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ON TYPES OF POLYNOMIALS AND HOLOMORPHIC FUNCTIONS ON BANACH SPACES

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1. INTRODUCTION

In 1966 L. Nachbin introduced the notion of a holomorphy type to consider certain types of polynomials (f.i. compact, nuclear, absolutely summing) in a uniform way [7, 8]. Holomorphy types with special properties were studied by S. Dineen in 1971 (cf. [4]). Using the well developped theory of linear operator ideals ([9]) various methods of the construction of holomorphy types were presented in [1].

These methods work also in the case of p-normed and quasinormed ideals. After introducing the basic notions the factorization method will be studied here in more details. The main result will be the Theorem 5.1.

Of special interest are multilinear operators of type $\mathscr{L}(\mathscr{S}_p)$ where $\mathscr{S}_p$ denotes the usual Schatten class of linear operators in Hilbert spaces. These multilinear operators can be characterized by the summablity of their eigenvalues or some other sort of associated sequences of reals (Proposition 4.1). The results will be applied to multilinear operators defined by kernels or convolutions and to holomorphic functions of ideal type.

2. CONTINUOUS POLYNOMIALS AND HOLOMORPHIC MAPPINGS

Let us recall some basic definitions of the theory of polynomials and holomorphic functions. We will use the notations of [5, 8]. Let $E, F, G, \ldots$ be complex Banach spaces. By $\mathscr{L}(E_m, \ldots, E_1; F)$ we denote the Banach space of *m-linear continuous mappings from $E_m \times \ldots \times E_1$ into F* equipped with the topology of uniform convergence on the product of the closed unit balls $U_{E_m} \times \ldots \times U_{E_1}$. We write $\mathscr{L}(^mE; F)$ instead of $\mathscr{L}(E, \ldots, E; F)$. A mapping $P : E \to F$ is called to be a *continuous m-homogeneous polynomial* if there is an $A \in \mathscr{L}(^mE; F)$ such that

$$P(x) = Ax \ldots x = A(x, \ldots, x) \quad for\ all \quad x \in E.$$

In this case we write $P = \widehat{A}$. The functional

$$||P|| = \sup_{||x|| \leq 1} ||P(x)||$$

defines a norm on the space $\mathscr{P}(^mE; F)$ of all continuous m-homogeneous polynomials from E into F. For $m = 0$ we put $\mathscr{P}(^0E; F) := F$. Let (P_m) be any sequence of polynomials of increasing degree with $P_m \in \mathscr{P}({}^m E; F), m \in \mathbf{N}_0$. Then the expression

$$\sum_{m=0}^{\infty} P_m(x - x_0)$$

is called to be the *power series at $x_0 \in E$ with the coefficients P_m*. Its *radius of convergence* is the largest $R, 0 \leq R \leq \infty$, such that the power series is uniformly convergent on every closed ball $\overline{U}_E(x_0, r)$ for $0 \leq r < R$. The radius of convergence can be computed by Cauchy-Hadamard's formula

$$R = (\lim_{m\to\infty} \sup ||P_m||^{1/m})^{-1}.$$

A mapping $f : E \to F$ is called to be *holomorphic at $x_0 \in E$* if there is a power series such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - x_0)$$

converges uniformly on some ball $U_E(x_0, r)$.

By $\mathscr{H}(E; F)$ we denote the vector space of all functions which are holomorphic in each point of E. Let us remark that on each infinite dimensional Banach space there are entire functions with finite radius of convergence.

The radius of convergence is infinite if and only if f maps bounded sets into bounded sets. Such functions are called *uniformly bounded*, and we denote by $\mathscr{H}_{ub}(E; F)$ the space of all of such functions.

3. OPERATOR IDEALS

Now let us study more special properties of polynomials and holomorphic functions as uniform boundedness. For this, Nachbin introduced the notion of holomorphy type ([4, 7, 8]). In our approach here we will use the well developped theory of linear operator ideals to express additional properties of functions and polynomials.

First, let us recall the definition of a p-normed operator ideal. For details we refer to [9]. By $\mathscr{L}$ we denote the class of all continuous linear operators acting between arbitrary Banach spaces. A subclass $\mathscr{A}$ (together with a functional $|| \cdot |\mathscr{A}|| : \mathscr{A} \to \mathbf{R}_+$) is called to be a ($p$-normed) operator ideal, if the components $\mathscr{A}(E; F) := \mathscr{A} \cap \mathscr{L}(E; F)$ fulfill the following conditions:

$I) I_{\mathbb{C}} \in \mathscr{A}$ (and $\|I_{\mathbb{C}}|\mathscr{A}\| = 1$).

