V-set, hence Postulate 2 can be obtained from previous axioms.

By Axiom 2, as made for V-sets using Postulate 1, we can define a new operation denoted, as before, $\%$. A singleton $\{\Psi\}$ is obtained as taking Φ as $\emptyset$, and it is $Set(\{\Psi\})$. Pair $\{\Phi, \Psi\}$ is defined as $\{\Phi\}\%\Psi$, and it is a Set. Unordered n-tuples of objects are defined by repeated applications of the operation $\%$ and are $Sets$. This approach avoids the use of «lists» and sets obtained from them. At this point we can introduce, as in [V], ordered pairs of objects [(1)] by the usual Kuratowski definition:

$$\langle\Phi, \Psi\rangle \ for \ \{\{\Phi\}, \{\Phi, \Psi\}\}.$$

Mathematical importance of V-sets, Λ -classes and ordered pair of Λ -classes, suggest us to introduce a new predicate $\mathscr{F}(\Phi)$, to be read: Φ *is a Fregean object,* for $V(\Phi) \vee \Lambda(\Phi) \vee (\exists X, Y)(\Phi \div \langle X, Y\rangle)$

[(1)] We think that usual set-theoretical operations such that (binary) union and intersection, difference and parts can be introduced for general sets with specific axioms, without loss of consistency.

Remark 12. For every Φ and Ψ, one has $Set(\langle \Phi, \Psi \rangle)$. If $\Phi \div \Psi' \wedge \Psi \div \Psi'$ and the objects Φ, Ψ, Φ' and Ψ' are such that equality relation be reflexive, symmetric and transitive, e.g. for $Sets$, Λ-classes and V-sets, by Axioms 2 and 12, we have $\{\Phi\} \div \{\Phi'\}, \{\Psi\} \div \{\Psi'\}$ and $\{\Phi, \Psi\} \div \{\Phi', \Psi'\}$, hence $(\forall \Phi, \Psi, \Phi', \Psi')((\Phi \div \Phi' \wedge \Psi \div \Psi') \rightarrow \langle \Phi, \Psi \rangle \div \langle \Phi', \Psi' \rangle)$. But not conversely, since we are not able to state that in case $\Sigma \div \Theta$ and $Set(\Theta)$, we have $Set(\Sigma)$.

This is a general problem of Vopěnka's treatment: equality for objects may not be substitutive over other predicates (even over membership!). The suitable axioms are the following

A13. $(\forall \Phi)(\forall \Psi)(\Phi \div \Psi \rightarrow (Set(\Psi) \equiv Set(\Phi))$;

A14. $(\forall \Phi)(\forall \Psi)(\Phi \div \Psi \rightarrow (Cls(\Psi) \equiv Cls(\Phi))$;

A15. $(\forall \Phi)(\forall \Psi)(\Phi \div \Psi \rightarrow (\Lambda(\Psi) \equiv \Lambda(\Phi))$;

Proposition 13. $(\forall \Phi, \Psi, \Phi', \Psi')((\Phi \div \Phi' \wedge \Psi \div \Psi') \equiv \langle \Phi, \Psi \rangle \div \langle \Phi', \Psi' \rangle)$;

Proof. Let $\langle \Phi, \Psi \rangle \div \langle \Phi', \Psi' \rangle$; by Axiom 12, $Set(\langle \Phi, \Psi \rangle) \wedge Set(\langle \Phi', \Psi' \rangle)$, thence, by Axiom 2, $(\forall \Theta)(\Theta \in \langle \Phi, \Psi \rangle \equiv \Theta \in \langle \Phi', \Psi' \rangle)$. Now from $\Theta \in \langle \Phi, \Psi \rangle$ it follows that $\Theta \div \{\Phi\} \vee \Theta \div \{\Phi, \Psi\}$. These two equalities, by Axiom 13, imply $Set(\Theta)$, hence, using Axiom 2, the remaining part of the claim is straightforward.