$II) S, T \in \mathscr{A}(E,F)$ implies $S + T \in \mathscr{A}(E;F)$

(and $\|S + T|\mathscr{A}\|^p \le \|S|\mathscr{A}\|^p + \|T|\mathscr{A}\|^p$).

$III) R \in \mathscr{L}(F;F_0), S \in \mathscr{A}(E;F)$, and $T \in L(E_0;E)$

implies $RST \in \mathscr{A}(E_0;F_0)$

(and $\|RST|\mathscr{A}\| \le \|R\|\ \|S|\mathscr{A}\|\ \|T\|$).

$IV)$ (All components are complete with respect to the topology generated by $\|.|\mathscr{A}\|$).

Definition 3.1. *Let $\mathscr{A}$ be a p-normed operator ideal. A multilinear mapping $M \in L(E_m, \ldots, E_1; F)$ is called to be of type $\mathscr{L}(\mathscr{A})$ if there are operators $T_i \in \mathscr{A}(E_i; G_i)$ and a mapping $M_0 \in \mathscr{L}(G_m, \ldots, G_1; F)$ such that*

$$M = M_0(T_m, \ldots, T_1).$$

In this case we set $\|M|\mathscr{L}(\mathscr{A})\| := \inf \|M_0\|\ \|T_m|\mathscr{A}\| \ldots \|T_1|\mathscr{A}\|$, where the infimum is taken over all possible factorizations of this kind. By $\mathscr{L}(\mathscr{A})(E_m, \ldots, E_1; F)$ we denote the set of all multilinear mappings of $\mathscr{L}(\mathscr{A})$ from $E_m \times \ldots \times E_1$ into F.

The type $\mathscr{L}(\mathscr{A})$ is an example for a so-called mltiideal introduced by A. Pietsch in [10] (cf. [1, 3]).

Proposition 3.1. *Let $\mathscr{A}$ be a p-normed operator ideal. Then $\mathscr{L}(\mathscr{A})(E_m, \ldots, E_1; F)$ is a complete p/m-normed space.*

Proof. In [2] it has been proven that $\mathscr{L}(\mathscr{A})(E_m, \ldots, E_1; F)$ is a quasinormed Banach space. Here we will prove that $\| \cdot |\mathscr{L}(\mathscr{A})\|$ is even a p/m-norm. Suppose that $M, N \in \mathscr{L}(\mathscr{A})(E_m, \ldots, E_1; F)$. Corresponding to $\varepsilon > 0$ there are $M_0 \in \mathscr{L}(G_{1m}, \ldots, G_{11}; F)$, $N_0 \in \mathscr{L}(G_{2m}, \ldots, G_{21}; F), S_i \in \mathscr{A}(E_i; G_{1i})$, and $T_i \in \mathscr{A}(E_i; G_{2i})$ such that $\|M_0\|, \|N_0\| \le 1 + \varepsilon, \|S_i|\mathscr{A}\| \le \|M|\mathscr{L}(\mathscr{A})\|^{1/m}, \|T_i|\mathscr{A}\| \le \|N|\mathscr{L}(\mathscr{A})\|^{1/m}, M = M_0(S_m, \ldots, S_1)$, and $N = N_0(T_m, \ldots, T_1)$. We put $E_{oi} := G_{1i} \times G_{2i}, \|(x_{1i}, x_{2i})\| = \|x_{1i}\| + \|x_{2i}\|$, where $x_{ji} \in G_{ji}$. Furthermore, $R_i := i_{1i}S_i + i_{2i}T_i, A := M_0(\pi_{1m}, \ldots, \pi_{11}) + N_0(\pi_{2m}, \ldots, \pi_2)$ where i_{ji} and π_{ji} are the canonical injections from G_{ji} into E_{oi} and projections from E_{oi} onto G_{ji}, respectively. Then we obtain $R_i \in \mathscr{A}(E_i; E_{oi}), \|R_i|\mathscr{A}\|^p \le \|S_i|\mathscr{A}\|^p + \|T_i|\mathscr{A}\|^p, A \in \mathscr{L}(E_{om}, \ldots, E_{o1}; F), \|A\| \le 1 + \varepsilon$, and $M + N = A(R_m, \ldots, R_1)$. Finally, it follows that $\|M + N|\mathscr{L}(\mathscr{A})\|^{p/m} = \|A(R_m, \ldots, R_1)|\mathscr{L}(\mathscr{A})\|^{p/m} \le$

$(1+\varepsilon)^{p/m}(\|S_m|\mathscr{A}\|^p + \|T_m|\mathscr{A}\|^p)^{1/m}\ldots(\|S_1|\mathscr{A}\|^p + \|T_1|\mathscr{A}\|^p)^{1/m} \leq (1+\varepsilon)^{p/m}(\|M|\mathscr{L}(\mathscr{A})\|^{p/m} + \|N|\mathscr{L}(\mathscr{A})\|^{p/m})$. This concludes the proof, since ε is arbitrary.

Definition 3.2. *Let $\mathscr{A}$ be a p-normed operator ideal. A polynomial $P \in \mathscr{P}(^mE;F)$ is called to be of type $\mathscr{L}(\mathscr{A})$ if there is a mapping $M \in \mathscr{L}(\mathscr{A})(^mE;F)$ such that $P = \widehat{M}$. In this case we put $\|P|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\| = \inf_{\hat{M}=P}\|M|\mathscr{L}(\mathscr{A})\|$.*

By $\mathscr{P}_{\mathscr{L}(\mathscr{A})}(^mE;F)$ we denote the complete p/m-normed space of all *m-homogeneous polynomials of type $\mathscr{L}(\mathscr{A})$ from E into F.*

An easy computation shows that the single generated multilinear mapping $M = a_m \otimes \ldots \otimes a_1 \otimes y, a_i \in E', y \in F$, belongs to $\mathscr{L}(\mathscr{A})(^mE;F)$ for every operator ideal $\mathscr{A}$. Moreover, we have $\|M\| = \|M|\mathscr{L}(\mathscr{A})\|$ by $\|I_{\mathbb{C}}|\mathscr{A}\| = 1$.

The following statement shows that each polynomial of type $\mathscr{L}(\mathscr{A})$ is the superposition of a polynomial and a single operator $R \in \mathscr{A}$ (cf. [6]).

Lemma 3.1. *Let $\mathscr{A}$ be a p-normed operator ideal. Then corresponding to any $P \in \mathscr{P}_{\mathscr{L}(\mathscr{A})}(^mE;F)$ and $\varepsilon > 0$ there are an operator $T \in \mathscr{A}(E;G)$ and a polynomial $Q \in \mathscr{P}(^mG;F)$ such that $P = QT, \|T|\mathscr{A}\| \leq 1$ and $\|Q\| \leq (1+\varepsilon)m^{(1/p-1)m}\|P|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|$.*

Proof. Suppose that $P \in \mathscr{P}_{\mathscr{L}(\mathscr{A})}(^mE;F)$ and $\varepsilon > 0$. Then there is an $M \in \mathscr{L}(\mathscr{A})(^mE;F)$ such that $\widehat{M} = P$ and $\|M|\mathscr{L}(\mathscr{A})\| \leq (1+\varepsilon)\|P|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|$. We choose $N \in \mathscr{L}(G_m,\ldots,G_1;F)$ and $T_i \in \mathscr{A}(E;G_i)$ with $N(T_m,\ldots,T_1) = M, \|N\| \leq (1+\varepsilon)\|P|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|$, and $\|T_i|\mathscr{A}\| \leq 1$. Let be $G := l_1^m(G_m,\ldots,G_1), T := m^{-1/p}\sum_{k=1}^m i_k T_k$, and $Q := m^{m/p}(N(\pi_m,\ldots,\pi_1))^\wedge$, where $i_k : G_k \to G$ and $\pi_k : G \to G_k$ are the canonical mappings. From this we obtain

$$T \in \mathscr{A}(E;G) \quad and \quad \|T|\mathscr{A}\|^p \leq m^{-1}\sum_{k=1}^m \|T_k|\mathscr{A}\|^p \leq 1,$$

$$\begin{aligned}\|Q\| &= \sup_{\Sigma\|x_i\|\leq 1} m^{m/p}\|(N\pi_m,\ldots,\pi_1))^\wedge((x_m,\ldots,x_1))\| \\ &= m^{m/p}\sup_{\Sigma\|x_i\|\leq 1}\|Nx_m\ldots x_1\| \leq m^{m/p}\|N\|\sup_{\Sigma\|x_i\|\leq 1}\|x_m\|\ldots\|x_1\| \\ &= m^{(1/p-1)m}\|N\| \leq (1+\varepsilon)m^{(1/p-1)m}\|P|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|.\end{aligned}$$

Moreover, we get

$$QT = m^{m/p}(N(\pi_m,\ldots,\pi_1))^\wedge T = m^{m/p}[N(m^{-1/p}T_m,\ldots,m^{-1/p}T_1)]^\wedge = \widehat{M} = P.$$

This concludes the proof.