Proposition 14. *(i)* $(\forall \Phi)(\forall \Psi)(\Phi \div \Psi \wedge \mathscr{F}(\Psi) \rightarrow \mathscr{F}(\Phi))$;

(ii) $(\forall \Phi)(\forall \Psi)(\Phi \div \Psi \wedge V(\Psi) \rightarrow V(\Phi))$;

(iii) $(\forall \Phi, \Psi)(\Phi \div \Psi \wedge \mathscr{F}(\Phi) \rightarrow \Psi \div \Phi)$; $(\forall \Phi, \Psi, \Theta)((\Phi \div \Psi \wedge \Psi \div \Theta \wedge \mathscr{F}(\Phi)) \rightarrow \Phi \div \Theta)$.

Proof. (i) Suppose $\Phi \div \Psi$ and $\mathscr{F}(\Psi)$. In case $V(\Psi)$, by Axiom 7, $Set(\Psi) \wedge \Lambda(\Psi)$. Using Remarks 1 and 7, Axioms 13 and 15, $Set(\Phi) \wedge \Lambda(\Phi)$; from Axiom 7, $V(\Phi)$. In case $\Lambda(\Psi)$, the claim follows from Axiom 15. If $(\exists X, Y)(\Psi \div \langle X, Y \rangle)$, then, by Axioms 12 and 13, $Set(\Psi)$ and $Set(\Phi)$; moreover for $Sets$ equality relation is transitive, by Remark 1. Hence $(\exists X, Y)(\Phi \div \langle X, Y \rangle)$.

(ii) Obtained in the proof of (i).

(iii) We show these properties only if $\mathscr{F}(\Phi)$ means $(\exists X, Y)(\Phi \div \langle X, Y \rangle)$. But in this case we have $Set(\Phi)$ and the claim follows from (i) and Remark 1.

Introducing abstraction principle we said that for suitable objects the full Frege's principle can be assumed. In fact we have

A16. For each formula $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)$:

$$(\forall \Theta)((\varphi(\Theta, \Psi_1, \Psi_2, \ldots, \Psi_n) \wedge \mathscr{F}(\Theta)) \rightarrow (\Theta \in \{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\})).$$

According to the axiom above, we assume a substitutivity condition in the form

A17. For each formula $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)$,

$$(\forall\Theta)(\forall\Psi)((\Theta \div \Psi \wedge \mathscr{F}(\theta) \wedge \Theta \in \{\Phi|\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n)\}) \rightarrow \Psi \in \{\Phi|\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n)\})).$$

Proposition 15. *(i) For each formula* $\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n)$,

$$(\forall\Phi)((\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n) \rightarrow \mathscr{F}(\Phi)) \rightarrow (\forall\Theta)(\Theta \in \{\Phi|\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n)\} \equiv \varphi(\Theta,\Psi_1,\Psi_2,\dots,\Psi_n))).$$

(ii) For each formula $\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n)$,

$$(\forall\Theta)(\Theta \in \{\Phi|\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n) \wedge \mathscr{F}(\Phi)\} \equiv \varphi(\Theta,\Psi_1,\Psi_2,\dots,\Psi_n) \wedge \mathscr{F}(\Theta))).$$

(iii) $(\forall x)(\forall y)(x \div y \rightarrow (\forall Z)(x \in Z \equiv y \in Z))$.
(iv) $(\forall x)(\forall y)(x \div y \rightarrow (\forall z)(x \in z \equiv y \in z))$.

Proof. (i) By Axiom 6, it is enough to prove that $\varphi(\Theta,\Psi_1,\Psi_2,\dots,\Psi_n) \rightarrow \Theta \in \{\Phi\}|\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n)\}$. Suppose $(\forall\Phi)((\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n) \rightarrow \mathscr{F}(\Phi))$, and $\varphi(\Theta,\Psi_1,\Psi_2,\dots,\Psi_n)$; by a tautology and Axiom 16, we obtain the claim.

(ii) It is a particular case of (i), since $(\varphi(\Phi,\Psi_1,\Psi_2,\dots,\Psi_n) \wedge \mathscr{F}(\Phi)) \rightarrow \mathscr{F}(\Phi)$.

(iii) Suppose $x \div y$ and $x \in Z$. By definition $\mathscr{F}(x)$. Using Remark 7 and Axiom 9, $x \in \{w|w \in Z\}$ and by Axiom 17, $y \in \{w|w \in Z\}$, and from Axiom 6, $y \in Z$.