4. MULTILINEAR OPERATORS ON HILBERT SPACES

In this section we study polynomials of type $\mathcal{L}(\mathcal{S}_p)$. This class is a multilinear generalization of the Schatten class operators $\mathcal{S}_p$. Recall that a compact operator mapping a Hilbert space H into some other Hilbert space K belongs to $\mathcal{S}_p$, iff the sequence of eigenvalues of $\sqrt{T^*T}$ is an element of $l_p (0 < p < \infty)$. In this case we have $\|T|\mathcal{S}_p\| = \|(\lambda_n(\sqrt{T^*T}))|l_p\|$.

Suppose now $0 < p \leq 2$. By [9, 15.5.4], an operator $T \in \mathcal{L}(H; K)$ belongs to $\mathcal{S}_p$ if and only if there is some orthonormal basis $\{e_\alpha\}$ of H such that $(\|Te_\alpha\|) \in l_p(A)$. In this case we have

$$\|T|\mathcal{S}_p\| \leq \|(\|Te_\alpha\|)|l_p(A)\|.$$

For each $x \in H$ let x^* be the functional on H defined via scalar product $\langle y, x^* \rangle = (y, x)$.

Proposition 4.1. *Let $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ be orthonormal bases of the Hilbert spaces H and K, respectively, and suppose that $M \in \mathcal{L}(^mH; K)$. If $((Me_{i_m} \dots e_{i_1}, f_j)) \in l_{p/m}(I^m \times J)$ then $M \in \mathcal{L}(\mathcal{S}_p)(^mH; K)$ holds true for $0 < p \leq 2$.*

Additionally, we have

$$\|M|\mathcal{L}(\mathcal{S}_p)\| \leq \|((Me_{i_m} \dots e_{i_1}, f_j))|l_{p/m}\|.$$

Proof. By A we denote the index set $I^m \times J$. We put $\lambda_\alpha := (Me_{i_m} \dots e_{i_1}, f_j)$ for all $\alpha = (i_m, \dots, i_1, j) \in A$. If $(\lambda_\alpha) \in l_{p/m}(A)$ then we have $(\mu_\alpha) := (\lambda_\alpha^{1/m}) \in l_p(A)$. Let $(g_\alpha)_{\alpha\in A}$ be an orthonormal basis of $l_2(A)$. By $\pi_k : A \to I$ we denote the map with $\pi_k(i_m, \dots, i_1, j) := i_k (k = 1, \dots, m)$. We define operators $T_k : H \to l_2(A)$ via $T_k := \sum_{\alpha\in A} \mu_\alpha e^*_{\pi_k(\alpha)} \otimes g_\alpha$.

Since $(\|T_k^* g_\alpha\|) = (|\mu_\alpha|) \in l_p(A)$, we get $T_k^* \in \mathcal{S}_p$. This yields $T_k \in \mathcal{S}_p(H; l_2(A))$ and $\|T_k|\mathcal{S}_p\| = \|T_k^*|\mathcal{S}_p\| \leq \|(\mu_\alpha)|l_p(A)\|$.

Furthermore we set $N := \sum_{\alpha\in A} g_\alpha^* \otimes \dots \otimes g_\alpha^* \otimes f_{\pi(\alpha)}$, where $\pi(i_m, \dots, i_1, j) := j$ for

$(i_m, \dots, i_1, j) \in A$. Since $N \in \mathscr{L}({}^m l_2(A); K)$ and $\|N\| \le 1$, we get

$$\begin{aligned} N(T_m, \dots, T_1) &= \left(\sum_{\alpha\in A} g_\alpha^* \otimes \dots \otimes g_\alpha^* \otimes f_{\pi(\alpha)}\right) \\ &= \left(\sum_{\alpha\in A} \mu_\alpha e^*_{\pi_m(\alpha)} \otimes g_\alpha, \dots, \sum_{\alpha\in A} \mu_\alpha e^*_{\pi_1(\alpha)} \otimes g_\alpha\right) \\ &= \sum_{\alpha\in A} (\mu_\alpha e^*_{\pi_m(\alpha)}) \otimes \dots \otimes (\mu_\alpha e^*_{\pi_1(\alpha)}) \otimes f_{\pi(\alpha)} \\ &= \sum_{\alpha\in A} \mu_\alpha^m e^*_{\pi_m(\alpha)} \otimes \dots \otimes e^*_{\pi_1(\alpha)} \otimes f_{\pi(\alpha)} = M. \end{aligned}$$

Moreover,

$$\|T_k|\mathscr{S}_p\| \le \|(\mu_\alpha)l_p\| = \|(\lambda_\alpha)|l_{p/m}\|^{1/p},$$

hence $M \in \mathscr{L}(\mathscr{S}_p)({}^m H; K)$ and

$$\|M|\mathscr{L}(\mathscr{S}_p)\| \le \|N\|\|T_m|\mathscr{S}_p\| \dots \|T_1|\mathscr{S}_p\| \le \|(\lambda_\alpha)|l_{p/m}\|.$$

This concludes the proof.