(iv) Trivial consequence of (iii), by Axiom 7.

The last two axioms allow us to treat $\mathscr{F}$-objects in a way very near to the practice of the working mathematician. In particular, it is possible to consider relations and functions whose elements are ordered pairs of Λ-classes. For instance, ordered pairs of classes are used in the definitions of orderings, well-ordering, orderings of type ω and Ω, countable classes, and codable classes. In this approach, the definitions of relations and functions can be generalized to recover the cases of classes, in the sense of Cls. For example: given a relation R, from the extended universe, for each given x one can consider

$$\downarrow x \; for \; \{y|\langle y,x\rangle \in R\}.$$

By Remark 4, one has $\Lambda(\downarrow x)$. Let R^* for $\{\Phi|(\exists x)(\Phi \div \langle x,\downarrow x\rangle)\}$: it is not a Λ-class, but it is a class by Axiom 6. We would like to prove that R^* is a function.

Proposition 16. *For every relation R that is a* Λ*-class, the class* R^* *is a function.*

Proof. Note that $\Lambda(\downarrow x')$ and $\Lambda(\downarrow x")$. By Axiom 6, $\langle y,\Psi\rangle \in R^* \wedge \langle y,\Sigma\rangle \in R^*$ imply $(\exists x')(\langle y,\Psi\rangle \div \langle x',\downarrow x'\rangle) \wedge (\exists x")(\langle y,\Sigma\rangle \div \langle x",\downarrow x"\rangle)$. Using Proposition 13, we get

$y \div x', \Psi \div \downarrow x'$ and $y \div x", \Sigma \div \downarrow x"$. By Remark 2, $x' \div x"$ and by Axiom 15, $\Lambda(\Psi)$ and $\Lambda(\Sigma)$. For all V-sets $z, w, \mathscr{F}(\langle z, w\rangle)$ and $\mathscr{F}(z)$; moreover by Axioms 6 and 16, $z \in\downarrow x' \equiv \langle z, x'\rangle \in R$. From Proposition 13, $\langle z, x'\rangle \div \langle z, x"\rangle$; by Proposition 14 (iii), $\langle z, x"\rangle \in R$ and by the point (i) of the same proposition $z \in\downarrow x"$, thence, from Axiom 9, these two Λ-classes having the same V-sets as elements are equal: $\downarrow x' \div \downarrow x"$. By Remark 7, $\Psi \div \Sigma$.

Remark 17. In the proposition above, a general meaning is assigned to the term «function», and there are examples of relations R (classes from the extended universe) such that R^* may not be a class from the extended universe.

From now on, we shall use traditional mathematical concepts in this general meaning, e.g. $Fnc(\Phi), Rel(\Psi), dom(\Phi), rng(\Sigma)$; moreover set Φ^{-1} for $\{\Theta|(\exists\Sigma)(\exists\Psi)(\Theta\div\langle\Sigma, \Psi\rangle\wedge \langle\Psi, \Sigma\rangle \in \Phi)\}$.

6. THE EXTENSIONAL CODING AXIOM OF TAI

We have prepared all the instruments we needed in order to formulate the extensional coding Axiom, which shall stress the similarity between this and the usual Axiom of choice.

A18. $(\forall R)((Rel(R) \wedge \Lambda(R)) \rightarrow (\exists\Phi)(Cls(\Phi) \wedge Fnc(\Phi) \wedge dom(\Phi) \div dom((R^*)^{-1}) \wedge \Phi \Subset (R^*)^{-1})))$.

Remark 18. Let Φ be the function obtained by Axiom 18, from a given relation R. For all $\Theta \in \Phi, \Theta \in (R^*)^{-1}$; it means that $(\exists\Sigma)(\exists\Psi)(\Theta \div \langle\Sigma, \Psi\rangle \wedge \langle\Psi, \Sigma\rangle \in R^*)$; by Axiom 6 and Proposition 13, $(\exists y)(\Theta \div \langle\downarrow y, y\rangle)$. Moreover if $\Psi \in dom(\Phi)$, there is an object $\Xi(\Xi \div \Phi(\Psi))$ such that $\langle\Psi, \Xi\rangle \in \Phi$, then there exists y such that $\langle\Psi, \Phi(\Psi)\rangle \div \langle\downarrow y, y\rangle$. From Proposition 13, $\Psi \div \downarrow y$, and $\Phi(\Psi) \div y$, thence $\Lambda(\Psi)$, by Axiom 15, and $V(\Phi(\Psi))$, by Proposition 14 (ii). This shows that $\Lambda(rng(\Phi))$.