Let us remark that for diagonal multilinear operators this condition is also necessary. Now, we consider multilinear integral mappings on $L_2[0, 2\pi]$ and ask the question under which conditions they belong to $\mathscr{L}(\mathscr{S}_p)$. The following corollary gives a sufficient condition. Let f be a measurable function on $[0, 2\pi]^{m+1}$. If

$$\int_0^{2\pi} \dots \int_0^{2\pi} f(t_0, t_1, \dots, t_m) x_m(t_m) \dots x_1(t_1) dt_m \dots dt_1$$

defines a bounded multilinear mapping from $(L_2[0, 2\pi])^m$ into $L_2[0, 2\pi]$ then we denote this mapping by M_f.

Corollary 4.1. *Let f be a measurable function on $[0, 2\pi]^{m+1}, 0 < p \le 2$, and suppose that the sequence of its Fourier coefficients*

$$\mathscr{F}f(k) := \left(\frac{1}{2\pi}\right)^{\frac{m+1}{2}} \int_0^{2\pi} \dots \int_0^{2\pi} f(t_0, \dots, t_m) e^{-i\langle k,t\rangle} dt_0 \dots dt_m$$

belongs to $l_{p/m}(\mathbf{Z}^{m+1})$. *Then* $M_f \in \mathscr{L}(\mathscr{S}_p)({}^mL_2[0,2\pi];L_2[0,2\pi])$.

Proof. The assertion follows from Proposition 4.1 since

$$\begin{aligned}
&(M_f e_{k_m}\dots e_{k_1}, e_{k_0})\\
&= \int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi} f(t_0,t_1,\dots,t_m)e_{k_m}(t_m)\dots e_{k_1}(t_1)\,dt_1\dots dt_m\overline{e_{k_0}(t_0)}\,dt_0\\
&= \left(\frac{1}{2\pi}\right)^{\frac{m+1}{2}}\int_0^{2\pi}\dots\int_0^{2\pi} f(t_0,\dots,t_m)e^{ik_mt_m}\dots e^{ik_1t_1}e^{-ik_0t_0}\,dt_m\dots dt_0\\
&= \mathscr{F}f(k_0,-k_1,\dots,-k_m).
\end{aligned}$$

In the following we consider multilinear mappings defined by convolution. Corresponding to $f \in L_1[0,2\pi]$ we set

$$\begin{aligned}
&(C_f x_m\dots x_1)(t_0)\\
&:= \left(\frac{1}{2\pi}\right)^{\frac{m-1}{2}}\int_0^{2\pi}\dots\int_0^{2\pi} f(t_0+t_1+\dots+t_m)x(t_m)\dots x(t_1)\,dt_m\dots dt_1.
\end{aligned}$$

Then a standard computation shows

$$C_f \in \mathscr{L}({}^mL_2[0,2\pi];L_2[0,2\pi]) \quad \textit{and} \quad \|C_f\| \le \|f|L_1\|.$$

The verification of the next lemma is easy

Lemma 4.1. *Let* $f \in L_1[0,2\pi]$. *Then the Fourier coefficients of* $g(t_0,\dots,t_m) = (2\pi)^{(1-m)/2}f(t_0+\dots+t_m)$ *are* $\mathscr{F}g(k) = \sqrt{2\pi}\mathscr{F}f(k_0)$ *for* $k = (k_0,\dots,k_0)$, *and* $\mathscr{F}g(k) = 0$ *elsewhere.*

By $W_r^l[0,2\pi]$ we denote the Sobolev spaces of periodic functions on the real line, where $l \in \mathbf{N}$ and $1 \le r \le \infty$ (cf. [11], 6.5.).

Proposition 4.2. *Let* $0 < p \le 2$, $1 < r < \infty$, *and* $l + \min\left(\frac{1}{r'},\frac{1}{2}\right) > \frac{m}{p}$. *Then* $f \in W_r^l[0,2\pi]$ *implies* $C_f \in \mathscr{L}(\mathscr{S}_p)({}^mL_2[0,2\pi];L_2[0,2\pi])$.

Proof. Suppose that the assumptions are fulfilled. Using [11], 6.5.11., it follows that $(\mathscr{F}f(k)) \in l_{s,w}(\mathbf{Z})$ with $\frac{1}{s} = l + \min\left(\frac{1}{r'},\frac{1}{2}\right)$ and $w = \min(r,2)$, where $l_{s,w}(\mathbf{Z})$

is the Lorentz sequence space. Since $s_1 < s_2$ implies $l_{s_1,w_1} \subseteq l_{s_2,w_2}$ for arbitrary w_1, w_2, we have $(\mathscr{F}f(k)) \in l_{s,w}(\mathbf{Z}) \subseteq l_{p/m,p/m}(\mathbf{Z}) = l_{p/m}(\mathbf{Z})$. With the notations as in Lemma 4.1., we get

$$\sum_{k_0 \in \mathbf{Z}} \cdots \sum_{k_m \in \mathbf{Z}} |\mathscr{F}g(k)|^{p/m} = (2\pi)^{p/2m} \sum_{k_0 \in \mathbf{Z}} |\mathscr{F}f(k_0)|^{p/m} < \infty.$$

Now, the statement follows from Corollary 4.1.

Let us formulate this statement especially for $r = 2$.

Corollary 4.2. *Let* $0 < p \leq 2$ *and* $l + \frac{1}{2} > \frac{p}{m}$. *Then* $f \in W_2^l[0,2\pi]$ *implies* $C_f \in \mathscr{L}(\mathscr{S}_p)({}^mL_2[0,2\pi]; L_2[0,2\pi])$.

5. HOLOMORPHIC FUNCTIONS OF TYPE $\mathscr{L}(\mathscr{A})$

In this section the main question is the following. Given any entire holomorphic function f with the Taylor expansions

$$f(x) = \sum_{m=0}^{\infty} P_m(x - x_0).$$

Suppose that all P_m are of type $\mathscr{L}(\mathscr{A})$ for some operator ideal $\mathscr{A}$. Does f admit a factorization $f = g \cdot T$ with some linear operator $T \in \mathscr{A}$ and some holomorphic function g? Does this property depend on the centre of this series? We will answer these questions in the following.

An operator ideal is called to be *closed* if it is a normed ideal with respect to the uniform norm. For such ideals the questions asked above have been answered by S. Geiß in [6]. Here we are interested in ideals defined by more geometric properties. This includes for instance the $\mathscr{S}_p$-ideals. These ideals are of course not closed because they contain all finite rank operators. But the uniform closure of this subset of operators give the set of all compact operators.

As we are dealing with norms quite different from the uniform norm, we have to use completely other methods than those used by Geiß to treat the problem.

Definition 5.1. *Let* $\mathscr{A}$ *be a* p*-normed operator ideal. A holomorphic mapping* f *from* E *into* F *is called to be of type* $\mathscr{L}(\mathscr{A})$ *at* $x_0 \in E$ *if*

$$P_m \in \mathscr{P}_{\mathscr{L}(\mathscr{A})}({}^mE; F) \quad \textit{for all} \quad m \in \mathbf{N}_0$$

and

$$\limsup_{m\to\infty} \|P_m|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|^{1/m} < \infty,$$

where

$$f(x) = \sum_{m=0}^{\infty} P_m(x - x_0) \quad on \quad U(x_0, r)$$

for some $r > 0$.

Example 5.1. Let $E = l_1, F = \mathbb{C}$. By $\mathscr{R}$ we denote the normed operator ideal of absolutely summing operators.

Then $f((x_i)) = \sum_{i=1}^{\infty} \frac{x_i}{1 - x_i}$ defines a holomorphic function on the open unit ball of l_1 and f is of type $\mathscr{L}(\mathscr{R})$ at 0.