We now prove that Vopěnka's extensional coding Axiom can be obtained from the formalization presented here.

Proposition 19. *Let $\varphi(X)$ be a property (i.e. a formula) of classes from the extended universe, and let $\langle K, R\rangle$ be a coding pair, with K and R classes from the extended universe and $Rel(R)$, which codes the class $\{X|\varphi(X)\}$. Then there is an extensional coding pair that codes the class $\{X|\varphi(X)\}$.*

Proof. Let Φ be the function obtained by Axiom 18, from the relation R. Consider the class $\Phi^\sim$ defined as $\{x|(\exists y)(\exists Z)(Z \in dom(\Phi) \wedge x \div \langle y, \phi(Z)\rangle \wedge y \in Z)\}$. By Remark 4, one has that $\Lambda(\Phi^\sim)$ and by definition $Rel(\Phi^\sim)$. Since $\langle K, R\rangle$ is a coding pair, which codes the class $\{X|\varphi(X)\}$ it means that for any class Z such that $\varphi(Z)$, there is a $y \in K$ such that $Z \div R^*(y)$. In other words, $\langle y, Z\rangle \in R^*$, since, by Axiom 16, $\mathscr{F}(\langle y, Z\rangle)$. Recall that $\mathscr{F}(\langle Z, y\rangle)$, too. By definition of inverse relation ad Axiom 16, $\langle Z, y\rangle \in (R^*)^{-1}$. This condition implies that $Z \in dom(\Phi)$, hence $\langle Z, \Phi(Z)\rangle \in \Phi$. From $\Phi \Subset (R^*)^{-1}$ it follows that $\langle Z, \Phi(Z)\rangle \in (R^*)^{-1}$ and $\langle\Phi(Z), Z\rangle \in R^*$. As in Proposition 16, $\langle\Phi(Z), Z\rangle \in R^*$ if and only if there is an x such that $x \div \Phi(Z)$ and $Z \div \downarrow x$. By Remark 18, $\Lambda(rng(\Phi))$. The pair $\langle rng(\Phi), \Phi^{\sim}\rangle$ is a coding pair. Indeed, for every class Z with $\varphi(Z)$, $\langle Z, \Phi(Z)\rangle \in \Phi$, thus for any $y \in Z$, $\langle y, \Phi(Z)\rangle \in \Phi^{\sim}$. This relation can be written as $y \in (\Phi^{\sim})^{*}(\Phi(Z))$. Recall that $\Lambda((\Phi^{\sim})^{*}(\Phi(Z)))$; hence $Z \subseteq (\Phi^{\sim})^{*}(\Phi(Z))$. But if $z \in (\Phi^{\sim})^{*}(\Phi(Z))$, then $\langle z, \Phi(Z)\rangle \in \Phi^{\sim}$, and $z \in Z$. In conclusion, by Axiom 9, $Z \div (\Phi^{\sim})^{*}(\Phi(Z))$, and $\langle rng(\Phi), \Phi^{\sim}\rangle$ codes $\{X|\varphi(X)\}$.

If Z', Z'' are such that $\varphi(Z')$, $\varphi(Z'')$ and $(\Phi^{\sim})^{*}(\Phi(Z')) \div (\Phi^{\sim})^{*}(\Phi(Z''))$, then trivially $Z' \div (\Phi^{\sim})^{*}(\Phi(Z')) \div (\Phi^{\sim})^{*}(\Phi(Z'')) \div Z''$: thus the coding pair is an extensional coding of $\{X|\varphi(X)\}$, which proves the assertion.

Proposition 19. Axioms 10 and 11, are used in TAI to prove, as in [V], that there exists a Λ-class that is a well-ordering of V.

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