To show this, we put

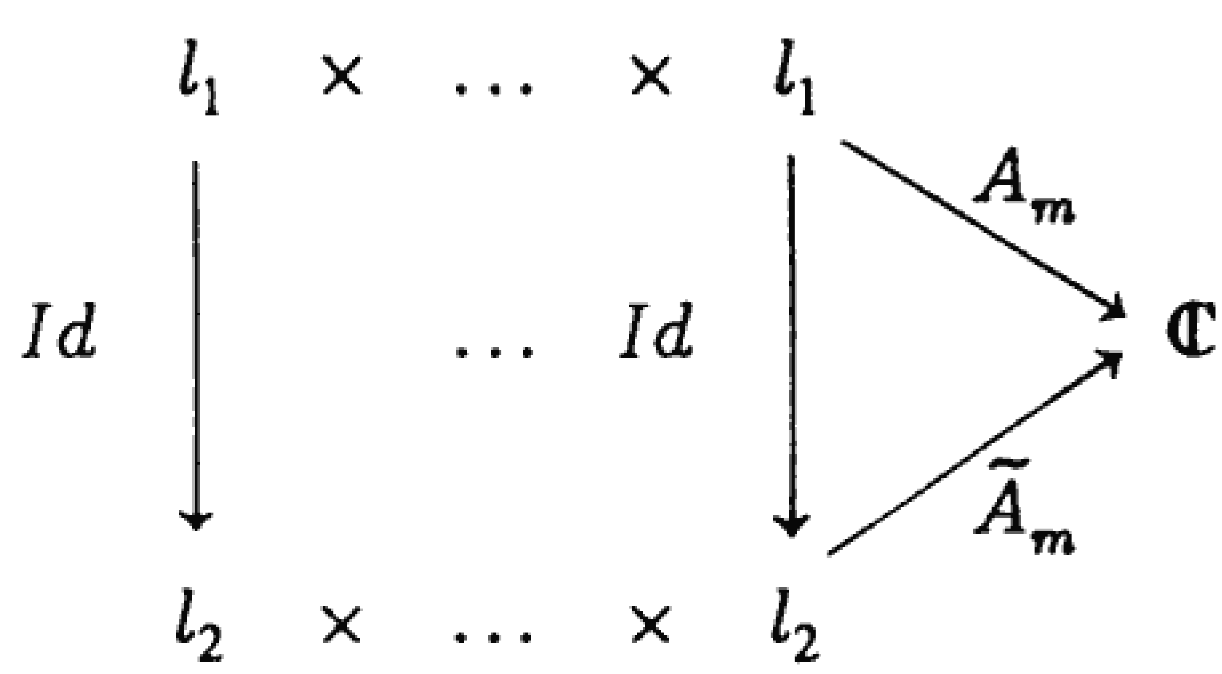
$$A_m := \sum_{i=1}^{\infty} e_i \otimes \ldots \otimes e_i \quad for \quad m \in \mathbf{N}.$$

Hence we get

$$A_m \in \mathscr{L}(^m l_1; \mathbb{C}) \quad and \quad ||A_m|| = 1.$$

Since $A_1 \in l_1' = l_\infty$ we have $A_1 \in \mathscr{L}(\mathscr{R})(l_1; \mathbb{C})$.

For $m > 1, A_m$ admits a factorization



Using the facts that $||\widetilde{A}_m|| = 1$ and $Id \in \mathscr{R}(l_1; l_2)$ with $||Id|\mathscr{R}(l_1; l_2)|| = c_G$, where c_G is the Grothendieck constant (cf. [11], 1.6.4), we have $A_m \in \mathscr{L}(\mathscr{R})(^m l_1; \mathbb{C})$ and $||A_m|\mathscr{L}(\mathscr{R})|| \leq c_G^m$, hence $\widehat{A}_m \in \mathscr{P}_{\mathscr{L}(\mathscr{R})}(^m l_1; \mathbb{C})$ and $||\widehat{A}_m|\mathscr{P}_{\mathscr{L}(\mathscr{R})}|| \leq c_G^m$.

So we have

$$g := \sum_{m=1}^{\infty} \widehat{A}_m \in \mathscr{H}(U_{l_1}; \mathbb{C})$$

and g is of type $\mathscr{L}(\mathscr{R})$ in 0.

Finally, for $x \in U_{l_1}$ the following holds true

$$g(x) = \sum_{m=1}^{\infty} \widehat{A}_m(x) = \sum_{m=1}^{\infty} \sum_{i=1}^{\infty} x_i^m = \sum_{i=1}^{\infty} \frac{x_i}{1 - x_i} = f(x).$$

Theorem 5.1. *Let $\mathcal{A}$ be a p-normed operator ideal for some $0 < p \leq 1$ and let $f \in \mathcal{H}(E;F)$. Suppose that there is an $x_0 \in E$ such that f is of type $\mathcal{L}(\mathcal{A})$ at x_0 and that*

$$(n^{(1-p)/p}\|P_n|\mathcal{P}_{\mathcal{L}(\mathcal{A})}\|^{1/n}) \in l_p.$$

This implies

(1) *there are an operator $T \in \mathcal{A}(E;G)$ and a mapping $g \in \mathcal{H}_{ub}(G;F)$ such that $f = g \circ T$.*

(2) *f is of type $\mathcal{L}(\mathcal{A})$ at every $x_1 \in E$.*

Proof. We put

$$\alpha_n := \|P_n|\mathcal{P}_{\mathcal{L}(\mathcal{A})}\|^{1/n}.$$

By assumption $(n^{(1-p)/p}\alpha_n) \in l_p$. Now, choose some sequences $(w_n) \in l_p, w_n > 0$, and $(z_n) \in c_0$ satisfying

$$w_n z_n \geq n^{(1-p)p}\alpha_n.$$

Corresponding to Lemma 3.1. there are $T_n \in \mathcal{A}(E;G_n)$ and $Q_n \in \mathcal{P}(^nG_n;F)$ with $P_n = Q_nT_n, \|T_n|\mathcal{A}\| \leq w_n$, and $\|Q_n\| \leq w_n^{-n}2n^{(1-p)n/p}\alpha_n^n$. We set $G := l_\infty(G_n)$ and $T := \sum_{n=1}^{\infty} i_nT_n$. Since $\sum_{n=1}^{\infty} \|i_nT_n|\mathcal{A}\|^p \leq \sum_{n=1}^{\infty} w_n^p < \infty$, we have $T \in \mathcal{A}(E;G)$.

Next, by $g(y) = f(x_0) + \sum_{n=1}^{\infty} Q_n(\pi_n(y - Tx_0))$ we define a power series from G into F. Since

$$\begin{aligned}(gT)(x) &= f(x_0) + \sum_{n=1}^{\infty} Q_n(\pi_n(Tx - Tx_0)) \\ &= f(x_0) + Q_n\left(\pi_n\left(\sum_{k=1}^{\infty} i_kT_k\right)(x - x_0)\right) \\ &= f(x_0) + \sum_{n=1}^{\infty} P_n(x - x_0) = f(x)\end{aligned}$$

and $(\|Q_n\pi_n\|^{1/n})$ tends to zero, (1) is proved.

To verify (2) we consider a factorization $f = gT$ as in (1). Since $g \in \mathcal{H}_{ub}(G;F)$, we get for any $y \in E$

$$g(y) = \sum_{m=0}^{\infty} \widehat{M}_m(y - Tx_1),$$

where M_m is a symmetric mapping belonging to $L({}^mG; F)$. Using the Taylor expansion of f in x_1, we get

$$\sum_{m=0}^{\infty} \overline{P}_m(x - x_1) = f(x) = (gT)(x) = \sum_{m=0}^{\infty} \widehat{M}_m(T(x - x_1)).$$

This implies $\overline{P}_m = \widehat{M}_m T$, i.e. $\overline{P}_m \in \mathscr{P}_{\mathscr{L}(\mathscr{A})}({}^mE; F)$ and

$$\begin{aligned}\lim_{m\to\infty} \sup \|\overline{P}_m|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|^{1/m} &= \lim_{m\to\infty} \sup \|M_m(T,\dots,T)|\mathscr{L}(\mathscr{A})\|^{1/m} \\ &= \lim_{m\to\infty} \sup \|M_m\|^{1/m}\|T|\mathscr{A}\| = 0.\end{aligned}$$

This concludes the proof.

In the normed case $(p = 1)$ this condition is especially simple

$$(a) \qquad \sum_{n=0}^{\infty} \|P_n|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|^{1/n} < \infty.$$

Example 5.2. Let $1 \leq q \leq \infty, E = l_q, F = \mathbb{C}$, and $\mathscr{A}$ be any normed operator ideal.

We put $P_m := \frac{1}{m^{2m}} [a_m \otimes \dots \otimes a_m]^\wedge$, where $a_m \in l_q'$ and $\|a_m\| = 1$.

From paragraph 3.2 it follows $P_m \in \mathscr{P}_{\mathscr{L}(\mathscr{A})}({}^mE; \mathbb{C})$ and $\|P_m|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\| = \frac{1}{m^{2m}}$. If f is defined by $f(x) = \sum_{m=0}^{\infty} P_m(x)$ then we get $f \in \mathscr{H}(l_q; \mathbb{C})$ and f is of type $\mathscr{L}(\mathscr{A})$ at 0. Since

$$(\|P_m|\mathscr{P}_{\mathscr{L}(\mathscr{A})}\|^{1/m}) \in l_1$$

the condition (a) is fulfilled. Hence there are an operator $T \in \mathscr{A}(l_q; G)$ and a function $g \in \mathscr{H}_{ub}(G; \mathbb{C})$ with $f = g \circ T$. This shows that f is of type $\mathscr{L}(\mathscr{A})$ at any point of l_q.

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